Chapter 1
Introduction

Let X be a nonempty set and $T : X \to X$ a mapping. We say that $x \in X$ is a fixed point of T if and only if $x = Tx$. The set of all fixed points of T is denoted by $F(T)$, i.e. $F(T) = \{x \in X : Tx = x\}$. If the mapping T does not have a fixed point, we often say that T is fixed point free. In general, the problem in fixed point theory is to find essential and sufficient conditions on space or a map for the existence of such a fixed point. The fixed point theory is the most important tool to solve a problem in many branches of science and a new technology. When a problem in science has transformed into a mathematical model such as equality, inequality, equality system and inequality system. There are two questions following.

1. Does the answer of model exist?
2. How can we find the answer?

In 1912, Brouwer [4.1] started to answer the first question. He proved that every continuous map on the closed unit ball of $\mathbb{R}^n$, has a fixed point.

Let E be a nonempty subset of a Banach space X. A mapping $T : E \to E$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in E$. It is called quasi-nonexpansive if $\|Tx - z\| \leq \|x - z\|$ for all $x \in E$ and for all $z \in F(T)$.

In 2008, Suzuki [28] introduced a condition on mappings, called condition (C) (sometimes called Suzuki's mapping), which is weaker than nonexpansiveness and stronger than quasi-nonexpansiveness. Let T be a mapping on a subset E of Banach space X. Then T is said to satisfy condition (C) if

$$\tfrac{1}{2}\|x - Tx\| \leq \|x - y\| \quad \text{implies} \quad \|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in E$. Moreover, he obtained some interesting fixed point theorems and convergence theorems for such mappings.

In 2008, Dhompongsa et al. [7] proved a fixed point theorem for mappings with condition (C) on a Banach space such that its asymptotic center in a bounded closed and convex subset of each bounded sequence is nonempty and compact.

In 2010, Nanjaras et al. [26] extended Suzuki's results on existence theorems and convergence theorems to a special kind of metric spaces, namely CAT(0) spaces, which will be defined in Section 2.2.

In 2008, Kohsaka and Takahashi [20] introduced a notion of nonspreading mappings on smooth, strictly convex and reflexive Banach spaces. In the case of Hilbert spaces, let E be a nonempty closed convex subset of a real Hilbert space H. A mapping $T : E \to E$ is said to be a nonspreading mapping if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2$$

for all $x, y \in E$. (For details, one can to refer [15].)

In 2011, Lin et al. [22] introduced generalized nonspreading mappings on CAT(0) spaces, called generalized hybrid mappings, which will be defined in Section 2.3.

In order to answer the second question, in 1953, Mann [23] defined an iteration as follows: let E be a compact convex subset of a Banach space X and $T : E \to E$ a continuous mapping. Let $x_1 \in E$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \geq 1. \tag{1.1}$$

He proved that the sequence $\{x_n\}$ converges to a fixed point of T under some suitable conditions.

In 1974, Ishikawa [16] defined a new iteration which is a generalization of (1.1) by:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T[\beta_n x_n + (1 - \beta_n)Tx_n], \quad n \geq 0. \tag{1.2}$$

He proved that if X is a Hilbert space and T is a Lipschitzian pseudo-contractive mapping, then the sequence $\{x_n\}$ converges to a fixed point of T under some suitable conditions.

In 1998, Takahashi and Tamura [31] discussed about weak convergence theorems for two nonexpansive mappings T_1, T_2 on E to itself as follows:

$$x_1 = x \in E,$$
$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_1\{\beta_n T_2 x_n + (1-\beta_n)x_n\}, \quad n \geq 1.$$

In 2007, Moudafi [24] also considered another iterative procedure for two nonexpansive mappings T_1 and T_2 defined on E in Hilbert space into itself:

$$x_1 = x \in E,$$
$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n\{\beta_n T_1 x_n + (1-\beta_n)T_2 x_n\}, \quad n \geq 1.$$

After that, Iemoto and Takahashi [15] considered weak convergence theorems for nonspreading mapping and nonexpansive mapping in the case of Hilbert spaces by using Mudafi's iterative cheme.

Motivated by Takahashi and Tamura [31], and Iemoto and Takahashi [15], Dhompongsa et al. [8] studied the iterative scheme for approximation of fixed points of two mappings which are nonspreading mappings and Suzuki's mappings in Hilbert spaces.

In 2010, Niwongsa and Panyanak [4.1] defined the Mann and Ishikawa iterations in the CAT(0) space. Let E be a nonempty convex subset of a CAT(0) space X, and $T : E \to E$. The Mann iterate sequence [23] is defined by $x_1 \in E$ and

$$x_{n+1} = t_n T x_n \oplus (1-t_n)x_n, \quad n = 1, 2, 3, ..., \tag{MS}$$

where $\{t_n\}$ is a sequence in $[0, 1]$.
The Ishikawa iterate sequence [16] is defined by $x_1 \in E$ and

$$x_{n+1} = t_n T\left[s_n T x_n \oplus (1-s_n)x_n\right] \oplus (1-t_n)x_n, \quad n = 1, 2, 3, ..., \tag{IS}$$

where $\{t_n\}$ and $\{s_n\}$ are sequences in $[0, 1]$.

The purpose of this thesis, motivated by [8], is to study existence theorems and convergence theorems for some generalized nonexpansive mappings and nonspreading mappings in CAT(0) spaces.

This thesis is divided into 4 chapters. Chapter 1 is an introduction to the research problems. In Chapter 2, we collect some basic definitions and results which are needed in later chapters. In Chapter 3, we prove some fixed points theorems and $\Delta-$convergence theorems for some generalized nonexpansive mappings and nonspreading mappings in CAT(0) spaces. Finally, we prove strong convergence theorems for some generalized nonexpansive mappings and nonspreading mappings in CAT(0) spaces.