Chapter 1

Introduction

In this chapter we introduce the importance and backgrounds of the thesis problems as follows: Firstly, we present the backgrounds of fixed point theory and quasi-contractive type mappings. Secondly, we introduce some iterative methods for finding a fixed point of nonlinear mappings. Finally, we recall the background of rate of convergence.

There are problems in many branches of science and applied science, the researchers in these fields convert these problems into mathematical models. Most of mathematical models are represented by equations or inequalities. So, there are mainly two important questions arising for solving those problems. The first one is the existence of solutions of those problems, and the second one is how can we find an exact solution or approximate those solutions of such problems.

Fixed point iteration procedures are mainly designed to be applied in solving exact solution or approximate the solutions of some problems for nonlinear mappings.

For a nonempty convex subset C of a normed space E and $T : C \to C$, the fixed point set of T is denoted by $F(T)$, where $F(T) = \{x \in C : Tx = x\}$.
A mapping T is called
(a) α *-contraction* if there is an $\alpha \in [0, 1)$ such that

$$d(Tx, Ty) \leq \alpha d(x, y) \tag{1.1.1}$$

for all $x, y \in C$, where $0 < \alpha < 1$.

(b) *Kannan mapping* [9] if there exists $0 < b < \frac{1}{2}$ such that

$$d(Tx, Ty) \leq b(d(x, Tx) + d(y, Ty)) \tag{1.1.2}$$

for all $x, y \in C$.
(c) *Chatterjea mapping* [7] if there exists $0 < c < \frac{1}{2}$ such that

$$d(Tx, Ty) \leq c(d(x, Ty) + d(y, Tx)) \tag{1.1.3}$$

for all $x, y \in C$.
(d) *Zamfirescu operator* [3, 5, 6] if for each $x, y \in C$, one of the contractive conditions (1.1.1)-(1.1.3) holds.

In 2005, Berinde [3, 5] introduced a new class of quasi-contractive type operators T on a normed space E satisfying the condition: there exist $0 < \delta < 1$ and $L \geq 0$ such that,

$$d(Tx, Ty) \leq \delta d(x, y) + Ld(Tx, x), \tag{1.1.4}$$

for all $x, y \in E$. He showed that this class of operators is wider than the class of Zamfirescu operators. In this thesis quasi-contractive type mean the mapping satisfies condition (1.1.4).

The following well-known fixed point theorems of above mappings are very important and they can be applied to the existence problem of a solution of Ordinary Differential Equation, Partial Differential Equation, integral equations, system of linear equations and others.

Theorem 1.1.1. *Let X be a complete metric space and let T be a contraction of X into itself. Then T has a unique fixed point x in X and for any $x_0 \in X$, the sequence $\{x_n\}$ generated by $x_{n+1} = Tx_n$ converges to x.*

Definition 1.1.2. Let (X, d) be a complete metric space and let T be a selfmap of X. If T has a unique fixed point, which can be obtained as the limit of the sequence $\{x_n\}$, where $x_{n+1} = Tx_n, n = 0, 1, 2, ...$ for any $x_0 \in X$, then T is called a *Picard operator*, and the iteration defined by $\{x_n\}$ is called *Picard iteration*.

Theorem 1.1.3. *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping for which there exist $b \in [0, \frac{1}{2})$ such that*

$$d(Tx, Ty) \leq b(d(x, Tx) + d(y, Ty)), \forall x, y \in X.$$

Then T is a picard operator.

Theorem 1.1.4. *[19] Let (X, d) be a complete metric space, and $T : X \to X$ a mapping for which there exist real number $a, b,$ and c satisfying $0 < a < 1, 0 < b, c < \frac{1}{2}$ such that for each pair $x, y \in X$ at least one of the following is true:*

(1) $d(Tx, Ty) \leq ad(x, y)$,

(2) $d(Tx, Ty) \leq b(d(x, Tx) + d(y, Ty))$,

(3) $d(Tx, Ty) \leq c(d(x, Ty) + d(y, Tx))$.

Then T has a unique fixed point q and the picard iteration $\{x_n\}_{n=1}^{\infty}$ defined by

$$x_{n+1} = Tx_n, n = 0, 1, 2, ...,$$

converges to q, for any $x_0 \in X$.

One of the most general contractive conditions for which a map T is a Picard operator is that of [8].

Definition 1.1.5. A selfmap T is called a *quasicontractive* if it satisfies

$$d(Tx, Ty) \leq \delta \; max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \qquad (1.1.5)$$

for each $x, y \in X$, where δ is a real number satisfying $0 \leq \delta < 1$.

Definition 1.1.6. Let (X, d) be a metric space. A map $T : X \to X$ is called weak contraction if there exist a constant $\delta \in (0, 1)$ and some $L \geq 0$ such that

$$d(Tx, Ty) \le \delta d(x, y) + Ld(y, Tx), \text{ for all } x, y \in X.$$

Theorem 1.1.7. *Let (X, d) be a complete metric space and $T : X \to X$ be a weak contraction with $\delta \in (0, 1)$ and some $L \ge 0$. Then*

(1) $F(T) \ne \emptyset$,

(2) *for any $x_0 \in X$, the Picard iteration $\{x_n\}_{n=1}^{\infty}$ converges to some $x^* \in F(T)$.*

Theorem 1.1.8. *Any mapping T satisfying the following:*
there exist $c \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \le c(d(x, Ty) + d(y, Tx)), \forall x, y \in X,$$

is a weak contraction.

Theorem 1.1.9. *Let (X, d) be a complete metric space and $T : X \to X$ a weak contraction for which there exist a constant $\theta \in (0, 1)$ and some $L_1 \ge 0$ such that*

$$d(Tx, Ty) \le \theta d(x, y) + L_1 d(x, Tx), \forall x, y \in X. \tag{1.1.6}$$

Then

1) T has a unique fixed point, i.e. $F(T) = \{x^*\}$;

2) the Picard iteration $\{x_n\}_{n=1}^{\infty}$ converges to x^*, for any $x_0 \in X$.

Remark 1.1.10. It is known that condition (1.1.6) alone does not ensure that T has a fixed point. But if T satisfying (1.1.6) has a fixed point, T is certainly unique.

It is well-known that the classical Mann, Ishikawa, Noor, S-iteration are very useful methods for approximating a fixed point of several nonlinear mappings. Recently, there are many authors introduced iteration methods for such purpose.

The following iteration methods are well known:
Mann's iteration is the sequence $\{u_n\}$ generated by

$$\begin{cases} u_1 \in C \\ u_{n+1} = (1 - a_n)u_n + a_n T u_n, n \ge 1 \end{cases} \tag{1.1.7}$$

where $\{a_n\}$ is a sequence in $[0,1]$.

Ishikawa's iteration is the sequence $\{v_n\}$ generated by

$$\begin{cases} v_1 \in C \\ y_n = (1-b_n)v_n + b_n T v_n, \\ v_{n+1} = (1-a_n)v_n + a_n T y_n, n \geq 1 \end{cases} \tag{1.1.8}$$

where $\{a_n\}, \{b_n\}$ are sequences in $[0,1]$.

The S-iteration is the sequence $\{s_n\}$ generated by

$$\begin{cases} s_1 \in C \\ y_n = (1-b_n)s_n + b_n T s_n, \\ s_{n+1} = (1-a_n)T s_n + a_n T y_n, n \geq 1 \end{cases} \tag{1.1.9}$$

where $\{a_n\}, \{b_n\}$ are sequences in $[0,1]$ is known as the Agarwal et al. [1] iteration process.

The Noor's iteration is the sequence $\{w_n\}$ generated by

$$\begin{cases} w_1 \in C \\ z_n = (1-c_n)w_n + c_n T w_n, \\ y_n = (1-b_n)w_n + b_n T z_n, \\ w_{n+1} = (1-a_n)w_n + a_n T y_n, n \geq 1 \end{cases} \tag{1.1.10}$$

where $\{a_n\}, \{b_n\}, \{c_n\}$ are sequences in $[0,1]$.

In 2006, Suantai [16] introduced the following three-step iteration process :

$$\begin{cases} x_1 \in C \\ z_n = (1-a_n)x_n + a_nTx_n, \\ y_n = (1-b_n-c_n)x_n + b_nTz_n + c_nTx_n, \\ x_{n+1} = (1-\alpha_n-\beta_n)x_n + \alpha_nTy_n + \beta_nTz_n, n \geq 1 \end{cases} \quad (1.1.11)$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}$ are sequences in $[0, 1]$.

Then in 2009, Shaini Pulickakunnel and Neeta Singh [13] modified that of Suantai to the following three-step iteration process :

$$\begin{cases} t_1 \in C \\ z_n = (1-a_n)t_n + a_nTt_n, \\ y_n = (1-b_n-c_n)t_n + b_nTz_n + c_nTt_n, \\ t_{n+1} = (1-\alpha_n-\beta_n-\gamma_n)t_n + \alpha_nTy_n + \beta_nTz_n + \gamma_nTt_n, n \geq 1 \end{cases} \quad (1.1.12)$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $[0, 1]$.

The following theorems showed that Mann, Ishikawa, S-iteration, Noor iteration and iteration defined by (1.1.11) can be used to approximate a fixed point of the operator satisfying quasi-contractive type.

Theorem 1.1.11. *[3] Let C be a nonempty closed convex subset of a normed space E. Let $T : C \to C$ be an operator satisfying quasi-contractive type Let $\{u_n\}$ be defined through the iterative process* (1.1.7)*. If $F(T) \neq \emptyset$ and $\sum a_n = \infty$, then $\{u_n\}$ converges strongly to the unique fixed point of T.*

Theorem 1.1.12. *[5] Let C be a nonempty closed convex subset of an arbitrary Banach space E. Let $T : C \to C$ be an operator satisfying quasi-contractive type with $F(T) \neq \emptyset$. Let $\{v_n\}$ be defined through the iterative process* (1.1.8) *with $v_1 \in C$, where $\{a_n\}, \{b_n\}$ are sequences of positive numbers in $[0, 1]$ with $\sum a_n = \infty$, then $\{v_n\}$ converges strongly to the fixed point of T.*

Theorem 1.1.13. *[12] Let C be a nonempty closed convex subset of an arbitrary Banach space E. and $T : C \to C$ be an operator satisfying quasi-contractive type with $F(T) \neq \emptyset$. Let $\{s_n\}$ be defined through the iterative process* (1.1.9) *with $s_1 \in C$, where $\{a_n\}, \{b_n\}$ are sequences of positive numbers in* $[0, 1]$ *with $\sum a_n = \infty$, then $\{s_n\}$ converges strongly to the fixed point of T.*

Theorem 1.1.14. *[10] Let C be a nonempty closed convex subset of normed space E. Let $T : C \to C$ be an operator satisfying quasi-contractive type with $F(T) \neq \emptyset$. Let $\{x_n\}$ be defined through the iterative process* (1.1.11) *with $x_1 \in C$, where $\{a_n\}, \{b_n\}, \{\alpha_n\}, \{\beta_n\}$ are sequences in* $[0, 1]$ *such that $b_n + c_n \in [0, 1]$ and $\sum(\alpha_n + \beta_n) = \infty$, then $\{x_n\}$ converges strongly to a fixed point of T.*

As (1.1.10) reduces to Noor's iteration by choosing $c_n = \beta_n = 0$, we have the following theorem.

Theorem 1.1.15. *[10] Let C be a nonempty closed convex subset of normed space E. Let $T : C \to C$ be an operator satisfying quasi-contractive type Let $\{w_n\}$ be defined by the iterative process* (1.1.10). *If $F(T) \neq \emptyset$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, then $\{w_n\}$ converges strongly to a fixed point of T.*

There are many authors defined several iteration methods for finding a fixed point of certain nonlinear mappings but there are few papers concentrating on comparison of the rate of convergence of those iteration methods.

In 2004, Berinde [6] gave the concept of the rate of convergence of iterative methods as follows:

Definition 1.1.16. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of positive numbers that converge to a and b, respectively. Assume that there exists

$$l = \lim_{n \to \infty} \frac{|a_n - a|}{|b_n - b|}.$$

If $l = 0$, then it is said that the sequence $\{a_n\}_{n=1}^{\infty}$ converges to a faster than $\{b_n\}_{n=1}^{\infty}$ to b.

Definition 1.1.17. Let C be a convex set, $T : C \to C$ and $\{u_n\}, \{v_n\} \subset C$ be two sequences of positive numbers converge to the same $q \in F(T)$, $\|u_n - q\| \leq a_n$ and $\|v_n - q\| \leq b_n$ where $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ are two sequences of positive numbers converging to zero. If $\{a_n\}_{n=1}^{\infty}$ converges faster than $\{b_n\}_{n=1}^{\infty}$, then we shall say that the fixed point iteration $\{u_n\}$ converges faster than $\{v_n\}$.

The purpose of this thesis are three folds. Firstly, we prove strong convergence of three-step iteration (1.1.12) for mappings satisfying quasi-contractive type. Secondly we compare the rate of convergence between Mann, Ishikawa, Noor, S-iteration and the three-step iterations and we give some numerical results for the rate of convergence of Mann, Ishikawa, Noor, S-iteration and the three-step iterations. Finally we prove strong convergence of S-iteration for quasicontractive mappings then we compare the rate of convergence between S-iteration and Ishikawa iteration.

This thesis is divided into 4 chapters. Chapter 1 is an introduction of this thesis. Chapter 2 is devoted to preliminaries, lemmas and propositions which will be used in the thesis. Chapter 3 the main results of this thesis.Chapter 4 the conclusion.