Chapter 3
$\Delta-$Convergence Theorems

This chapter is divided in to 2 sections. In section 1, we prove existence theorems for some generalized nonexpansive mappings and nonspreading mappings in CAT(0) spaces. In section 2, we prove $\Delta-$convergence theorems for some generalized nonexpansive mappings and nonspreading mappings in CAT(0) spaces.

3.1 Existence Theorems

Theorem 3.1.1. Let E be a nonempty bounded closed convex subset of a complete CAT(0) space X, and let $T : E \to E$ satisfies condition (C) and $S : E \to E$ is a nonspreading mapping. Let T and S are commuting mappings on E. Then T and S have a common fixed point.

Proof. By Lemma 2.3.7, we have $F(T) \neq \emptyset$. By the assumption T and S are commuting mappings on E, we have $Sx = S(Tx) = T(Sx)$ and hence $Sx \in F(T)$ for all $x \in F(T)$. So $S : F(T) \to F(T)$. By Remark 2.4.3 and E is bounded, we can use Lemma 2.4.4, then we have $F(S) \neq \emptyset$. So there exists $y \in F(S)$ such that $y = Sy \in F(T)$. So $y \in F(T) \cap F(S)$. □

3.2 $\Delta-$Convergence Theorems

We defined $\omega_w(\{x_n\}) := \cup A(\{u_n\})$ where the union is taken over any subsequence $\{u_n\}$ of $\{x_n\}$. In order to prove our main theorem the following facts are needed.

Lemma 3.2.1 ([22])**.** Let E be a nonempty closed convex subset of a complete CAT(0) space X, and let $T : E \to X$ be a generalized hybrid mapping. If $\{x_n\}$ is a bounded sequence in E such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and $\{d(x_n, v)\}$ converges for all $v \in F(T)$, then $\omega_w(\{x_n\}) \subset F(T)$. Furthermore, $\omega_w(\{x_n\})$ consists of exactly one point.

Remark 3.2.2 ([22])**.** The conclusion of Lemma 3.2.1 is still true if $T : E \to X$ is any one of nonexpansive mapping, nonspreading mapping, TJ1 mapping, TJ2 mapping, and hybrid mapping.(For other mapping, one can also refer [22].)

We need the following lemmas for complete the proof of main results.

Lemma 3.2.3. Let E be a nonempty closed convex subset of a complete CAT(0) space X, and let $T : E \to X$ satisfies condition (C). If $\{x_n\}$ is a bounded sequence in E such that $\lim_{n\to\infty} d(Tx_n, x_n) = 0$ and $\{d(x_n, v)\}$ converges for all $v \in F(T)$, then $\omega_w(\{x_n\}) \subset F(T)$. Furthermore, $\omega_w(\{x_n\})$ consists of exactly one point.

Proof. By the assumption $\{x_n\}$ is a bounded sequence in E such that $\lim_{n\to\infty} d(Tx_n, x_n) = 0$. Let $u \in \omega_w(\{x_n\})$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $\mathrm{A}(\{u_n\}) = \{u\}$. By Lemma 2.2.12 and 2.2.13 there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta-\lim_{n\to\infty} u_n = v \in E$. Since $\lim_{n\to\infty} d(Tv_n, v_n) = 0$, then $v \in F(T)$ by Lemma 2.3.8. By the assumption $\{d(x_n, v)\}$ converges for all $v \in F(T)$, then $u = v \in F(T)$ by Lemma 2.2.14. This shows that $\omega_w(\{x_n\}) \subset F(T)$. Next, we show that $\omega_w(\{x_n\})$ consists of exactly one point. Let $\mathrm{A}(\{x_n\}) = \{x\}$ and $\{u_n\}$ be a subsequence of $\{x_n\}$ with $\mathrm{A}(\{u_n\}) = \{u\}$. Since $u \in \omega_w(\{x_n\}) \subset F(T)$, we have seen that $u = v \in F(T)$ then $\{d(x_n, u)\}$ converges. By Lemma 2.2.14, $x = u$. $\square$

Now, we define the sequence $\{x_n\}$ by

$$(A)\begin{cases} x_1 \in E, \\ x_{n+1} = \alpha_n S y_n \oplus (1-\alpha_n)x_n, \end{cases}$$

where $y_n = \beta_n T x_n \oplus (1-\beta_n)x_n$ for all $n \in N$ and we prove the following lemma which, in fact, forms a major part of the proofs of both Δ and strong convergence theorems for a sequence $\{x_n\}$.

Lemma 3.2.4. Let E be a nonempty closed convex subset of a complete CAT(0) space X, and let $T : E \to E$ satisfies condition (C) and $S : E \to E$ is a non-spreading mapping such that $F(T) \cap F(S) \neq \emptyset$. Let $\{x_n\}$ be defined as (A). Then $\lim_{n\to\infty} d(x_n, w)$ exists for all $w \in F(T) \cap F(S)$.

Proof. Let $\{x_n\}$ be a sequence defined by (A) and $w \in F(T) \cap F(S)$. Then $d(Tx, w) \leq d(x, w)$ and $d(Sy, w) \leq d(y, w)$ for all $x, y \in E$. By Lemma 2.2.9(ii), we have

$$\begin{aligned} d^2(y_n, w) &= d^2(\beta_n T x_n \oplus (1-\beta_n)x_n, w) \\ &\leq \beta_n d^2(Tx_n, w) + (1-\beta_n)d^2(x_n, w) - \beta_n(1-\beta_n)d^2(Tx_n, x_n) \\ &\leq \beta_n d^2(x_n, w) + (1-\beta_n)d^2(x_n, w) - \beta_n(1-\beta_n)d^2(Tx_n, x_n) \\ &= d^2(x_n, w) - \beta_n(1-\beta_n)d^2(Tx_n, x_n) \qquad (3.1) \\ &\leq d^2(x_n, w) \end{aligned}$$

and

$$\begin{aligned} d^2(x_{n+1}, w) &= d^2(\alpha_n S y_n \oplus (1-\alpha_n)x_n, w) \\ &\leq \alpha_n d^2(Sy_n, w) + (1-\alpha_n)d^2(x_n, w) - \alpha_n(1-\alpha_n)d^2(Sy_n, x_n) \\ &\leq \alpha_n d^2(y_n, w) + (1-\alpha_n)d^2(x_n, w) - \alpha_n(1-\alpha_n)d^2(Sy_n, x_n) \qquad (3.2) \\ &\leq \alpha_n d^2(x_n, w) + (1-\alpha_n)d^2(x_n, w) - \alpha_n(1-\alpha_n)d^2(Sy_n, x_n) \\ &\leq d^2(x_n, w) - \alpha_n(1-\alpha_n)d^2(Sy_n, x_n) \qquad (3.3) \\ &\leq d^2(x_n, w). \end{aligned}$$

So $\{d(x_n, w)\}$ is bounded and decreasing sequence.

Hence $\lim_{n\to\infty} d(x_n, w)$ exists. □

Lemma 3.2.5. Let E be a nonempty closed convex subset of a complete CAT(0) space X, and let $T : E \to E$ satisfies condition (C) and $S : E \to E$ is a nonspreading mapping such that $F(T) \cap F(S) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $(0, 1)$. Let $\{x_n\}$ be defined as (A). If $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$, then $\lim_{n\to\infty} d(y_n, x_n) = 0$ and $\lim_{n\to\infty} d(y_n, w)$ exist.

Proof. Let $\{x_n\}$ be a sequence defined by (A) and $w \in F(T) \cap F(S)$. By Lemma 3.2.4 $\lim_{n\to\infty} d(x_n, w)$ exists. Since $d(y_n, w) \leq d(x_n, w) \leq d(x_1, w)$, so $\{x_n\}$ and $\{y_n\}$ are boundeds.

By (3.3), we have

$$d^2(x_{n+1}, w) \leq d^2(x_n, w) - \alpha_n(1-\alpha_n)d^2(Sy_n, x_n).$$

Then

$$\alpha_n(1-\alpha_n)d^2(Sy_n, x_n) \leq d^2(x_n, w) - d^2(x_{n+1}, w).$$

Since $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, so there exist $k > 0$ and $\exists N \in \mathbb{N}$ such that $\alpha_n(1-\alpha_n) \geq k$ for all $n \geq N$, so

$$\begin{aligned}\limsup_{n\to\infty} kd^2(Sy_n, x_n) &\leq \limsup_{n\to\infty} \alpha_n(1-\alpha_n)d^2(Sy_n, x_n)\\ &\leq \limsup_{n\to\infty} \left(d^2(x_n, w) - d^2(x_{n+1}, w)\right)\\ &= 0.\end{aligned}$$

Hence $0 \leq \liminf_{n\to\infty} d^2(Sy_n, x_n) \leq \limsup_{n\to\infty} d^2(Sy_n, x_n) \leq 0$.

$$\text{Then } \lim_{n\to\infty} d^2(Sy_n, x_n) = 0. \quad \text{This implies that } \lim_{n\to\infty} d(Sy_n, x_n) = 0. \tag{3.4}$$

By (3.2), we have

$$d^2(x_{n+1}, w) \leq \alpha_n d^2(y_n, w) + (1-\alpha_n)d^2(x_n, w) - \alpha_n(1-\alpha_n)d^2(Sy_n, x_n).$$

Then

$$\alpha_n[d^2(x_n, w) - d^2(y_n, w)] \leq d^2(x_n, w) - d^2(x_{n+1}, w).$$

Since $\alpha_n(1-\alpha_n) < \alpha_n$ so $\liminf_{n\to\infty} \alpha_n > 0$.
Using the same argument we have $\lim_{n\to\infty}(d^2(x_n, w) - d^2(y_n, w)) = 0.$
By (3.1), we get

$$d^2(y_n, w) \le d^2(x_n, w) - \beta_n(1-\beta_n)d^2(Tx_n, x_n).$$

Then

$$\beta_n(1-\beta_n)d^2(Tx_n, x_n) \le d^2(x_n, w) - d^2(y_n, w).$$

Since $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$. Using the same argument we have

$$\lim_{n\to\infty} d^2(Tx_n, x_n) = 0. \quad \text{This implies that} \quad \lim_{n\to\infty} d(Tx_n, x_n) = 0. \tag{3.5}$$

Hence

$$\limsup_{n\to\infty} d(y_n, x_n) = \limsup_{n\to\infty} \beta_n d(Tx_n, x_n) \le \limsup_{n\to\infty} d(Tx_n, x_n) = 0.$$

So $\lim_{n\to\infty} d(y_n, x_n) = 0$. Since $\lim_{n\to\infty}(d^2(x_n, w) - d^2(y_n, w)) = 0$ and $\lim_{n\to\infty} d(x_n, w)$ exists, then $\lim_{n\to\infty} d(y_n, w)$ exists. $\square$

Now we are ready to prove $\Delta-$convergence theorem for a sequence $\{x_n\}$.

Theorem 3.2.6. Let E be a nonempty closed convex subset of a complete CAT(0) space X, and let $T : E \to E$ satisfies condition (C) and $S : E \to E$ is a nonspreading mapping such that $F(T) \cap F(S) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $(0, 1)$. Let $\{x_n\}$ be defined as (A). If $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$, then $\Delta-\lim_n x_n = w \in F(T) \cap F(S)$.

Proof. Let $\{x_n\}$ be a sequence defined by (A) and $w \in F(T) \cap F(S)$. By Lemma 3.2.4, we have $\lim_{n\to\infty} d(x_n, w)$ exists. Then $\{x_n\}$ is bounded. By Lemma 3.2.5, we have $\lim_{n\to\infty} d(y_n, x_n) = 0$ and $\lim_{n\to\infty} d(y_n, w)$ exists. Then $\{y_n\}$ is also bounded.
By (3.5) we have $\lim_{n\to\infty} d(Tx_n, x_n) = 0$ and by (3.4) we have $\lim_{n\to\infty} d(Sy_n, x_n) = 0$. Since $d(Sy_n, y_n) \le d(Sy_n, x_n) + d(x_n, y_n)$, then $\lim_{n\to\infty} d(Sy_n, y_n) = 0$. By Lemma 3.2.3 and Remark 3.2.2, there exist $\bar{x}, \bar{y} \in E$ such that $\omega_w(\{x_n\}) = \{\bar{x}\} \subset F(T)$ and $\omega_w(\{y_n\}) = \{\bar{y}\} \subset F(S)$. So, $\Delta-\lim_n x_n = \bar{x}$ and $\Delta-\lim_n y_n = \bar{y}$. By Lemma 2.2.15, $\bar{x} = \bar{y}$. $\square$

Now, we define the sequence $\{z_n\}$ by

$$(B)\begin{cases} z_1 \in E, \\ z_{n+1} = \alpha_n T y'_n \oplus (1-\alpha_n) z_n, \end{cases}$$

where $y'_n = \beta_n S z_n \oplus (1-\beta_n) z_n$ for all $n \in N$ and we prove the following lemma which, in fact, forms a major part of the proofs of both Δ and strong convergence theorems for a sequence $\{z_n\}$.

Lemma 3.2.7. Let E be a nonempty closed convex subset of a complete CAT(0) space X, and let $T : E \to E$ satisfies condition (C) and $S : E \to E$ is a non-spreading mapping such that $F(T) \cap F(S) \neq \emptyset$. Let $\{z_n\}$ be defined as (B). Then $\lim_{n\to\infty} d(z_n, v)$ exists for all $v \in F(T) \cap F(S)$.

Proof. Let $\{z_n\}$ be a sequence defined by (B) and $v \in F(T) \cap F(S)$. Then $d(Sz, v) \leq d(z, v)$ and $d(Ty', v) \leq d(y', v)$ for all $z, y' \in E$. By Lemma 2.2.9(ii), we have

$$\begin{aligned} d^2(y'_n, v) &= d^2(\beta_n S z_n \oplus (1-\beta_n) z_n, v) \\ &\leq \beta_n d^2(Sz_n, v) + (1-\beta_n) d^2(z_n, v) - \beta_n(1-\beta_n) d^2(Sz_n, z_n) \\ &\leq \beta_n d^2(z_n, v) + (1-\beta_n) d^2(z_n, v) - \beta_n(1-\beta_n) d^2(Sz_n, z_n) \\ &= d^2(z_n, v) - \beta_n(1-\beta_n) d^2(Sz_n, z_n) \qquad (3.6) \\ &\leq d^2(z_n, v) \end{aligned}$$

and

$$\begin{aligned} d^2(z_{n+1}, v) &= d^2(\alpha_n T y'_n \oplus (1-\alpha_n) z_n, v) \\ &\leq \alpha_n d^2(Ty'_n, v) + (1-\alpha_n) d^2(z_n, v) - \alpha_n(1-\alpha_n) d^2(Ty'_n, z_n) \\ &\leq \alpha_n d^2(y'_n, v) + (1-\alpha_n) d^2(z_n, v) - \alpha_n(1-\alpha_n) d^2(Ty'_n, z_n) \qquad (3.7) \\ &\leq \alpha_n d^2(z_n, v) + (1-\alpha_n) d^2(z_n, v) - \alpha_n(1-\alpha_n) d^2(Ty'_n, z_n) \\ &\leq d^2(z_n, v) - \alpha_n(1-\alpha_n) d^2(Ty'_n, z_n) \qquad (3.8) \\ &\leq d^2(z_n, v). \end{aligned}$$

So $\{d(z_n, v)\}$ is bounded and decreasing sequence.

Hence $\lim_{n\to\infty} d(z_n, v)$ exists. □

Lemma 3.2.8. Let E be a nonempty closed convex subset of a complete CAT(0) space X, and let $T : E \to E$ satisfies condition (C) and $S : E \to E$ is a non-spreading mapping such that $F(T) \cap F(S) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $(0, 1)$. Let $\{z_n\}$ be defined as (B). If $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$, then $\lim_{n\to\infty} d(y'_n, z_n) = 0$ and $\lim_{n\to\infty} d(y'_n, v)$ exists.

Proof. Let $\{z_n\}$ be a sequence defined by (B) and $v \in F(T) \cap F(S)$. By Lemma 3.2.7 $\lim_{n\to\infty} d(z_n, v)$ exists. Since $d(y'_n, v) \leq d(z_n, v) \leq d(z_1, v)$, so $\{z_n\}$ and $\{y'_n\}$ are boundeds.

By (3.8), we have

$$d^2(z_{n+1}, v) \leq d^2(z_n, v) - \alpha_n(1-\alpha_n)d^2(Ty'_n, z_n).$$

Then $\alpha_n(1-\alpha_n)d^2(Ty'_n, z_n) \leq d^2(z_n, v) - d^2(z_{n+1}, v)$.

Since $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, so there exist $k > 0$ and $N \in \mathbb{N}$ such that $\alpha_n(1-\alpha_n) \geq k > 0$ for all $n \geq N$, so

$$\begin{aligned}
\limsup_{n\to\infty} kd^2(Ty'_n, z_n) &\leq \limsup_{n\to\infty} \alpha_n(1-\alpha_n)d^2(Ty'_n, z_n)\\
&\leq \limsup_{n\to\infty}\{d^2(z_n, v) - d^2(z_{n+1}, v)\}\\
&= 0.
\end{aligned}$$

Hence $0 \leq \liminf_{n\to\infty} d^2(Ty'_n, z_n) \leq \limsup_{n\to\infty} d^2(Ty'_n, z_n) \leq 0$.

Then $\lim_{n\to\infty} d^2(Ty'_n, z_n) = 0$. This implies that $\lim_{n\to\infty} d(Ty'_n, z_n) = 0$. (3.9)

By (3.7), we have

$$d^2(z_{n+1}, v) \leq \alpha_n d^2(y'_n, v) + (1-\alpha_n)d^2(z_n, v) - \alpha_n(1-\alpha_n)d^2(Ty'_n, z_n).$$

Then $\alpha_n[d^2(z_n, v) - d^2(y'_n, v)] \leq d^2(z_n, v) - d^2(z_{n+1}, v)$.

Since $\alpha_n(1-\alpha_n) < \alpha_n$ so $\liminf_{n\to\infty} \alpha_n > 0$.

Using the same argument we have $\lim_{n\to\infty}(d^2(z_n,v)-d^2(y'_n,v))=0$.
By (3.6), we have $d^2(y'_n,v)\le d^2(z_n,v)-\beta_n(1-\beta_n)d^2(Sz_n,z_n)$.
Then $\beta_n(1-\beta_n)d^2(Sz_n,z_n)\le d^2(z_n,v)-d^2(y'_n,v)$.
Since $\liminf_{n\to\infty}\beta_n(1-\beta_n)>0$. Using the same argument we have

$$\lim_{n\to\infty} d^2(Sz_n,z_n)=0. \quad \text{This implies that} \quad \lim_{n\to\infty} d(Sz_n,z_n)=0. \tag{3.10}$$

Hence

$$\limsup_{n\to\infty} d(y'_n,z_n)=\limsup_{n\to\infty}\beta_n d(Sz_n,z_n)\le \limsup_{n\to\infty} d(Sz_n,z_n)=0.$$

So $\lim_{n\to\infty} d(y'_n,z_n)=0$. Since $\lim_{n\to\infty}(d^2(z_n,v)-d^2(y'_n,v))=0$ and $\lim_{n\to\infty} d(z_n,v)$ exists, then $\lim_{n\to\infty} d(y'_n,v)$ exists. □

Now we are ready to prove $\Delta-$convergence theorem for a sequence $\{z_n\}$.

Theorem 3.2.9. Let E be a nonempty closed convex subset of a complete CAT(0) space X, and let $T:E\to E$ satisfies condition (C) and $S:E\to E$ is a non-spreading mapping such that $F(T)\cap F(S)\neq\emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $(0,1)$. Let $\{z_n\}$ be defined as (B). If $\liminf_{n\to\infty}\alpha_n(1-\alpha_n)>0$ and $\liminf_{n\to\infty}\beta_n(1-\beta_n)>0$, then $\Delta-\lim_n z_n=v\in F(T)\cap F(S)$.

Proof. Let $\{z_n\}$ be a sequence defined by (B) and $v\in F(T)\cap F(S)$. By Lemma 3.2.7, we have $\lim_{n\to\infty} d(z_n,v)$ exists. Then $\{z_n\}$ is bounded. By Lemma 3.2.8, we have $\lim_{n\to\infty} d(y'_n,z_n)=0$ and $\lim_{n\to\infty} d(y'_n,v)$ exists. Then $\{y'_n\}$ is also bounded.
By (3.10) we have $\lim_{n\to\infty} d(Sz_n,z_n)=0$ and by (3.9) we have $\lim_{n\to\infty} d(Ty'_n,z_n)=0$. Since $d(Ty'_n,y'_n)\le d(Ty'_n,z_n)+d(z_n,y'_n)$, then $\lim_{n\to\infty} d(Ty'_n,y'_n)=0$. By Lemma 3.2.3 and Remark 3.2.2, there exist $\bar z,\bar y\in E$ such that $\omega_w(\{z_n\})=\{\bar z\}\subset F(S)$ and $\omega_w(\{y'_n\})=\{\bar y\}\subset F(T)$. So, $\Delta-\lim_n z_n=\bar z$ and $\Delta-\lim_n y'_n=\bar y$. By Lemma 2.2.15, $\bar z=\bar y$. □