CHAPTER 1
INTRODUCTION

Let Y be a nonempty subset of a set X and $T(X)$ be the full transformation semigroup on the set X. Consider a subset of $T(X)$ given by

$$S(X,Y) = \{\alpha \in T(X) : Y\alpha \subseteq Y\}$$

where $Y\alpha$ is the range of $\alpha|_Y$, we have $S(X,Y)$ is a subsemigroup of $T(X)$. In fact, if $Y = X$, then $S(X,Y) = T(X)$. So we can think of $S(X,Y)$ as a generalization of $T(X)$.

In 1952, Vagner [9] defined the natural partial order on an inverse semigroup S by

$$a \leq b \text{ if and only if } a = eb \text{ for some } e \in E(S),$$

where $E(S)$ is the set of all idempotents in S, and he proved that $\leq$ is compatible on both sides with multiplication. In 1980, R. E. Hartwig [1] and K. S. S. Nambooripad [7] independently extended that partial order to a regular semigroup S by

$$a \leq b \text{ if and only if } a = eb = bf \text{ for some } e, f \in E(S).$$

In general $\leq$ is not compatible with multiplication on a regular semigroup S. In 1986, Mitsch [6] further extended the natural partial order to any semigroup S by

$$a \leq b \text{ if and only if } a = xb = by,\ xa = a \text{ for some } x, y \in S^1,$$

where S^1 is a monoid obtained from S by adjoining an identity if necessary. If S is a monoid, then $S^1 = S$.

Later G. Kowol and H. Mitsch [4] characterized the natural partial order on the full transformation semigroup in terms of images and kernels and determine the minimal and maximal elements of $T(X)$. Also the compatible elements of $T(X)$ were characterized by M. P. O. Marques-Smith and R. P. Sullivan in [5]. In 2011, P. Honyam and J. Sanwong [2] described Green's relations and ideals on $S(X,Y)$.

The purposes of this thesis are:

(1) To characterize the natural partial order on $S(X, Y)$.

(2) To give a necessary and sufficient condition for elements in $S(X, Y)$ to be minimal or maximal.

(3) To find elements of $S(X, Y)$ which are compatible with $\leq$ on $S(X, Y)$.

(4) To count the numbers of minimal and maximal elements of $S(X, Y)$ when X is a finite set.

This thesis comprises of four chapters. Chapter 1 is for the introduction. In Chapter 2, we list some well-known results, definitions and notations that will be used throughout this thesis. In Chapter 3, we present the characterization of the natural partial order on $S(X, Y)$, and give a necessary and sufficient condition for elements in $S(X, Y)$ to be minimal or maximal. Moreover, we find elements of $S(X, Y)$ which are compatible with $\leq$ on $S(X, Y)$, and count the numbers of minimal and maximal elements of $S(X, Y)$ when X is a finite set. Finally, Chapter 4 serves as the conclusion.