Chapter 2

Basic Concepts

The purpose of this chapter is to explain certain notations, terminologies and elementary results used throughout the thesis. Although details are included in some cases, many of the fundamental principles of functional analysis are merely stated without proof.

2.1 Basic concepts of limits

Definition 2.1.1 ([32])**.** A sequence is a function whose domain is the set $\mathbb{N}$ of natural numbers.

Definition 2.1.2 ([32])**.** A sequence whose range is a subset on the set of real numbers is called a real sequence, i.e. a real sequence s is a function $s : \mathbb{N} \to \mathbb{R}$.

Definition 2.1.3 ([32])**.** A sequence $\{x_n\}$ is said to be bounded above (below) if there exists a real number M such that $x_n \leq M (x_n \geq M)$ for all $n \in \mathbb{N}$. A sequence is said to be bounded if it is bounded both above and below.
Or a sequence $\{x_n\}$ is bounded if and only if there exists $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$.

Definition 2.1.4 ([32])**.** A sequence $\{x_n\}$ is said to converge to $x \in \mathbb{R}$ if given $\epsilon > 0$, there exists a positive integer m such that $n \geq m$ implies $|x_n - x| < \epsilon$. We call the limit of $\{x_n\}$ and write $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

Theorem 2.1.5 ([32])**.** If a sequence has a limit, this limit is unique.

Theorem 2.1.6 ([32])**.** Every convergent sequence is bounded.

Theorem 2.1.7 ([32])**.** Let $\{x_n\}$ and $\{y_n\}$ be two sequences of real numbers such that $\lim\limits_{n\to\infty} x_n = x$ and $\lim\limits_{n\to\infty} y_n = y$, then $\lim\limits_{n\to\infty} (x_n \pm y_n) = x \pm y$.

Theorem 2.1.8 ([32]). If $\{x_n\}$ is a convergent sequence such that $\lim_{n\to\infty} x_n = x$ and $c \in \mathbb{R}$, then $\lim_{n\to\infty} cx_n = cx$.

Theorem 2.1.9 ([32]). Let $\{x_n\}$ be a sequence such that $\{x_n\} \geq 0$ and $\lim_{n\to\infty} x_n = x$, then $x \geq 0$.

Definition 2.1.10 ([1]). Let $\{x_n\}$ be a sequence of real numbers. We say that $\{x_n\}$ is increasing if it satisfies the inequalities

$$x_1 \leq x_2 \leq \cdots \leq x_n \leq x_{n+1} \leq \cdots .$$

We say that $\{x_n\}$ is decreasing if it satisfies the inequalities

$$x_1 \geq x_2 \geq \cdots \geq x_n \geq x_{n+1} \geq \cdots .$$

We say that $\{x_n\}$ is monotone if it is either increasing or it is decreasing.

Theorem 2.1.11 ([1]). A monotone sequence of real numbers is convergent if and only if it is bounded. Further:

(1) If $\{x_n\}$ is a bounded increasing sequence, then $\lim_{n\to\infty} x_n = \sup_n\{x_n\}$.

(2) If $\{y_n\}$ is a bounded decreasing sequence, then $\lim_{n\to\infty} y_n = \inf_n\{y_n\}$.

Definition 2.1.12 ([1]). Let $\{x_n\}$ be a sequence of real numbers and let $r_1 < r_2 < \cdots < r_n < \cdots$ be a strictly increasing sequence of natural numbers. Then the sequence $\{x_{r_n}\}$ in $\mathbb{R}$ given by

$$(x_{r_1}, x_{r_2}, x_{r_3}, \ldots, x_{r_n}, \ldots)$$

is called a subsequence of $\{x_n\}$.

Theorem 2.1.13 ([32]). If $\{x_n\}$ converges to x, then every subsequence of $\{x_n\}$ converges to x.

Theorem 2.1.14 ([32]). Every sequence has a monotone subsequence.

Theorem 2.1.15 ([32]). Every bounded sequence has a convergent subsequence.

Definition 2.1.16 ([1]). A sequence $\{x_n\}$ is said to be a Cauchy sequence if for every $\epsilon > 0$ there is a natural number N such that all natural numbers $n, m \geq N$, the terms x_n, x_m satisfy $|x_n - x_m| < \epsilon$.

Theorem 2.1.17 ([32]). If $\{x_n\}$ is a bounded sequence, then $\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$.

Theorem 2.1.18 ([32]). If $\{x_n\}$ is a sequence such that

$$\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = x$$

for some $x \in \mathbb{R}$, then $\{x_n\}$ converges to x.

Theorem 2.1.19 ([32]). If $\{x_n\}$ and $\{y_n\}$ are bounded sequences such that for all $n \in \mathbb{N}$, $x_n \leq y_n$, then $\limsup_{n\to\infty} x_n \leq \limsup_{n\to\infty} y_n$ and $\liminf_{n\to\infty} x_n \leq \liminf_{n\to\infty} y_n$.

Theorem 2.1.20 ([32]). If $\{x_n\}$ and $\{y_n\}$ are bounded sequences, then

(1) $\limsup_{n\to\infty}(x_n + y_n) \leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n$,

(2) $\liminf_{n\to\infty}(x_n + y_n) \geq \liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n$.

Theorem 2.1.21 ([32]). Let $\{x_n\}$ be a bounded sequence.

(1) If $\limsup_{n\to\infty} x_n = M$, then for any $\epsilon > 0$

- $x_n < M + \epsilon$ for all except finitely many values of n,
- $x_n > M - \epsilon$ for infinitely many values of n.

(2) If $\liminf_{n\to\infty} x_n = m$, then for any $\epsilon > 0$

- $x_n > m - \epsilon$ for all except finitely many values of n,
- $x_n < m + \epsilon$ for infinitely many values of n.

Definition 2.1.22 ([1]). If $E \subset \mathbb{R}$, then the function $f : E \to \mathbb{R}$ is said to be increasing (decreasing) on E if whenever $x_1, x_2 \in E$ and $x_1 \leq x_2$ ($x_1 \leq x_2$) then $f(x_1) \leq f(x_2)$ ($f(x_1) \geq f(x_2)$).

Definition 2.1.23 ([1]). If $E \subset \mathbb{R}$, then the function $f : E \to \mathbb{R}$ is said to be strictly increasing (strictly decreasing) on E if whenever $x_1, x_2 \in E$ and $x_1 < x_2$ ($x_1 < x_2$) then $f(x_1) < f(x_2)$ ($f(x_1) > f(x_2)$).

2.2 CAT(0) spaces

Definition 2.2.1. Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or more briefly, a geodesic from x to y) is a map $c : [0, l] \to X$ such that $c(0) = x, c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$.

The image α of c is called a *geodesic* (*or metric*) *segment* joining x to y. When it is unique this geodesic segment is denote by $[x, y]$.

Definition 2.2.2. The space (X, d) is said to be a *geodesic space* if every two points of X are joined by a geodesic, and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$.

Definition 2.2.3. Let (X, d) be a *geodesic space.* A subset $Y \subseteq X$ is said to be *convex* if Y includes every geodesic segment joining any two of its points.

Definition 2.2.4. Let (X, d) be a geodesic space. A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ subset of X consists of three points x_1, x_2, x_3 in X (the *vertices* of $\triangle$) and a geodesic segment between each pair of vertices (the *edges* $\triangle$). A *comparison triangle* for the geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\bar{\triangle}(x_1, x_2, x_3) := \triangle(\bar{x_1}, \bar{x_2}, \bar{x_3})$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\bar{x_i}, \bar{x_j}) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

Definition 2.2.5. A geodesic space (X, d) is said to be a *CAT(0) space* if all geodesic triangles satisfy the following comparison axiom.

CAT(0) : Let $\triangle$ be a geodesic triangle in X and let $\bar{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the *CAT(0) inequality* if for all $x, y \in \triangle$ and all comparison points $\bar{x}, \bar{y} \in \bar{\triangle}$,

$$d(x, y) \leq d_{\mathbb{E}^2}(\bar{x}, \bar{y}). \tag{2.1}$$

Complete CAT(0) spaces are often called Hadamard spaces. Examples of CAT(0) spaces include, among others, Hilbert spaces, classical hyperbolic spaces, Euclidean buildings, $\mathbb{R}$-trees, etc.

If $x, y \in X$, and $t \in [0,1]$ then we use the notation $(1-t)x \oplus ty$ for the point z in $[x, y]$ which

$$d(z, x) = td(x, y) \text{ and } d(z, y) = (1-t)d(x, y). \tag{2.2}$$

Remark 2.2.6. Let X be a CAT(0) space and let $x, y \in X$ such that $x \neq y$ and $s, t \in [0,1]$. Then $(1-t)x \oplus ty = (1-s)x \oplus sy$ if and only if $s = t$.

At this point we collect some elementary facts about CAT(0) spaces.

Lemma 2.2.7 ([2, p.163]). A geodesic space (X, d) is a CAT(0) space if and only if for $x, y_1, y_2 \in X$ and if y_0 is the midpoint of the segment $[y_1, y_2]$ then

$$d^2(x, y_0) \leq \frac{1}{2}d^2(x, y_1) + \frac{1}{2}d^2(x, y_2) - \frac{1}{4}d^2(y_1, y_2). \tag{CN}$$

This is the (CN) inequality of Bruhat and Tits [5].

Lemma 2.2.8. Let (X, d) be a CAT(0) space. Then

(i) (X, d) is uniquely geodesic (see [2, p.160]).

(ii) For $x, y \in X$ and $t \in [0,1]$, there exists a unique point $z \in [x, y]$ such that

$$d(x, z) = td(x, y) \quad \text{and} \quad d(y, z) = (1-t)d(x, y). \tag{2.3}$$

We use the notation $(1-t)x \oplus ty$ for the unique point z satisfying (2.3). (see [11, p.2573]).

(iii) Let p, x, y be points of X, let $\alpha \in [0,1]$, and let m_1 and m_2 denote, respectively, the point of $[p, x]$ and $[p, y]$ satisfying

$$d(p, m_1) = \alpha d(p, x) \quad \textit{and} \quad d(p, m_2) = \alpha d(p, y).$$

Then $d(m_1, m_2) \leq \alpha d(x, y)$ (see [18, Lemma 3]).

(iv) Let $x, y \in X, x \neq y$ and $z, w \in [x, y]$ such that $d(x, z) = d(x, w)$. Then $z = w$. (see [11, Lemma 2.1]).

Lemma 2.2.9 ([11, p.163]). Let X be a CAT(0) space. Then for all $x, y, z \in X$ and $t \in [0,1]$.

(i) $d\big((1-t)x \oplus ty, z\big) \leq (1-t)d(x, z) + td(y, z)$.

(ii) $d^2\big((1-t)x \oplus ty, z\big) \leq (1-t)d^2(x, z) + td^2(y, z) - t(1-t)d^2(x, y)$.

Definition 2.2.10. Let X be a complete CAT(0) space, let $\{x_n\}$ be a bounded sequence in X and for $x \in X$ set

$$r\left(x, \{x_n\}\right) = \limsup_{n \to \infty} d\left(x, x_n\right).$$

The *asymptotic radius* $r\left(\{x_n\}\right)$ of $\{x_n\}$ is given by

$$r\left(\{x_n\}\right) = \inf\left\{r\left(x, \{x_n\}\right) : x \in X\right\}$$

and the *asymptotic center* $A\left(\{x_n\}\right)$ of $\{x_n\}$ is the set

$$A\left(\{x_n\}\right) = \left\{x \in X : r\left(x, \{x_n\}\right) = r\left(\{x_n\}\right)\right\}.$$

It is known from Proposition 7 of [10] that in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point.

We now give the definition and collect some basic properties of $\Delta-$convergence and introduce important related concepts which will be used in our work.

Definition 2.2.11 ([19, 21])**.** A sequence $\{x_n\}$ in a complete CAT(0) space X is said to $\Delta-$*converges* to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta-\lim_{n \to \infty} x_n = x$ and call x the $\Delta-$limit of $\{x_n\}$.

Notice that given $\{x_n\} \subset X$ such that $\{x_n\}$ $\Delta-$converges to x and given $y \in X$ with $y \neq x$,

$$\limsup_{n} d\left(x_n, x\right) < \limsup_{n} d\left(x_n, y\right).$$

Thus X satisfies a condition which is known in Banach space theory as *the Opial property.*

Lemma 2.2.12 ([19, p.3690])**.** Every bounded sequence in a complete CAT(0) space always has a $\Delta-$convergent subsequence.

Lemma 2.2.13 ([9])**.** If E is a closed convex subset of a complete CAT(0) space and if $\{x_n\}$ is a bounded sequence in E, then the asymptotic center of $\{x_n\}$ is in E.

Lemma 2.2.14 ([11])**.** Let E be a nonempty closed convex subset of a CAT(0) space (X,d). Let $\{x_n\}$ be a bounded sequence in X with $A(\{x_n\}) = \{x\}$, and let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$. Suppose that $\lim\limits_{n\to\infty} d(x_n,u)$ exists. Then $x = u$.

Proof. Suppose that $x \neq u$. By the uniqueness of asymptotic centers,

$$\begin{aligned}\limsup_{n\to\infty}\{d(u_n,u)\} &< \limsup_{n\to\infty}\{d(u_n,x)\}\\ &\le \limsup_{n\to\infty}\{d(x_n,x)\}\\ &< \limsup_{n\to\infty}\{d(x_n,u)\}\\ &= \limsup_{n\to\infty}\{d(u_n,u)\},\end{aligned}$$

a contradiction, and hence $x = u$. $\square$

Lemma 2.2.15 ([22])**.** Let X be a CAT(0) space. Let $\{x_n\}$ and $\{y_n\}$ be two bounded sequences in X with $\lim\limits_{n\to\infty} d(y_n,x_n) = 0$. If $\Delta-\lim\limits_{n\to\infty} x_n = x$, then $\Delta-\lim\limits_{n\to\infty} y_n = x$.

Proof. Since $\Delta-\lim\limits_{n} x_n = x$, we know that

$$r(\{x_n\}) = r(x,\{x_{n_k}\}) = \limsup_{k\to\infty} d(\{x_{n_k}\},x)$$

for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Now, take any subsequence $\{y_{n_k}\}$ of $\{y_n\}$ and let $\{y_{n_k}\}$ be fixed. Then there exists $y \in X$ such that $A(\{y_{n_k}\}) = \{y\}$. Hence

$$\begin{aligned}\limsup_{k\to\infty} d(y_{n_k},y) &\le \limsup_{k\to\infty} d(y_{n_k},x)\\ &\le \limsup_{k\to\infty} d(y_{n_k},x_{n_k}) + \limsup_{k\to\infty} d(x_{n_k},x)\\ &= \limsup_{k\to\infty} d(x_{n_k},x)\\ &= r(\{x_n\})\\ &\le \limsup_{k\to\infty} d(x_{n_k},y)\\ &\le \limsup_{k\to\infty} d(y_{n_k},y).\end{aligned}$$

So, $\limsup_{k\to\infty} d(y_{n_k}, y) = \limsup_{k\to\infty} d(y_{n_k}, x)$. And this implies that $x \in A(\{y_{n_k}\})$. Since $A(\{y_{n_k}\}) = \{y\}$, $x = y$. So, $A(\{y_{n_k}\}) = \{x\}$ for every subsequence $\{y_{n_k}\}$ of $\{y_n\}$. Therefore $\Delta-\lim_n y_n = x$. □

2.3 A mapping satisfying condition (C)

In 2010, Nanjaras et al. [4.1] extended Suzuki results on fixed point theorems and convergence theorems to CAT(0) spaces.

Definition 2.3.1 ([26])**.** Let E be a nonempty subset of a complete CAT(0) space X. Then T is said to satisfy condition (C) if

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \text{ implies } d(Tx, Ty) \leq d(x, y)$$

for all $x, y \in E$.

Proposition 2.3.2 ([26])**.** Every nonexpansive mapping satisfies condition (C) but the converse is not true.

The condition is weaker than nonexpansiveness and stronger than quasi-nonexpansiveness see Proposition 2.3.3 and 2.3.4 bellow.

Proposition 2.3.3 ([26])**.** Let E be a nonempty subset of a CAT(0) space X. Suppose $T : E \to E$ is a nonexpansive mapping, then T satisfies condition (C).

Proposition 2.3.4 ([26])**.** Let E be a nonempty subset of a CAT(0) space X. Suppose $T : E \to E$ satisfies condition (C) and has a fixed point, then T is quasi-nonexpansive mapping.

Proof. Let $p \in F(T)$ and $x \in E$. Since $\frac{1}{2}d(p, Tp) = 0 \leq d(p, x)$, we have

$$d(p, Tx) = d(Tp, Tx) \leq d(p, x).$$

The proof is complete. □

Example 2.3.5 ([28])**.** Define a mapping T on [0,5] by

$$Tx = \begin{cases} 0 & if\ x \neq 5, \\ 1 & if\ x = 5. \end{cases}$$

Then T satisfies condition (C), but T is not nonexpansive.

Proof. If $x < y$ and $(x, y) \in ([0,5] \times [0,5]) \setminus ((4,5) \times 5)$, then $d(Tx, Ty) \neq d(x, y)$ holds. If $x \in (4,5)$ and $y = 5$, then

$$\frac{1}{2}d(x, Tx) = \frac{x}{2} > 1 > d(x, y) \text{ and } \frac{1}{2}d(y, Ty) = 1 > d(x, y)$$

holds. Thus T satisfies condition (C). However, since T is not continuous, T is not nonexpansive. □

Example 2.3.6 ([28])**.** Define a mapping T on [0, 5] by

$$Tx = \begin{cases} 0 & if\ x \neq 5, \\ 4 & if\ x = 5. \end{cases}$$

Then $F(T) \neq \emptyset$ and T is a quasi-nonexpansive, but T does not satisfy condition (C).

Proof. It is clear that $F(T) = \{0\} \neq \emptyset$ and T is a quasi-nonexpansive. However, since

$$\frac{1}{2}d(5, T5) = \frac{1}{2} \leq 1 = d(5, 4) \text{ and } d(T5, T4) = 4 > 1 = d(5, 4)$$

holds. Thus, T does not satisfy condition (C). □

Lemma 2.3.7 ([26])**.** Let E be a nonempty bounded closed convex subset of a complete CAT(0) space X. Suppose that $T : E \to E$ satisfies condition (C). Then $F(T)$ is nonempty closed, convex and hence contractible.

Lemma 2.3.8 ([26])**.** Let E be a nonempty closed convex subset of a complete CAT(0) space X, and suppose that $T : E \to E$ satisfies condition (C). If $\{x_n\}$ is a sequence in E such that $\lim_{n\to\infty} d(Tx_n, x_n) = 0$ and $\Delta-\lim_{n\to\infty} x_n = z$ for some $z \in X$, then $z \in E$ and $z = Tz$.

2.4 Generalized hybrid mappings

In 2011, Lin et al. [4.1] introduced generalized hybrid mappings on CAT(0) spaces.

Definition 2.4.1 ([22])**.** Let E be a nonempty closed convex subset of a CAT(0) space X. We say $T : E \to X$ is a generalized hybrid mapping if there exist $a_1 : E \to [0,1]$, $a_2, a_3 : E \to [0,1)$ such that
P(1) $d^2(Tx, Ty) \leq a_1(x)\, d^2(x, y) + a_2(x)\, d^2(Tx, y)$
$+ a_3(x)d^2(x, Ty) + k_1(x)d^2(Tx, x) + k_2(x)d^2(Ty, y)$ for all $x, y \in E$,
P(2) $a_1(x) + a_2(x) + a_3(x) \leq 1$ for all $x, y \in E$,
P(3) $2k_1(x) < 1 - a_2(x)$ and $k_2(x) < 1 - a_3(x)$ for all $x, y \in E$.

They also gave the definition of nonspreading mappings on CAT(0) spaces.

Definition 2.4.2 ([22])**.** Let E be a nonempty closed convex subset of a complete CAT(0) space X. A mapping $T : E \to E$ is said to be a nonspreading mapping if

$$2d^2(Tx, Ty) \leq d^2(Tx, y) + d^2(Ty, x)$$

for all $x, y \in E$.

Remark 2.4.3. If T is a nonspreading mapping then T is one case of a generalized hybrid mapping. In Definition 2.4.1, $a_1(x) = 0$, $a_2(x) = a_3(x) = \dfrac{1}{2}$ and $k_1(x) = k_2(x) = 0$, then T is a nonspreading mapping for all $x \in E$.

They also gave the following result for a generalized hybrid mapping on CAT(0) spaces.

Lemma 2.4.4 ([22])**.** Let E be a nonempty closed convex subset of a complete CAT(0) space X, and let $T : E \to E$ be a generalized hybrid mapping with $k_1(x) = k_2(x) = 0$ for all $x \in E$. Then $\{T^n x\}$ is a bounded for some $x \in E$ if and only if $F(T) \neq \emptyset$.

Proposition 2.4.5 ([22])**.** Let E be a nonempty closed convex subset of a complete CAT(0) space (X, d), and let $T : E \to X$ be a generalized hybrid mapping with $F(T) \neq \emptyset$. Then $F(T)$ is a closed convex subset of E.

Lemma 2.4.6 ([22]). Let E be a nonempty closed convex subset of a complete CAT(0) space X, and let $T : E \to X$ be a generalized hybrid mapping. Let $\{x_n\}$ be a bounded sequence in E with $\Delta - \lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Then $x \in E$ and $Tx = x$.

The class of nonspreading mappings and the class of mappings satisfying condition (C) are different.

Example 2.4.7 ([8]). Define a mapping T on [0,3] by

$$Tx = \begin{cases} 0 & if\ x \neq 3, \\ 2 & if\ x = 3. \end{cases}$$

T does not satisfy condition (C). But T is a nonspreading.

Proof. If $x = 3$ and $y \neq 3$, we have

$$2d^2(Tx, Ty) = 8 < 9 = d^2(Ty, x).$$

It is easy to see in the other cases that $2d^2(Tx, Ty) \leq d^2(Tx, y) + d^2(Ty, x)$. Thus, T is a nonspreading. Since

$$\frac{1}{2}d(3, T3) = \frac{1}{2} \leq 1 = d(3,2) \text{ and } d(T3, T2) = 2 > 1 = d(3,2)$$

holds. Thus T does not satisfy condition (C). □

Example 2.4.8 ([8]). Define a mapping T on [0, 1] by

$$Tx = 1 - x \quad \forall x \in [0,1].$$

Thus T is a nonexpansive and hence it satisfies condition (C). But T is not a nonspreading. In fact, if $x = 0$ and $y = 1$, we have

$$2d^2(Tx, Ty) = 2 > 0 = d^2(Ty, x) + d^2(y, Tx).$$