CHAPTER 3
MAIN RESULTS

In this chapter, we present the characterization of the natural partial order on $S(X,Y)$, and give necessary and sufficient conditions for elements in $S(X,Y)$ to be minimal or maximal. Moreover, we find elements of $S(X,Y)$ which are compatible with $\leq$ on $S(X,Y)$, and count the numbers of minimal and maximal elements of $S(X,Y)$ when X is a finite set.

3.1 Characterizations

In this section, we give necessary and sufficient conditions for $\alpha \leq \beta$ where $\alpha, \beta \in S(X,Y)$.

Theorem 3.1.1 *Let $\alpha, \beta \in S(X,Y)$. Then $\alpha \leq \beta$ if and only if α, β satisfy the following conditions:*

(i) *$X\alpha \subseteq X\beta$ and $Y\alpha \subseteq Y\beta$;*

(ii) *π_β refines π_α and $\pi_\beta(Y)$ refines $\pi_\alpha(Y)$;*

(iii) *for each $x \in X$, $x\beta \in X\alpha$ implies $x\alpha = x\beta$.*

Proof. Assume that $\alpha \leq \beta$. Then there exist $\gamma, \mu \in S(X,Y)$ such that $\alpha = \gamma\beta = \beta\mu$ and $\alpha = \alpha\mu$ by Lemma 2.4.3. From $\alpha = \gamma\beta$, we have $X\alpha \subseteq X\beta$ and $Y\alpha \subseteq Y\beta$ by Lemma 2.4.1 and from $\alpha = \beta\mu$, we get π_β refines π_α and $\pi_\beta(Y)$ refines $\pi_\alpha(Y)$ by Lemma 2.4.2. If $x\beta \in X\alpha$, then $x\beta = z\alpha$ for some $z \in X$ and therefore $x\alpha = x\beta\mu = z\alpha\mu = z\alpha = x\beta$.

Conversely, assume that the conditions hold. From (i) and (ii), there exist $\gamma, \mu \in S(X,Y)$ such that $\alpha = \gamma\beta = \beta\mu$ by Lemma 2.4.1 and Lemma 2.4.2. For each $x \in X$, we have $x\alpha = x\gamma\beta = y\beta$ for some $y \in X$, so $y\beta \in X\alpha$. By (iii), we get $y\alpha = y\beta$ and hence $x\alpha = y\beta = y\alpha = y\beta\mu = x\alpha\mu$, that is, $\alpha = \alpha\mu$. Therefore, $\alpha \leq \beta$ by Lemma 2.4.3. ■

Example 3.1.2 Let $X = \{1, 2, 3, 4, 5, 6\}$ and $Y = \{1, 2, 3\}$. We define $\alpha, \beta \in S(X, Y)$ by

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 2 & 4 & 4 & 6 \end{pmatrix}, \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 4 & 6 \end{pmatrix}.$$

Then there are $\gamma, \mu \in S(X, Y)$ such that

$$\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 2 & 4 & 4 & 6 \end{pmatrix}, \mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 2 & 4 & 5 & 6 \end{pmatrix},$$

and $\alpha = \gamma\beta = \beta\mu$, $\alpha = \alpha\mu$ which follow that $\alpha \leq \beta$. In addition, we can check that $\alpha \leq \beta$ by using Theorem 3.1.1 as below.

(i) $X\alpha = \{1, 2, 4, 6\} \subseteq \{1, 2, 3, 4, 6\} = X\beta$ and $Y\alpha = \{1, 2\} \subseteq \{1, 2, 3\} = Y\beta$;

(ii) Since $\pi_\beta = \{\{1\}, \{2\}, \{3\}, \{4, 5\}, \{6\}\}$, $\pi_\alpha = \{\{1\}, \{2, 3\}, \{4, 5\}, \{6\}\}$, $\pi_\beta(Y) = \{\{1\}, \{2\}, \{3\}\}$, and $\pi_\alpha(Y) = \{\{1\}, \{2, 3\}\}$, we have π_β refines π_α and $\pi_\beta(Y)$ refines π_α(Y);

(iii) $1\beta, 2\beta, 4\beta, 5\beta, 6\beta \in X\alpha$ and $1\alpha = 1\beta, 2\alpha = 2\beta, 4\alpha = 4\beta, 5\alpha = 5\beta$, $6\alpha = 6\beta$. ■

Corollary 3.1.3 *Let $\alpha, \beta \in T(X)$. Then $\alpha \leq \beta$ if and only if α, β satisfy the following conditions:*

(i) $X\alpha \subseteq X\beta$;

(ii) *π_β refines π_α;*

(iii) *for each $x \in X$, $x\beta \in X\alpha$ implies $x\alpha = x\beta$.*

Proof. By taking $Y = X$, we obtain $S(X, Y) = T(X)$, $Y\alpha = X\alpha$, $Y\beta = X\beta$ and $\pi_\beta(Y) = \pi_\beta$, $\pi_\alpha(Y) = \pi_\alpha$. Thus the proof is complete by Theorem 3.1.1. ■

3.2 Minimal and Maximal Elements

In this section, we give necessary and sufficient conditions for elements in $S(X, Y)$ to be minimal or maximal elements.

Theorem 3.2.1 *Let $\alpha \in S(X, Y)$. Then α is a minimal element if and only if α is a constant map.*

Proof. Assume that α is not a constant map. Then $|X\alpha| > 1$. Choose $y \in Y\alpha$ and define $\beta \in S(X, Y)$ by $x\beta = y$ for all $x \in X$. Then $\alpha \neq \beta$. We have $X\beta = \{y\} \subseteq Y\alpha \subseteq X\alpha$, $Y\beta = \{y\} \subseteq Y\alpha$. Since $\pi_\beta = \{X\} = \pi_\beta(Y)$, π_α refines π_β and $\pi_\alpha(Y)$ refines $\pi_\beta(Y)$. For each $x \in X$, if $x\alpha \in X\beta = \{y\}$, implies $x\alpha = y = x\beta$. Thus $\beta \leq \alpha$ and $\alpha \neq \beta$ by Theorem 3.1.1. Hence α is not minimal.

On the other hand, assume that α is a constant map with image $\{y\}$. Let $\beta \in S(X, Y)$ be such that $\beta \leq \alpha$. By (i) of Theorem 3.1.1, we get $X\beta \subseteq X\alpha = \{y\}$. Then $\beta = \alpha$. Hence α is minimal. ■

Example 3.2.2 Let $X = \{1, 2, 3, 4\}$ and $Y = \{1, 2\}$. Consider

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 2 \end{pmatrix}.$$

Then we have $\alpha, \beta \in S(X, Y)$ are the only two minimal elements by Theorem 3.2.1. ■

Lemma 3.2.3 *Let $\alpha \in S(X, Y)$. If α is injective or α is surjective, then α is a maximal element.*

Proof. Assume that α is injective. Let $\beta \in S(X, Y)$ be such that $\alpha \leq \beta$. Since $\alpha \leq \beta$, we have α, β satisfy conditions (i) - (iii) of Theorem 3.1.1. Let $y \in Y$. Then $y\alpha \in Y\alpha \subseteq Y\beta$. Thus $y\alpha = y'\beta$ for some $y' \in Y$. Since $y'\beta = y\alpha \in Y\alpha \subseteq X\alpha$, we get $y'\alpha = y'\beta$. Then $y\alpha = y'\beta = y'\alpha$. From α is injective, we have $y = y'$. So $y\alpha = y\beta$ for all $y \in Y$. That is $X\alpha \subseteq X\beta$ and $Y\alpha = Y\beta$, hence $X\alpha \setminus Y\alpha \subseteq X\beta \setminus Y\beta$.

From π_β refines π_α and α is injective, we get β is also injective. Since α and β are injective, $X\alpha \setminus Y\alpha = (X \setminus Y)\alpha$ and $X\beta \setminus Y\beta = (X \setminus Y)\beta$. This implies that $(X \setminus Y)\alpha \subseteq (X \setminus Y)\beta$. Now, let $x \in X \setminus Y$. Then $x\alpha \in (X \setminus Y)\alpha \subseteq (X \setminus Y)\beta$. So $x\alpha = z\beta$ for some $z \in X \setminus Y$. Since $z\beta = x\alpha \in X\alpha$, we have $z\alpha = z\beta$ by (iii) of Theorem 3.1.1. Thus $x\alpha = z\beta = z\alpha$. Since α is injective, $x = z$. So $x\alpha = x\beta$. Therefore $\alpha = \beta$, that is, α is maximal.

Next, we consider the case α is surjective. Then $X\alpha = X = X\beta$. By (iii) of Theorem 3.1.1, we get $x\alpha = x\beta$ for all $x \in X$. Thus $\alpha = \beta$ and so α is a maximal element. ■

Lemma 3.2.4 *Let $\alpha \in S(X, Y)$. If $Y \subseteq X\alpha$ and $\alpha : X \setminus Y \to X \setminus Y\alpha$ is injective, then α is a maximal element.*

Proof. Assume that $Y \subseteq X\alpha$ and $\alpha : X\setminus Y \to X\setminus Y\alpha$ is injective. Let $\beta \in S(X, Y)$ be such that $\alpha \leq \beta$. Then α, β satisfy conditions (i) - (iii) of Theorem 3.1.1. From $Y\beta \subseteq Y \subseteq X\alpha$, we get $y\alpha = y\beta$ for all $y \in Y$. Next, we show that $(X\setminus Y)\alpha \subseteq (X\setminus Y)\beta$. Suppose that there is $x \in (X\setminus Y)\alpha$ such that $x \notin (X\setminus Y)\beta$. Since $(X \setminus Y)\alpha \subseteq X\alpha \subseteq X\beta$, we have $x \in X\beta$. From $X\beta = Y\beta \cup (X \setminus Y)\beta$, we get $x \in Y\beta$ or $x \in (X \setminus Y)\beta$. Since $x \notin (X \setminus Y)\beta$, we have $x \in Y\beta$. Then there is $y' \in Y$ such that $x = y'\beta$. From $y'\beta = x \in (X \setminus Y)\alpha$, $y'\alpha = y'\beta$ by (iii) of Theorem 3.1.1. Then $x = y'\beta = y'\alpha$. Thus $x = y'\alpha \in Y\alpha$ which is a contradiction since $x \in (X \setminus Y)\alpha \subseteq X \setminus Y\alpha$ by assumption. Hence there is no $x \in (X \setminus Y)\alpha$ such that $x \notin (X \setminus Y)\beta$, that is $(X \setminus Y)\alpha \subseteq (X \setminus Y)\beta$.

Let $x \in X\setminus Y$. Then $x\alpha \in (X\setminus Y)\alpha \subseteq (X\setminus Y)\beta$. Thus $x\alpha = x'\beta$ for some $x' \in X \setminus Y$. Since $x'\beta = x\alpha \in X\alpha$, we get $x'\alpha = x'\beta$ by (iii) of Theorem 3.1.1. Hence $x\alpha = x'\beta = x'\alpha$. Since $\alpha : X \setminus Y \to X \setminus Y\alpha$ is injective, we have $x = x'$. So $x\alpha = x\beta$ for all $x \in X \setminus Y$. Therefore $\alpha = \beta$, that is, α is a maximal element. ■

Lemma 3.2.5 *Let $\alpha \in S(X, Y)$. If $|y\alpha^{-1}| = 1$ for all $y \in X\alpha \cap Y$ and $X \setminus Y \subseteq (X \setminus Y)\alpha$, then α is a maximal element.*

Proof. Assume that the conditions hold. Let $\beta \in S(X, Y)$ be such that $\alpha \leq \beta$. Since $\alpha \leq \beta$, we have α, β satisfy conditions (i) - (iii) of Theorem 3.1.1. Let $y \in Y$.

Then $y\alpha \in Y\alpha \subseteq Y\beta$. Thus $y\alpha = y'\beta$ for some $y' \in Y$. Since $y'\beta = y\alpha \in Y\alpha \subseteq X\alpha$, we get $y'\alpha = y'\beta$ by (iii) of Theorem 3.1.1. Then $y\alpha = y'\beta = y'\alpha$. Since $y\alpha \in X\alpha \cap Y$ and $y, y' \in (y\alpha)\alpha^{-1}$, we get $y = y'$ by assumption. Thus $y\alpha = y\beta$ for all $y \in Y$. Now, let $x \in X \setminus Y$. Since $X\beta \subseteq X = (X \setminus Y) \cup (Y\alpha) \cup (Y \setminus Y\alpha)$, we consider three possibilities: If $x\beta = x' \in X \setminus Y \subseteq (X \setminus Y)\alpha \subseteq X\alpha$, then $x\alpha = x\beta$. If $x\beta = x' \in Y\alpha \subseteq X\alpha$, then $x\alpha = x\beta$. If $x\beta = x' \in Y \setminus Y\alpha$, then $x'\beta^{-1} \in \pi_\beta(Y)$ since $x' \in X\beta \cap Y$. Since $\alpha \leq \beta$, we get $\pi_\beta(Y)$ refines $\pi_\alpha(Y)$, that is there exists $z \in X\alpha \cap Y$ such that $x'\beta^{-1} \subseteq z\alpha^{-1}$. Since $|z\alpha^{-1}| = 1$ and $x \in x'\beta^{-1}$, it follows that $x'\beta^{-1} = \{x\} = z\alpha^{-1}$. Then $x\beta = x'$ and $x\alpha = z$. Thus $z = x\alpha \in X\alpha \subseteq X\beta$ which implies that $z \in X\beta \cap Y$ and that $z\beta^{-1} \in \pi_\beta(Y)$. Again, since $\pi_\beta(Y)$ refines $\pi_\alpha(Y)$, we have $z\beta^{-1} \subseteq u\alpha^{-1}$ for some $u \in X\alpha \cap Y$. From $|u\alpha^{-1}| = 1$, we get $z\beta^{-1} = \{u'\} = u\alpha^{-1}$ for some $u' \in X$. Hence $u'\beta = z$ and $u'\alpha = u$. Since $u'\beta = z \in X\alpha$, it follows that $u'\beta = u'\alpha$ and that $z = u$. Since $z\alpha^{-1} = \{x\}$, $u\alpha^{-1} = \{u'\}$ and $z = u \in X\alpha \cap Y$, we have $x = u'$ by the assumption. So $x' = x\beta = u'\beta = z$ and hence $x\alpha = z = x' = x\beta$. In any cases, we have $x\alpha = x\beta$ for all $x \in X \setminus Y$.

So, $\alpha = \beta$ and therefore α is a maximal element. ■

Example 3.2.6 Let X be the set of all natural numbers and Y the set of all positive even integers. Consider

$$\alpha = \begin{pmatrix} 2 & 4 & 6 & 8 & 10 & 12 & \ldots & 1 & 3 & 5 & 7 & 9 & 11 & \ldots \\ 4 & 6 & 8 & 10 & 12 & 14 & \ldots & 3 & 5 & 7 & 9 & 11 & 13 & \ldots \end{pmatrix},$$

$$\beta = \begin{pmatrix} 2 & 4 & 6 & 8 & 10 & 12 & \ldots & 1 & 3 & 5 & 7 & 9 & 11 & \ldots \\ 4 & 4 & 6 & 8 & 10 & 12 & \ldots & 2 & 2 & 1 & 1 & 3 & 5 & \ldots \end{pmatrix},$$

$$\gamma = \begin{pmatrix} 2 & 4 & 6 & 8 & 10 & 12 & \ldots & 1 & 3 & 5 & 7 & 9 & 11 & \ldots \\ 4 & 4 & 6 & 8 & 10 & 12 & \ldots & 2 & 3 & 5 & 7 & 9 & 11 & \ldots \end{pmatrix},$$

$$\mu = \begin{pmatrix} 2 & 4 & 6 & 8 & 10 & 12 & \ldots & 1 & 3 & 5 & 7 & 9 & 11 & \ldots \\ 6 & 8 & 10 & 12 & 14 & 16 & \ldots & 2 & 1 & 1 & 3 & 5 & 7 & \ldots \end{pmatrix}.$$

Since α is injective and β is surjective, we have α and β are maximal elements by Lemma 3.2.3. Since $Y \subseteq X\gamma$ and $\gamma : X \setminus Y \to X \setminus Y\gamma$ is injective, we have γ is a maximal element by Lemma 3.2.4. Also, since $|y\mu^{-1}| = 1$ for all $y \in X\mu \cap Y$ and $X \setminus Y \subseteq (X \setminus Y)\mu$, we have μ is a maximal element by Lemma 3.2.5. ■

In order to prove Thoerem 3.2.9, the following two lemmas are needed.

Lemma 3.2.7 *Let $\alpha \in S(X,Y)$. If α is a maximal element and $|y\alpha^{-1}| > 1$ for some $y \in X\alpha \cap Y$, then $Y \subseteq X\alpha$.*

Proof. Assume that α is a maximal element and $|y\alpha^{-1}| > 1$ for some $y \in X\alpha \cap Y$. Suppose that $Y \nsubseteq X\alpha$. Then there is $z \in Y$ such that $z \notin X\alpha$. Since $|y\alpha^{-1}| > 1$, there exist $a, b \in y\alpha^{-1}$ such that $a \neq b$, that is $a\alpha = b\alpha = y \in X\alpha \cap Y$. We define $\beta \in S(X,Y)$ by

$$x\beta = \begin{cases} x\alpha & \text{if } x \neq b, \\ z & \text{if } x = b. \end{cases}$$

Since $X\beta = X\alpha \dot{\cup} \{z\}$, we obtain $\alpha \neq \beta$ and $X\alpha \subseteq X\beta$. If $z \in Y\beta$, then $Y\beta = Y\alpha \dot{\cup} \{z\}$. But, if $z \notin Y\beta$, then $Y\beta = Y\alpha$. It follows that $Y\alpha \subseteq Y\beta$. Since $z\beta^{-1} = \{b\} \subseteq y\alpha^{-1}$ and $u\beta^{-1} \subseteq u\alpha^{-1}$ for all $u \in X\beta \setminus \{z\}$, we have π_β refines π_α and $\pi_\beta(Y)$ refines $\pi_\alpha(Y)$. If $x \in X$ and $x\beta \in X\alpha$, then $x \neq b$, so $x\beta = x\alpha$ by the definition of β. Then $\alpha \leq \beta$ by Theorem 3.1.1. This implies $\alpha < \beta$ which contradicts the maximality of α. Therefore, $Y \subseteq X\alpha$ as required. ■

Lemma 3.2.8 *Let $\alpha \in S(X,Y)$. If α is a maximal element, α is not injective and $X \setminus Y \nsubseteq (X \setminus Y)\alpha$, then $Y \subseteq X\alpha$.*

Proof. Assume that α is a maximal element, α is not injective and $X \setminus Y \nsubseteq (X \setminus Y)\alpha$. Suppose that $Y \nsubseteq X\alpha$. Then there is $z \in Y$ such that $z \notin X\alpha$. Since α is not injective, there exist $a, b \in X$ such that $a \neq b$ and $a\alpha = b\alpha$.

Case $a\alpha = b\alpha \in Y$: We define $\beta \in S(X,Y)$ by

$$x\beta = \begin{cases} x\alpha & \text{if } x \neq b, \\ z & \text{if } x = b. \end{cases}$$

Since $X\beta = X\alpha\dot{\cup}\{z\}$, we obtain $\alpha \neq \beta$ and $X\alpha \subseteq X\beta$. If $z \in Y\beta$, then $Y\beta = Y\alpha\dot{\cup}\{z\}$. If $z \notin Y\beta$, then $Y\beta = Y\alpha$. It follows that $Y\alpha \subseteq Y\beta$. Since $z\beta^{-1} = \{b\} \subseteq (b\alpha)\alpha^{-1}$ and $u\beta^{-1} \subseteq u\alpha^{-1}$ for all $u \in X\beta \setminus \{z\}$, we have π_β refines π_α. Let $A \in \pi_\beta(Y)$. Then $A = y\beta^{-1}$ for some $y \in X\beta \cap Y$. If $z = y \in X\beta \cap Y$, then $A = z\beta^{-1} = \{b\} \subseteq (b\alpha)\alpha^{-1}$ where $b\alpha \in X\alpha \cap Y$. If $z \neq y \in X\beta \cap Y$, then $A = y\beta^{-1} \subseteq y\alpha^{-1}$ where $y \in X\alpha \cap Y$. Hence $\pi_\beta(Y)$ refines $\pi_\alpha(Y)$. If $x \in X$ and $x\beta \in X\alpha$, then $x \neq b$, so $x\beta = x\alpha$ by the definition of β. Then $\alpha \leq \beta$ by Theorem 3.1.1. This implies $\alpha < \beta$.

Case $a\alpha = b\alpha \in X \setminus Y$: Then $a, b \in X \setminus Y$. Since $X \setminus Y \nsubseteq (X \setminus Y)\alpha$, there is $z' \in X \setminus Y$ such that $z' \notin (X \setminus Y)\alpha$. Since $Y\alpha \subseteq Y$ and $z' \notin Y$, we get $z' \notin Y\alpha$. Thus $z' \notin X\alpha$. We define $\beta \in S(X,Y)$ by

$$x\beta = \begin{cases} x\alpha & \text{if } x \neq b, \\ z' & \text{if } x = b. \end{cases}$$

Since $X\beta = X\alpha\dot{\cup}\{z'\}$, we obtain $\alpha \neq \beta$ and $X\alpha \subseteq X\beta$. Since $b \notin Y$, we get $Y\alpha = Y\beta$. Since $z'\beta^{-1} = \{b\} \subseteq (b\alpha)\alpha^{-1}$ and $u\beta^{-1} \subseteq u\alpha^{-1}$ for all $u \in X\beta \setminus \{z'\}$, we have π_β refines π_α. Let $A \in \pi_\beta(Y)$. Then $A = y\beta^{-1}$ for some $y \in X\beta \cap Y$. Since $z' \notin Y$, we have $z' \neq y \in X\beta \cap Y$, so $A = y\beta^{-1} \subseteq y\alpha^{-1}$ where $y \in X\alpha \cap Y$. Hence $\pi_\beta(Y)$ refines $\pi_\alpha(Y)$. If $x \in X$ and $x\beta \in X\alpha$, then $x \neq b$, so $x\beta = x\alpha$ by the definition of β. Then $\alpha \leq \beta$ by Theorem 3.1.1. This implies $\alpha < \beta$.

In any cases, we have $\alpha < \beta$ which contradicts the maximality of α. Therefore, $Y \subseteq X\alpha$ as required. ■

Theorem 3.2.9 *Let $\alpha \in S(X,Y)$. Then α is a maximal element if and only if one of the following statements holds.*

(i) *α is injective.*

(ii) *α is surjective.*

(iii) *$Y \subseteq X\alpha$ and $\alpha : X \setminus Y \to X \setminus Y\alpha$ is injective.*

(iv) *$|y\alpha^{-1}| = 1$ for all $y \in X\alpha \cap Y$ and $X \setminus Y \subseteq (X \setminus Y)\alpha$.*

Proof. Assume that α is a maximal element of $S(X,Y)$ under $\leq$. We prove that one of the conditions (i)-(iv) holds by supposing that (i),(ii) and (iv) are false. That is there are two cases arise:

I. α is not injective, α is not surjective and $|y\alpha^{-1}| > 1$ for some $y \in X\alpha \cap Y$, or II. α is not injective, α is not surjective and $X \setminus Y \nsubseteq (X \setminus Y)\alpha$.

If I. occurs, then by Lemma 3.2.7, we have $Y \subseteq X\alpha$. Now, we show that $\alpha : X \setminus Y \to X \setminus Y\alpha$ by supposing that there is $a \in X \setminus Y$ such that $a\alpha \in Y\alpha$. Let $b = a\alpha$. Since $b \in Y\alpha$, we get $b = y\alpha$ for some $y \in Y$. Since α is not surjective and $Y \subseteq X\alpha$, there is $z \in X \setminus Y$ such that $z \notin X\alpha$. We define $\beta \in S(X,Y)$ by

$$x\beta = \begin{cases} x\alpha & \text{if } x \neq a, \\ z & \text{if } x = a. \end{cases}$$

Since $X\beta = X\alpha \dot{\cup} \{z\}$, we obtain $\alpha \neq \beta$ and $X\alpha \subseteq X\beta$. Since $a \notin Y$, we get $Y\alpha = Y\beta$. Since $z\beta^{-1} = \{a\} \subseteq (a\alpha)\alpha^{-1}$ and $u\beta^{-1} \subseteq u\alpha^{-1}$ for all $u \in X\beta \setminus \{z\}$, we have π_β refines π_α and $\pi_\beta(Y)$ refines $\pi_\alpha(Y)$. If $x \in X$ and $x\beta \in X\alpha$, then $x \neq a$, so $x\beta = x\alpha$ by the definition of β. Then $\alpha \leq \beta$ by Theorem 3.1.1. This implies $\alpha < \beta$ which is a contradiction. Hence $\alpha : X \setminus Y \to X \setminus Y\alpha$.

Next, we show that $\alpha : X \setminus Y \to X \setminus Y\alpha$ is injective. Suppose that there exist $u, v \in X \setminus Y$ such that $u \neq v$ and $u\alpha = v\alpha$. We define $\gamma \in S(X,Y)$ by

$$x\gamma = \begin{cases} x\alpha & \text{if } x \neq v, \\ z & \text{if } x = v. \end{cases}$$

Since $X\gamma = X\alpha \dot{\cup} \{z\}$, we obtain $\alpha \neq \gamma$ and $X\alpha \subseteq X\gamma$. Since $v \notin Y$, we get $Y\alpha = Y\gamma$. Since $z\gamma^{-1} = \{v\} \subseteq (v\alpha)\alpha^{-1}$ and $w\gamma^{-1} \subseteq w\alpha^{-1}$ for all $w \in X\gamma \setminus \{z\}$, we have π_γ refines π_α and $\pi_\gamma(Y)$ refines $\pi_\alpha(Y)$. If $x \in X$ and $x\gamma \in X\alpha$, then $x \neq v$, so $x\gamma = x\alpha$ by the definition of γ. Then $\alpha \leq \gamma$ by Theorem 3.1.1. This implies $\alpha < \gamma$ which is a contradiction. Hence $\alpha : X \setminus Y \to X \setminus Y\alpha$ is injective.

If II. occurs, then by Lemma 3.2.8, we also get $Y \subseteq X\alpha$. And by the same proof as given for case I, we obtain that $\alpha : X \setminus Y \to X \setminus Y\alpha$.

Finally, we show that $\alpha : X \setminus Y \to X \setminus Y\alpha$ is injective. Suppose that there exist $u', v' \in X \setminus Y$ such that $u' \neq v'$ and $u'\alpha = v'\alpha$. Since $X \setminus Y \nsubseteq (X \setminus Y)\alpha$, there is $z' \in X \setminus Y$ such that $z' \notin (X \setminus Y)\alpha$. Since $Y\alpha \subseteq Y$ and $z' \notin Y$, we get $z' \notin Y\alpha$. Thus $z' \notin X\alpha$. We define $\sigma \in S(X, Y)$ by

$$x\sigma = \begin{cases} x\alpha & \text{if } x \neq v', \\ z' & \text{if } x = v'. \end{cases}$$

Since $X\sigma = X\alpha \dot{\cup} \{z'\}$, we obtain $\alpha \neq \sigma$ and $X\alpha \subseteq X\sigma$. Since $v' \notin Y$, we get $Y\alpha = Y\sigma$. Since $z'\sigma^{-1} = \{v'\} \subseteq (v'\alpha)\alpha^{-1}$ and $u\sigma^{-1} \subseteq u\alpha^{-1}$ for all $u \in X\sigma \setminus \{z'\}$, we have π_σ refines π_α and $\pi_\sigma(Y)$ refines $\pi_\alpha(Y)$. If $x \in X$ and $x\sigma \in X\alpha$, then $x \neq v'$, so $x\sigma = x\alpha$ by the definition of σ. Then $\alpha \leq \sigma$ by Theorem 3.1.1. This implies $\alpha < \sigma$ which is a contradiction. Hence $\alpha : X \setminus Y \to X \setminus Y\alpha$ is injective.

In both cases, we get $Y \subseteq X\alpha$ and $\alpha : X \setminus Y \to X \setminus Y\alpha$ is injective. Therefore, we obtain (iii).

The converse is true by Lemma 3.2.3, Lemma 3.2.4 and Lemma 3.2.5. ■

Next, we give a necessary and sufficient condition for elements in $S(X, Y)$ to be a minimum element.

Theorem 3.2.10 *$S(X, Y)$ has a minimum element if and only if $|Y| = 1$.*

Proof. Assume that $S(X, Y)$ has a minimum element, say γ. Let $a, b \in Y$ and α, β constant maps in $S(X, Y)$ with images $\{a\}$ and $\{b\}$, respectively. By Theorem 3.2.1, α and β are minimal elements. Since γ is minimum, $\alpha = \gamma = \beta$. Then $a = b$. Hence $|Y| = 1$.

Conversely, let $Y = \{y\}$ and α a constant map in $S(X,Y)$ with image $\{y\}$. Let $\beta \in S(X,Y)$. We show that $\alpha \le \beta$. Since $\beta \in S(X,Y)$, $Y\beta \subseteq Y = \{y\}$. Then $Y\beta = \{y\}$. Thus $Y\alpha = \{y\} = Y\beta$ and $X\alpha = \{y\} = Y\beta \subseteq X\beta$. Since $\pi_\alpha = \pi_\alpha(Y) = \{X\}$, we get π_β refines π_α and $\pi_\beta(Y)$ refines $\pi_\alpha(Y)$. Let $x\beta \in X\alpha = \{y\}$. Then $x\beta = y = x\alpha$. Therefore $\alpha \le \beta$, and so α is a minimum element. ■

Lemma 3.2.11 *If $|Y| \ge 2$, then $S(X,Y)$ has neither maximum element nor minimum element.*

Proof. Assume that $|Y| \ge 2$. By Theorem 3.2.10, we have $S(X,Y)$ has no minimum element. Next, we show that $S(X,Y)$ has no maximum element. Let α be an identity map. Then α is injective. By Lemma 3.2.3, we have α is maximal. Since $|Y| \ge 2$, there exist $a, b \in Y$ such that $a \ne b$. We define $\beta \in S(X,Y)$ by

$$x\beta = \begin{cases} b & \text{if } x = a, \\ a & \text{if } x = b, \\ x & \text{if } x \notin \{a,b\}. \end{cases}$$

Then $\alpha \ne \beta$ and β is injective. By Lemma 3.2.3, we get β is maximal. Suppose that $S(X,Y)$ has a maximum element, say γ. Then $\alpha, \beta \le \gamma$ and hence $\alpha = \gamma = \beta$ since α and β are maximal elements. This contradicts the fact that $\alpha \ne \beta$. Hence $S(X,Y)$ has no minimum elements. ■

Theorem 3.2.12 *$S(X,Y)$ has a maximum element if and only if $|Y| = 1$ and $|X| \le 2$.*

Proof. Assume that $S(X,Y)$ has a maximum element, say γ. By Lemma 3.2.11, we have $|Y| = 1$, so let $Y = \{a\}$. Suppose that $|X| > 2$. Then there exist $b, c \in X$ such that a, b, c are all distinct. Let α be an identity map on X. Then α is injective. By Lemma 3.2.3, we have α is maximal. We define $\beta \in S(X,Y)$ by

$$x\beta = \begin{cases} c & \text{if } x = b, \\ b & \text{if } x = c, \\ x & \text{if } x \notin \{b,c\}. \end{cases}$$

Then $\alpha \neq \beta$ and β is injective. Again by Lemma 3.2.3, we have β is maximal. Since γ is maximum, we have $\alpha = \gamma = \beta$ which is a contradiction. Hence $|X| \leq 2$.

Conversely, if $|Y| = |X| = 1$, then we let $X = Y = \{a\}$ (say). Thus

$$S(X,Y) = \left\{ \begin{pmatrix} a \\ a \end{pmatrix} \right\}.$$

Then $\begin{pmatrix} a \\ a \end{pmatrix}$ is a maximum element of $S(X,Y)$. If $|Y| = 1$ and $|X| = 2$, then we let $X = \{a, b\}$ and $Y = \{a\}$. Thus

$$S(X,Y) = \left\{ \begin{pmatrix} a & b \\ a & a \end{pmatrix}, \begin{pmatrix} a & b \\ a & b \end{pmatrix} \right\}.$$

Since $\begin{pmatrix} a & b \\ a & a \end{pmatrix} \leq \begin{pmatrix} a & b \\ a & b \end{pmatrix}$, we have $\begin{pmatrix} a & b \\ a & b \end{pmatrix}$ is maximum of $S(X,Y)$. ■

3.3 Compatibility

In this section, we find elements of $S(X,Y)$ which are left compatible elements or right compatible elements with $\leq$ on $S(X,Y)$.

Lemma 3.3.1 *Let $\gamma \in S(X,Y)$. If γ is a left compatible element with $\leq$ on $S(X,Y)$ then $Y\gamma = Y$.*

Proof. Suppose that $Y\gamma \subsetneq Y$. Then there exists $y \in Y \setminus Y\gamma$. Since $Y\gamma \neq \emptyset$ and $Y\gamma \subseteq Y$, we have $|Y| > 1$. Then there exists $z \in Y$ such that $z \neq y$. Let $\alpha \in S(X,Y)$ be a constant map with image $\{y\}$. We define $\beta \in S(X,Y)$ by

$$x\beta = \begin{cases} y & \text{if } x = y, \\ z & \text{if } x \neq y. \end{cases}$$

So, the following properties hold.

(i) $X\alpha = \{y\} \subseteq \{y,z\} = X\beta$ and $Y\alpha = \{y\} \subseteq \{y,z\} = Y\beta$.

(ii) Let $A \in \pi_\beta$. Then $A = z\beta^{-1}$ or $A = y\beta^{-1}$ since $\{y,z\} = X\beta$.

If $A = z\beta^{-1}$, then $A = X \setminus \{y\} \subseteq X = y\alpha^{-1} \in \pi_\alpha$.

If $A = y\beta^{-1}$, then $A = \{y\} \subseteq X = y\alpha^{-1} \in \pi_\alpha$. Then π_β refines π_α.

Let $A \in \pi_\beta(Y)$. Then $A = z\beta^{-1}$ or $A = y\beta^{-1}$ since $\{y,z\} = X\beta \cap Y$.

If $A = z\beta^{-1}$, then $A = X \setminus \{y\} \subseteq X = y\alpha^{-1} \in \pi_\alpha(Y)$.

If $A = y\beta^{-1}$, then $A = \{y\} \subseteq X = y\alpha^{-1} \in \pi_\alpha(Y)$. Then $\pi_\beta(Y)$ refines $\pi_\alpha(Y)$.

(iii) If $x\beta \in X\alpha = \{y\}$, then $x\beta = y = x\alpha$.

Hence α and β satisfy (i)-(iii) of Theorem 3.1.1, so we conclude that $\alpha \leq \beta$. Since $Y\gamma\alpha = \{y\} \nsubseteq \{z\} = Y\gamma\beta$, we get $\gamma\alpha \nleq \gamma\beta$. Hence γ is not a left compatible element. ■

Theorem 3.3.2 *Let $\gamma \in S(X,Y)$. Then γ is a left compatible element with $\leq$ on $S(X,Y)$ if and only if $Y\gamma = Y$ and ($X\gamma = Y$ or $X\gamma = X$).*

Proof. Assume that γ is a left compatible element. By Lemma 3.3.1, we get $Y\gamma = Y$. Suppose that $X\gamma \neq Y$ and $X\gamma \neq X$. Then there exists $y \in X \setminus X\gamma$. If $y \in Y$, we get $y \in Y\gamma \subseteq X\gamma$ which is a contradiction. Thus $y \in X \setminus Y$ and this implies $|X \setminus Y| \geq 1$. Since $Y \neq \emptyset$, there exists $z \in Y$ such that $z \neq y$.

Case $|X \setminus Y| = 1$: Then $X \setminus Y = \{y\}$. Since $y\gamma \in X$, we get $y\gamma = k$ for some $k \in X$. Since $y \in X \setminus X\gamma$ and $k \in X\gamma$, we have $k \neq y$. This implies $y\gamma = k \in Y$. Thus $X\gamma = Y$ which is a contradiction.

Case $|X \setminus Y| > 1$: Since $X\gamma \neq Y$, there exists $s \in X$ such that $s\gamma \notin Y$. If $s \in Y$, then $s\gamma \in Y\gamma = Y$ which is a contradiction. Thus $s \in X \setminus Y$. Let $s\gamma = s'$. Since $s' \in X\gamma$ and $y \in X \setminus X\gamma$, we obtain $y \neq s'$. Since $y, s' \in X \setminus Y$ and $y \neq s'$ and $z \in Y$, we have z, y, s' are all distinct. We define α and $\beta \in S(X,Y)$ by

$$x\alpha = \begin{cases} y & \text{if } x \in \{s', y\}, \\ z & \text{if } x \in X \setminus \{s', y\}, \end{cases}$$

$$x\beta = \begin{cases} s' & \text{if } x = s', \\ y & \text{if } x = y, \\ z & \text{if } x \in X \setminus \{s', y\}. \end{cases}$$

We show that $\alpha \leq \beta$.

(i) $X\alpha = \{y, z\} \subseteq \{s', y, z\} = X\beta$ and $Y\alpha = \{z\} = Y\beta$.

(ii) Let $A \in \pi_\beta$. Then $A = s'\beta^{-1}$ or $A = y\beta^{-1}$ or $A = z\beta^{-1}$ since $\{s', y, z\} = X\beta$. If $A = s'\beta^{-1}$, then $A = \{s'\} \subseteq \{s', y\} = y\alpha^{-1} \in \pi_\alpha$.
If $A = y\beta^{-1}$, then $A = \{y\} \subseteq \{s', y\} = y\alpha^{-1} \in \pi_\alpha$.
If $A = z\beta^{-1}$, then $A = X \setminus \{s', y\} = z\alpha^{-1} \in \pi_\alpha$. Then π_β refines π_α.

Let $A \in \pi_\beta(Y)$. Then $A = z\beta^{-1}$ since $\{z\} = X\beta \cap Y$.
Thus $A = z\beta^{-1} = X \setminus \{s', y\} = z\alpha^{-1} \in \pi_\alpha(Y)$. Then $\pi_\beta(Y)$ refines $\pi_\alpha(Y)$.

(iii) Let $x\beta \in X\alpha = \{z, y\}$. If $x\beta = z$, then $x\beta = z = x\alpha$ by definition of α and β. If $x\beta = y$, then $x\beta = y = x\alpha$ by definition of α and β. Hence α and β satisfy (i)-(iii) of Theorem 3.1.1, so we conclude that $\alpha \leq \beta$. Since $y \neq s'$, we obtain $X\gamma\alpha = \{z, y\} \nsubseteq \{z, s'\} = X\gamma\beta$. We have $\gamma\alpha \nleq \gamma\beta$ which contradicts the left compatible element of γ.

Therefore, $X\gamma = Y$ or $X\gamma = X$.

To prove the converse, we first prove that if π_β refines π_α, then $\pi_{\gamma\beta}$ refines $\pi_{\gamma\alpha}$. Let $A \in \pi_{\gamma\beta}$. Then $A = y(\gamma\beta)^{-1}$ for some $y \in X\gamma\beta$. Let $x \in A = y(\gamma\beta)^{-1}$. Then $x\gamma\beta = y$. Thus $x\gamma \in y\beta^{-1} \in \pi_\beta$. Since π_β refines π_α, there exists $z \in X\alpha$ such that $y\beta^{-1} \subseteq z\alpha^{-1}$. Thus $x\gamma \in y\beta^{-1} \subseteq z\alpha^{-1}$. Then $x\gamma\alpha = z$. Hence $x \in z(\gamma\alpha)^{-1} \in \pi_{\gamma\alpha}$. That is $\pi_{\gamma\beta}$ refines $\pi_{\gamma\alpha}$. Similarly, we can prove that if $\pi_\beta(Y)$ refines $\pi_\alpha(Y)$, then $\pi_{\gamma\beta}(Y)$ refines $\pi_{\gamma\alpha}(Y)$.

Now, assume that $Y\gamma = Y$ and ($X\gamma = Y$ or $X\gamma = X$). Let $\alpha, \beta \in S(X,Y)$ be such that $\alpha \leq \beta$. Then π_β refines π_α and $\pi_\beta(Y)$ refines $\pi_\alpha(Y)$. Thus $\pi_{\gamma\beta}$ refines $\pi_{\gamma\alpha}$ and $\pi_{\gamma\beta}(Y)$ refines $\pi_{\gamma\alpha}(Y)$. We consider two cases. If $Y\gamma = Y = X\gamma$, then $X\gamma\alpha = Y\gamma\alpha = Y\alpha \subseteq Y\beta = Y\gamma\beta = X\gamma\beta$ and if $x\gamma\beta \in X\gamma\alpha = Y\alpha \subseteq X\alpha$, we obtain $(x\gamma)\beta = (x\gamma)\alpha$. If $Y\gamma = Y$ and $X\gamma = X$, then $X\gamma\alpha = X\alpha \subseteq X\beta = X\gamma\beta$, $Y\gamma\alpha = Y\alpha \subseteq Y\beta = Y\gamma\beta$ and if $x\gamma\beta \in X\gamma\alpha = X\alpha$, we obtain $(x\gamma)\beta = (x\gamma)\alpha$.

Therefore $\gamma\alpha$ and $\gamma\beta$ satisfy (i)-(iii) of Theorem 3.1.1, so we conclude that $\gamma\alpha \leq \gamma\beta$. Hence γ is a left compatible element. ■

Example 3.3.3 Let $X = \{1,2,3,4,5,6\}, Y = \{1,2,3,4\}$. We define $\alpha, \beta \in S(X,Y)$ by

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 2 & 1 & 3 & 4 & 4 \end{pmatrix}, \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 2 & 1 & 3 & 5 & 6 \end{pmatrix}.$$

We see that $Y\alpha = \{1,2,3,4\} = Y = X\alpha$, $Y\beta = \{1,2,3,4\} = Y$ and $X\beta = \{1,2,3,4,5,6\} = X$. Thus α and β are left compatible elements with $\leq$ on $S(X,Y)$ by Theorem 3.3.2. ■

Lemma 3.3.4 *Let $\gamma \in S(X,Y)$. If γ is a constant map, then γ is a right compatible element.*

Proof. Assume that γ is a constant map with image $\{y\}$. Let $\alpha, \beta \in S(X,Y)$ be such that $\alpha \leq \beta$. Since $X\alpha$ and $X\beta$ are nonempty sets, we have $X\alpha\gamma = \{y\} = X\beta\gamma$. This implies $\alpha\gamma = \beta\gamma$. So $\alpha\gamma \leq \beta\gamma$. Hence γ is a right compatible element. ■

Lemma 3.3.5 *Let $\alpha, \beta, \gamma \in S(X, Y)$ be such that π_β refines π_α, $\pi_\beta(Y)$ refines $\pi_\alpha(Y)$, and γ is injective. Then the following statements hold.*

(i) *$\pi_{\beta\gamma}$ refines $\pi_{\alpha\gamma}$.*

(ii) *If $(X \setminus Y)\gamma \subseteq X \setminus Y$, then $\pi_{\beta\gamma}(Y)$ refines $\pi_{\alpha\gamma}(Y)$.*

(iii) *If $(X \setminus Y)\gamma \subseteq Y \setminus Y\gamma$, then $\pi_{\beta\gamma}(Y)$ refines $\pi_{\alpha\gamma}(Y)$.*

Proof. (i) Let $A \in \pi_{\beta\gamma}$. Then $A = z(\beta\gamma)^{-1}$ for some $z \in X\beta\gamma$. Let $x \in A = z(\beta\gamma)^{-1}$. Then $x\beta\gamma = z$. Since $x\beta \in X\beta$, we have $(x\beta)\beta^{-1} \in \pi_\beta$. Since π_β refines π_α, there exists $y \in X\alpha$ such that $x \in (x\beta)\beta^{-1} \subseteq y\alpha^{-1}$. Since γ is a function, there exists $z' \in X\gamma$ such that $y\gamma = z'$. We show that $z(\beta\gamma)^{-1} = (x\beta)\beta^{-1}$.

$$\begin{aligned} s \in z(\beta\gamma)^{-1} &\Leftrightarrow s\beta\gamma = z = x\beta\gamma \\ &\Leftrightarrow s\beta = x\beta \qquad \text{(Since } \gamma \text{ is injective)} \\ &\Leftrightarrow s \in (x\beta)\beta^{-1}. \end{aligned}$$

Next, we show that $z'(\alpha\gamma)^{-1} = y\alpha^{-1}$.

$$\begin{aligned} t \in z'(\alpha\gamma)^{-1} &\Leftrightarrow t\alpha\gamma = z' = y\gamma \\ &\Leftrightarrow t\alpha = y \qquad \text{(Since } \gamma \text{ is injective)} \\ &\Leftrightarrow t \in y\alpha^{-1}. \end{aligned}$$

Since $z' = y\gamma \in X\alpha\gamma$, we get $z'(\alpha\gamma)^{-1} \in \pi_{\alpha\gamma}$. We choose $B = z'(\alpha\gamma)^{-1}$. Then $A = z(\beta\gamma)^{-1} = (x\beta)\beta^{-1} \subseteq y\alpha^{-1} = z'(\alpha\gamma)^{-1} = B$. Hence $\pi_{\beta\gamma}$ refines $\pi_{\alpha\gamma}$.

(ii) Assume that $(X \setminus Y)\gamma \subseteq X \setminus Y$. Let $A \in \pi_{\beta\gamma}(Y)$. Then $A = z(\beta\gamma)^{-1}$ for some $z \in X\beta\gamma \cap Y$. Since $X\beta\gamma \subseteq X\gamma = Y\gamma \cup (X \setminus Y)\gamma$ and $z \in X\beta\gamma$, we get $z \in Y\gamma$ or $z \in (X \setminus Y)\gamma$. If $z \in (X \setminus Y)\gamma \subseteq X \setminus Y$, then $z \notin Y$ which is a contradiction. Hence $z \in Y\gamma$. So there exists $y \in Y$ such that $y\gamma = z$. Let $x \in A = z(\beta\gamma)^{-1}$. Then $x\beta\gamma = z = y\gamma$. Since γ is injective, we get $x\beta = y$. From $y = x\beta \in X\beta \cap Y$, so $y\beta^{-1} \in \pi_\beta(Y)$. Since $\pi_\beta(Y)$ refines $\pi_\alpha(Y)$, there exists $y' \in X\alpha \cap Y$ such that $x \in y\beta^{-1} \subseteq y'\alpha^{-1}$. Since $y' \in Y$, we have $y'\gamma = z'$ for some $z' \in Y$. Then $z' = y'\gamma \in X\alpha\gamma$. Thus $z' \in X\alpha\gamma \cap Y$. We show that

$z(\beta\gamma)^{-1} = y\beta^{-1}$.

$$\begin{aligned} u \in z(\beta\gamma)^{-1} &\Leftrightarrow u\beta\gamma = z = y\gamma \\ &\Leftrightarrow u\beta = y \qquad \text{(Since } \gamma \text{ is injective)} \\ &\Leftrightarrow u \in y\beta^{-1}. \end{aligned}$$

Next, we show that $z'(\alpha\gamma)^{-1} = y'\alpha^{-1}$.

$$\begin{aligned} v \in z'(\alpha\gamma)^{-1} &\Leftrightarrow v\alpha\gamma = z' = y'\gamma \\ &\Leftrightarrow v\alpha = y' \qquad \text{(Since } \gamma \text{ is injective)} \\ &\Leftrightarrow v \in y'\alpha^{-1}. \end{aligned}$$

Since $z' = y'\gamma \in X\alpha\gamma \cap Y$, we get $z'(\alpha\gamma)^{-1} \in \pi_{\alpha\gamma}(Y)$. We choose $B = z'(\alpha\gamma)^{-1}$. Then $A = z(\beta\gamma)^{-1} = y\beta^{-1} \subseteq y'\alpha^{-1} = z'(\alpha\gamma)^{-1} = B$. Hence $\pi_{\beta\gamma}(Y)$ refines $\pi_{\alpha\gamma}(Y)$.

(iii) Assume that $(X \setminus Y)\gamma \subseteq Y \setminus Y\gamma$. Let $A \in \pi_{\beta\gamma}(Y)$. Then $A = z(\beta\gamma)^{-1}$ for some $z \in X\beta\gamma \cap Y$. Let $x \in A = z(\beta\gamma)^{-1}$. Then $x\beta\gamma = z$. Since $x\beta \in X\beta$, we have $(x\beta)\beta^{-1} \in \pi_\beta$. Since π_β refines π_α, there exists $y' \in X\alpha$ such that $x \in (x\beta)\beta^{-1} \subseteq y'\alpha^{-1}$. Since γ is a function, there exists $z' \in X\gamma$ such that $y'\gamma = z'$. Since $X\gamma = Y\gamma \cup (X \setminus Y)\gamma \subseteq Y\gamma \cup (Y \setminus Y\gamma) = Y$, we get $z' \in Y$. We show that $z(\beta\gamma)^{-1} = (x\beta)\beta^{-1}$.

$$\begin{aligned} u \in z(\beta\gamma)^{-1} &\Leftrightarrow u\beta\gamma = z = x\beta\gamma \\ &\Leftrightarrow u\beta = x\beta \qquad \text{(Since } \gamma \text{ is injective)} \\ &\Leftrightarrow u \in (x\beta)\beta^{-1}. \end{aligned}$$

Next, we show that $z'(\alpha\gamma)^{-1} = y'\alpha^{-1}$.

$$\begin{aligned} v \in z'(\alpha\gamma)^{-1} &\Leftrightarrow v\alpha\gamma = z' = y'\gamma \\ &\Leftrightarrow v\alpha = y' \qquad \text{(Since } \gamma \text{ is injective)} \\ &\Leftrightarrow v \in y'\alpha^{-1}. \end{aligned}$$

Since $z' = y'\gamma \in X\alpha\gamma \cap Y$, we get $z'(\alpha\gamma)^{-1} \in \pi_{\alpha\gamma}(Y)$. We choose $B = z'(\alpha\gamma)^{-1}$. Then $A = z(\beta\gamma)^{-1} = (x\beta)\beta^{-1} \subseteq y'\alpha^{-1} = z'(\alpha\gamma)^{-1} = B$. Hence $\pi_{\beta\gamma}(Y)$ refines $\pi_{\alpha\gamma}(Y)$. ■

Lemma 3.3.6 *Let $\gamma \in S(X,Y)$. If γ is injective and $(X \setminus Y)\gamma \subseteq X \setminus Y$, then γ is a right compatible element.*

Proof. Assume that γ is injective and $(X \setminus Y)\gamma \subseteq X \setminus Y$. Let $\alpha, \beta \in S(X,Y)$ be such that $\alpha \leq \beta$. We show that $\alpha\gamma \leq \beta\gamma$.

(i) Since $X\alpha \subseteq X\beta$ and $Y\alpha \subseteq Y\beta$, we have $X\alpha\gamma \subseteq X\beta\gamma$ and $Y\alpha\gamma \subseteq Y\beta\gamma$.

(ii) By Lemma 3.3.5(i), we have $\pi_{\beta\gamma}$ refines $\pi_{\alpha\gamma}$, and by Lemma 3.3.5(ii), we obtain that $\pi_{\beta\gamma}(Y)$ refines $\pi_{\alpha\gamma}(Y)$.

(iii) Let $x\beta\gamma \in X\alpha\gamma$. Then $x\beta\gamma = y\gamma$ for some $y \in X\alpha$. Since γ is injective, we get $x\beta = y$. Since $x\beta = y \in X\alpha$, we have $x\beta = x\alpha$ since $\alpha \leq \beta$. Thus $x\beta\gamma = x\alpha\gamma$.

Therefore $\alpha\gamma$ and $\beta\gamma$ satisfy (i)-(iii) of Theorem 3.1.1, so we conclude that $\alpha\gamma \leq \beta\gamma$. Hence γ is a right compatible element. ■

Theorem 3.3.7 *Let X be a nonempty set and $Y \subseteq X$ such that $|Y| = 1$. Then $\gamma \in S(X,Y)$ is a right compatible element if and only if one of the following statements holds.*

(i) *γ is a constant map.*

(ii) *γ is injective.*

Proof. Assume that γ is a right compatible element. Since $|Y| = 1$, we let $Y = \{y\}$. We show that γ is a constant map or γ is injective by supposing that this is false. Then γ is not a constant map and γ is not injective. Since γ is not a constant map, there exists $a \in X \setminus Y$ such that $a\gamma \neq y$. Since γ is not injective, there exist $b, c \in X$ such that $b \neq c$ and $b\gamma = c\gamma$. From $|Y| = 1$, we conclude that b and c can not both belong to Y. Therefore, we consider the following cases.

Case $b \in Y$ and $c \in X \setminus Y$: Since $b \in Y = \{y\}$, we have $b = y$. Since $c\gamma = b\gamma = y\gamma = y$ and $a\gamma \neq y$, we obtain $a\gamma \neq c\gamma$. This implies $a \neq c$. Since $a, c \in X \setminus Y$ and $a \neq c$ and $y \in Y$, we have y, a, c are all distinct. Let $\alpha \in S(X,Y)$ be such that

$$x\alpha = \begin{cases} y & \text{if } x \in Y, \\ a & \text{if } x \in X \setminus Y. \end{cases}$$

Let $\beta \in S(X,Y)$ be such that

$$x\beta = \begin{cases} y & \text{if } x \in Y, \\ a & \text{if } x = c, \\ c & \text{if } x \in X \setminus (Y \cup \{c\}). \end{cases}$$

Then $\alpha \leq \beta$. Thus

$$x\alpha\gamma = \begin{cases} y & \text{if } x \in Y, \\ a\gamma & \text{if } x \in X \setminus Y, \end{cases}$$

and

$$x\beta\gamma = \begin{cases} y & \text{if } x \in X \setminus \{c\}, \\ a\gamma & \text{if } x = c. \end{cases}$$

From $\pi_{\beta\gamma} = \{\{c\}, X \setminus \{c\}\}$ and $\pi_{\alpha\gamma} = \{\{y\}, X \setminus \{y\}\}$, we see that $\pi_{\beta\gamma}$ does not refine $\pi_{\alpha\gamma}$. Then $\alpha\gamma \nleq \beta\gamma$.

Case $b, c \in X \setminus Y$ and $b\gamma = y$: If $a = b$, then $a\gamma = b\gamma = y$ which is a contradiction. Then $a \neq b$. Also, if $a = c$, then $a\gamma = c\gamma = b\gamma = y$ which is a contradiction. Then $a \neq c$. Hence a, b, c are all distinct. Since $a, b, c \in X \setminus Y$ and $y \in Y$, we obtain y, a, b, c are all distinct. Let $\alpha \in S(X,Y)$ be such that

$$x\alpha = \begin{cases} y & \text{if } x \in Y, \\ b & \text{if } x = b, \\ a & \text{if } x \in X \setminus (Y \cup \{b\}). \end{cases}$$

Let $\beta \in S(X,Y)$ be such that

$$x\beta = \begin{cases} y & \text{if } x \in Y, \\ b & \text{if } x = b, \\ a & \text{if } x = c, \\ c & \text{if } x \in X \setminus (Y \cup \{b, c\}). \end{cases}$$

Then $\alpha \leq \beta$. Thus

$$x\alpha\gamma = \begin{cases} y & \text{if } x \in Y \cup \{b\}, \\ a\gamma & \text{if } x \in X \setminus (Y \cup \{b\}), \end{cases}$$

and

$$x\beta\gamma = \begin{cases} y & \text{if } x \in X \setminus \{c\}, \\ a\gamma & \text{if } x = c. \end{cases}$$

From $\pi_{\beta\gamma} = \{\{c\}, X \setminus \{c\}\}$ and $\pi_{\alpha\gamma} = \{\{y, b\}, X \setminus \{y, b\}\}$, we see that $\pi_{\beta\gamma}$ does not refine $\pi_{\alpha\gamma}$. Then $\alpha\gamma \not\leq \beta\gamma$.

Case $b, c \in X \setminus Y$ and $b\gamma \neq y$: Since $b, c \in X \setminus Y$ and $b \neq c$ and $y \in Y$, we obtain y, b, c are all distinct. Let $\alpha \in S(X, Y)$ be such that

$$x\alpha = \begin{cases} y & \text{if } x \in Y \cup \{b\}, \\ c & \text{if } x \in X \setminus (Y \cup \{b\}). \end{cases}$$

Let $\beta \in S(X, Y)$ be such that

$$x\beta = \begin{cases} y & \text{if } x \in Y, \\ b & \text{if } x = b, \\ c & \text{if } x \in X \setminus (Y \cup \{b\}). \end{cases}$$

Then $\alpha \leq \beta$. Thus

$$x\alpha\gamma = \begin{cases} y & \text{if } x \in Y \cup \{b\}, \\ b\gamma & \text{if } x \in X \setminus (Y \cup \{b\}), \end{cases}$$

and

$$x\beta\gamma = \begin{cases} y & \text{if } x \in Y, \\ b\gamma & \text{if } x \in X \setminus Y. \end{cases}$$

From $\pi_{\beta\gamma} = \{\{y\}, X \setminus \{y\}\}$ and $\pi_{\alpha\gamma} = \{\{y, b\}, X \setminus \{y, b\}\}$, we see that $\pi_{\beta\gamma}$ does not refine $\pi_{\alpha\gamma}$. Then $\alpha\gamma \not\leq \beta\gamma$.

In any cases, we get that γ is not a right compatible element which is a contradiction. Therefore, γ is a constant map or γ is injective.

Conversely, assume that γ is a constant map or γ is injective. Let $Y = \{y\}$. If γ is injective, then $x\gamma \neq y$ for all $x \in X \setminus Y$. That is $(X \setminus Y)\gamma \subseteq X \setminus Y$. By Lemma 3.3.6, we get γ is a right compatible element. If γ is a constant map, then by Lemma 3.3.4, we obtain γ is a right compatible element. ∎

Example 3.3.8 Let $X = \{1, 2, 3, 4, 5, 6\}, Y = \{1\}$. We define $\alpha, \beta \in S(X, Y)$ by

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 4 & 5 & 6 & 2 \end{pmatrix}.$$

We see that α is a constant map and β is injective. Thus α and β are right compatible elements with $\leq$ on $S(X, Y)$ by Theorem 3.3.7. ■

Lemma 3.3.9 *Let $\gamma \in S(X, Y)$. If γ is injective and $(X \setminus Y)\gamma \subseteq Y \setminus Y\gamma$, then γ is a right compatible element.*

Proof. Assume that γ is injective and $(X \setminus Y)\gamma \subseteq Y \setminus Y\gamma$. Let $\alpha, \beta \in S(X, Y)$ be such that $\alpha \leq \beta$. We show that $\alpha\gamma \leq \beta\gamma$.

(i) Since $X\alpha \subseteq X\beta$ and $Y\alpha \subseteq Y\beta$, we have $X\alpha\gamma \subseteq X\beta\gamma$ and $Y\alpha\gamma \subseteq Y\beta\gamma$.

(ii) By Lemma 3.3.5(i), we have $\pi_{\beta\gamma}$ refines $\pi_{\alpha\gamma}$, and by Lemma 3.3.5(iii), we obtain that $\pi_{\beta\gamma}(Y)$ refines $\pi_{\alpha\gamma}(Y)$.

(iii) Let $x\beta\gamma \in X\alpha\gamma$. Then $x\beta\gamma = y\gamma$ for some $y \in X\alpha$. Since γ is injective, we get $x\beta = y$. Since $x\beta = y \in X\alpha$, we have $x\beta = x\alpha$. Thus $x\beta\gamma = x\alpha\gamma$.

Therefore $\alpha\gamma$ and $\beta\gamma$ satisfy (i)-(iii) of Theorem 3.1.1, so we conclude that $\alpha\gamma \leq \beta\gamma$. Hence γ is a right compatible element. ■

Lemma 3.3.10 *Let $\alpha, \beta, \gamma \in S(X, Y)$ be such that π_β refines π_α, $\pi_\beta(Y)$ refines $\pi_\alpha(Y)$, and $\gamma|_Y$ is a constant map. Then the following statements hold.*

(i) *If $\gamma : X \setminus Y \to X \setminus Y$ is injective, then $\pi_{\beta\gamma}$ refines $\pi_{\alpha\gamma}$ and $\pi_{\beta\gamma}(Y)$ refines $\pi_{\alpha\gamma}(Y)$.*

(ii) *If $\gamma : X \setminus Y \to Y \setminus Y\gamma$ is injective, then $\pi_{\beta\gamma}$ refines $\pi_{\alpha\gamma}$ and $\pi_{\beta\gamma}(Y)$ refines $\pi_{\alpha\gamma}(Y)$.*

Proof. Let $\gamma|_Y$ be a constant map with image $\{y\}$.

(i) Assume that $\gamma : X \setminus Y \to X \setminus Y$ is injective. Let $A \in \pi_{\beta\gamma}$. Then $A = z(\beta\gamma)^{-1}$ for some $z \in X\beta\gamma$.

Case $z = y$: We show that $y(\beta\gamma)^{-1} \subseteq y(\alpha\gamma)^{-1}$. Let $s \in y(\beta\gamma)^{-1}$. Then $s\beta\gamma = y \in Y$. If $s\beta \in X \setminus Y$, then $s\beta\gamma \in X \setminus Y$ by $(X \setminus Y)\gamma \subseteq X \setminus Y$ which is a contradiction, so $s\beta \in Y$. Since $s\beta \in X\beta \cap Y$, we have $(s\beta)\beta^{-1} \in \pi_\beta(Y)$. Since $\pi_\beta(Y)$ refines $\pi_\alpha(Y)$, there exists $k \in X\alpha \cap Y$ such that $s \in (s\beta)\beta^{-1} \subseteq k\alpha^{-1}$. Then $s\alpha = k$. Thus $s\alpha\gamma = k\gamma = y$ since $k \in Y$. So $s \in y(\alpha\gamma)^{-1}$. This implies $y(\beta\gamma)^{-1} \subseteq y(\alpha\gamma)^{-1}$. Since $y \in X\alpha\gamma$, we have $y(\alpha\gamma)^{-1} \in \pi_{\alpha\gamma}$. We choose $B = y(\alpha\gamma)^{-1}$. Thus $A = z(\beta\gamma)^{-1} = y(\beta\gamma)^{-1} \subseteq y(\alpha\gamma)^{-1} = B$.

Case $z \neq y$: Let $t \in A = z(\beta\gamma)^{-1}$. Then $t\beta\gamma = z$. Since $t\beta \in X\beta$, we have $(t\beta)\beta^{-1} \in \pi_\beta$. Since π_β refines π_α, there exists $\ell \in X\alpha$ such that $t \in (t\beta)\beta^{-1} \subseteq \ell\alpha^{-1}$. Since γ is a function, there exists $z' \in X\gamma$ such that $\ell\gamma = z'$. We show that $z(\beta\gamma)^{-1} = (t\beta)\beta^{-1}$. Let $u \in z(\beta\gamma)^{-1}$. Then $u\beta\gamma = z = t\beta\gamma$. Since $z \neq y$, we have $u\beta$ and $t\beta \notin Y$. Then $u\beta$ and $t\beta \in X \setminus Y$. Since $\gamma|_{X\setminus Y}$ is injective, we get $u\beta = t\beta$. Then $u \in (t\beta)\beta^{-1}$. That is $z(\beta\gamma)^{-1} \subseteq (t\beta)\beta^{-1}$. On the other hand, let $v \in (t\beta)\beta^{-1}$. Then $v\beta = t\beta$. Thus $v\beta\gamma = t\beta\gamma = z$. Hence $v \in z(\beta\gamma)^{-1}$. That is $(t\beta)\beta^{-1} \subseteq z(\beta\gamma)^{-1}$. Therefore $z(\beta\gamma)^{-1} = (t\beta)\beta^{-1}$. Now, we show that $\ell\alpha^{-1} \subseteq z'(\alpha\gamma)^{-1}$. Let $w \in \ell\alpha^{-1}$. Then $w\alpha = \ell$. Thus $w\alpha\gamma = \ell\gamma = z'$. Then $w \in z'(\alpha\gamma)^{-1}$. That is $\ell\alpha^{-1} \subseteq z'(\alpha\gamma)^{-1}$. Since $z' = \ell\gamma \in X\alpha\gamma$, we get $z'(\alpha\gamma)^{-1} \in \pi_{\alpha\gamma}$. We choose $B = z'(\alpha\gamma)^{-1}$. Then $A = z(\beta\gamma)^{-1} = (t\beta)\beta^{-1} \subseteq \ell\alpha^{-1} \subseteq z'(\alpha\gamma)^{-1} = B$.

By both cases, we have $\pi_{\beta\gamma}$ refines $\pi_{\alpha\gamma}$.

Now, we prove $\pi_{\beta\gamma}(Y)$ refines $\pi_{\alpha\gamma}(Y)$. Let $A \in \pi_{\beta\gamma}(Y)$. Then $A = z(\beta\gamma)^{-1}$ for some $z \in X\beta\gamma \cap Y$. We show that $X\beta\gamma \cap Y = \{y\}$. Let $x \in X\beta\gamma \cap Y$. Then $x = k\beta\gamma$ for some $k \in X$. If $k\beta \in X \setminus Y$, then $x = k\beta\gamma \in X \setminus Y$ which is a contradiction. Then $k\beta \in Y$. Thus $k\beta\gamma = x = y$. On the other hand, from $y \in Y$, so $y\beta \in Y\beta$. Thus $y\beta\gamma \in Y\beta\gamma \subseteq Y\gamma = \{y\}$. So $y = y\beta\gamma \in Y\beta\gamma \subseteq X\beta\gamma$. Then $y \in X\beta\gamma \cap Y$. Hence $X\beta\gamma \cap Y = \{y\}$. This implies $z = y$. Next, we show that $y(\beta\gamma)^{-1} \subseteq y(\alpha\gamma)^{-1}$. Let $u \in y(\beta\gamma)^{-1}$. Then $u\beta\gamma = y$. Since $u\beta \in X\beta \cap Y$, we have $(u\beta)\beta^{-1} \in \pi_\beta(Y)$. Since $\pi_\beta(Y)$ refines $\pi_\alpha(Y)$, there exists $\ell \in X\alpha \cap Y$ such that $u \in (u\beta)\beta^{-1} \subseteq \ell\alpha^{-1}$. Then $u\alpha = \ell$. Since $\ell \in Y$, we get $u\alpha\gamma = \ell\gamma = y$. Thus $u \in y(\alpha\gamma)^{-1}$. That is $y(\beta\gamma)^{-1} \subseteq y(\alpha\gamma)^{-1}$. Since $y = u\alpha\gamma \in X\alpha\gamma \cap Y$, we get $y(\alpha\gamma)^{-1} \in \pi_{\alpha\gamma}(Y)$. We choose $B = y(\alpha\gamma)^{-1}$. Then $A = z(\beta\gamma)^{-1} = y(\beta\gamma)^{-1} \subseteq y(\alpha\gamma)^{-1} = B$. Therefore $\pi_{\beta\gamma}(Y)$ refines $\pi_{\alpha\gamma}(Y)$.

(ii) Assume that $\gamma : X \setminus Y \to Y \setminus Y\gamma$ is injective. Let $A \in \pi_{\beta\gamma}$. Then $A = z(\beta\gamma)^{-1}$ for some $z \in X\beta\gamma$.

Case $z = y$: We show that $y(\beta\gamma)^{-1} \subseteq y(\alpha\gamma)^{-1}$. Let $s \in y(\beta\gamma)^{-1}$. Then $s\beta\gamma = y$. If $s\beta \in X \setminus Y$, we obtain $s\beta\gamma \in Y \setminus Y\gamma = Y \setminus \{y\}$ by $(X \setminus Y)\gamma \subseteq Y \setminus Y\gamma$. Then $s\beta\gamma \neq y$ which is a contradiction, thus $s\beta \in Y$. Since $s\beta \in X\beta \cap Y$, we have $(s\beta)\beta^{-1} \in \pi_\beta(Y)$. Since $\pi_\beta(Y)$ refines $\pi_\alpha(Y)$, there exists $k \in X\alpha \cap Y$ such that $s \in (s\beta)\beta^{-1} \subseteq k\alpha^{-1}$. Then $s\alpha = k$. Thus $s\alpha\gamma = k\gamma = y$ since $k \in Y$. Then $s \in y(\alpha\gamma)^{-1}$. This implies $y(\beta\gamma)^{-1} \subseteq y(\alpha\gamma)^{-1}$. Since $y \in X\alpha\gamma$, we get $y(\alpha\gamma)^{-1} \in \pi_{\alpha\gamma}$. We choose $B = y(\alpha\gamma)^{-1}$. Thus $A = z(\beta\gamma)^{-1} = y(\beta\gamma)^{-1} \subseteq y(\alpha\gamma)^{-1} = B$.

Case $z \neq y$: By the same proof as given in (i) case $z \neq y$, we have $A \subseteq B$ for some $B \in \pi_{\alpha\gamma}$.

By both cases, we have $\pi_{\beta\gamma}$ refines $\pi_{\alpha\gamma}$.

Now, we prove $\pi_{\beta\gamma}(Y)$ refines $\pi_{\alpha\gamma}(Y)$. Let $A \in \pi_{\beta\gamma}(Y)$. Then $A = z(\beta\gamma)^{-1}$ for some $z \in X\beta\gamma \cap Y$. From $X\beta\gamma \subseteq X\gamma = Y\gamma \cup (X \setminus Y)\gamma \subseteq \{y\} \cup (Y \setminus Y\gamma)$, then $z \in \{y\} \cup (Y \setminus Y\gamma)$.

Case $z = y$: We show that $y(\beta\gamma)^{-1} \subseteq y(\alpha\gamma)^{-1}$. Let $s \in y(\beta\gamma)^{-1}$. Then $s\beta\gamma = y$. Since $s\beta \in X\beta \cap Y$, we have $(s\beta)\beta^{-1} \in \pi_\beta(Y)$. Since $\pi_\beta(Y)$ refines $\pi_\alpha(Y)$, there exists $k \in X\alpha \cap Y$ such that $s \in (s\beta)\beta^{-1} \subseteq k\alpha^{-1}$. Then $s\alpha = k$. Thus $s\alpha\gamma = k\gamma = y$ since $k \in Y$. Then $s \in y(\alpha\gamma)^{-1}$. This implies $y(\beta\gamma)^{-1} \subseteq y(\alpha\gamma)^{-1}$. Since $y \in X\alpha\gamma \cap Y$, we have $y(\alpha\gamma)^{-1} \in \pi_{\alpha\gamma}(Y)$. We choose $B = y(\alpha\gamma)^{-1}$. Thus $A = z(\beta\gamma)^{-1} = y(\beta\gamma)^{-1} \subseteq y(\alpha\gamma)^{-1} = B$.

Case $z \neq y$: Then $z \in Y \setminus Y\gamma$. Let $t \in A = z(\beta\gamma)^{-1}$. Then $t\beta\gamma = z$. Since $t\beta \in X\beta$, we have $(t\beta)\beta^{-1} \in \pi_\beta$. Since π_β refines π_α, there exists $\ell \in X\alpha$ such that $t \in (t\beta)\beta^{-1} \subseteq \ell\alpha^{-1}$. Since γ is a function, there exists $z' \in X\gamma$ such that $\ell\gamma = z'$. Since $X\gamma = Y\gamma \cup (X \setminus Y)\gamma \subseteq \{y\} \cup (Y \setminus Y\gamma) \subseteq Y$, we get $z' \in Y$. We show that $z(\beta\gamma)^{-1} = (t\beta)\beta^{-1}$. Let $u \in z(\beta\gamma)^{-1}$. Then $u\beta\gamma = z = t\beta\gamma$. Since $z \neq y$, we have $u\beta$ and $t\beta \notin Y$. Then $u\beta$ and $t\beta \in X \setminus Y$. Since $\gamma : X \setminus Y \to Y \setminus Y\gamma$ is injective, we get $u\beta = t\beta$. Then $u \in (t\beta)\beta^{-1}$. That is $z(\beta\gamma)^{-1} \subseteq (t\beta)\beta^{-1}$.

On the other hand, let $v \in (t\beta)\beta^{-1}$. Then $v\beta = t\beta$. Thus $v\beta\gamma = t\beta\gamma = z$. Hence $v \in z(\beta\gamma)^{-1}$. That is $(t\beta)\beta^{-1} \subseteq z(\beta\gamma)^{-1}$. Therefore $z(\beta\gamma)^{-1} = (t\beta)\beta^{-1}$. We show that $\ell\alpha^{-1} \subseteq z'(\alpha\gamma)^{-1}$. Let $w \in \ell\alpha^{-1}$. Then $w\alpha = \ell$. Thus $w\alpha\gamma = \ell\gamma = z'$. Then $w \in z'(\alpha\gamma)^{-1}$. That is $\ell\gamma^{-1} \subseteq z'(\alpha\gamma)^{-1}$. Since $z' = \ell\gamma \in X\alpha\gamma \cap Y$, we get $z'(\alpha\gamma)^{-1} \in \pi_{\alpha\gamma}(Y)$. We choose $B = z'(\alpha\gamma)^{-1}$. Then $A = z(\beta\gamma)^{-1} = (t\beta)\beta^{-1} \subseteq \ell\alpha^{-1} \subseteq z'(\alpha\gamma)^{-1} = B$. Therefore $\pi_{\beta\gamma}(Y)$ refines $\pi_{\alpha\gamma}(Y)$. ■

Lemma 3.3.11 *Let $\gamma \in S(X,Y)$. If $\gamma|_Y$ is a constant map and $\gamma : X \setminus Y \to X \setminus Y$ is injective, then γ is a right compatible element.*

Proof. Assume that $\gamma|_Y$ is a constant map with image $\{y\}$ and $\gamma : X \setminus Y \to X \setminus Y$ is injective. Let $\alpha, \beta \in S(X,Y)$ be such that $\alpha \leq \beta$. We show that $\alpha\gamma \leq \beta\gamma$.

(i) Since $X\alpha \subseteq X\beta$ and $Y\alpha \subseteq Y\beta$, we have $X\alpha\gamma \subseteq X\beta\gamma$ and $Y\alpha\gamma \subseteq Y\beta\gamma$.

(ii) By Lemma 3.3.10(i), we get $\pi_{\beta\gamma}$ refines $\pi_{\alpha\gamma}$ and $\pi_{\beta\gamma}(Y)$ refines $\pi_{\alpha\gamma}(Y)$.

(iii) Let $x\beta\gamma \in X\alpha\gamma$. Then $x\beta\gamma = y'\gamma$ for some $y' \in X\alpha$.

Case $x\beta \in X \setminus Y$: Since $\gamma : X \setminus Y \to X \setminus Y$, we get $x\beta\gamma = g$ for some $g \in X \setminus Y$. So $x\beta\gamma = g = y'\gamma$. This implies $y' \in X \setminus Y$. Since $\gamma : X \setminus Y \to X \setminus Y$ is injective, we have $x\beta = y'$. Since $x\beta = y' \in X\alpha$, we have $x\beta = x\alpha$. Thus $x\beta\gamma = x\alpha\gamma$.

Case $x\beta \in Y$: Then $x\beta\gamma = y$. If $x \in Y$, then $x\alpha \in Y\alpha \subseteq Y$. Then $x\alpha\gamma = y$. Hence $x\beta\gamma = y = x\alpha\gamma$. If $x \in X \setminus Y$, we know that $x\beta \in Y$ so $x\beta = g$ for some $g \in Y$. Since $g \in X\beta \cap Y$, we get $x \in g\beta^{-1} \in \pi_\beta(Y)$. Suppose that $x\alpha \in X \setminus Y$. Then $x\alpha = h$ for some $h \in X \setminus Y$. Then $h\alpha^{-1}$ contains x. Since α is a function, there is no element in $\pi_\alpha(Y)$ which contain x. Thus $g\beta^{-1} \nsubseteq A$ for all $A \in \pi_\alpha(Y)$. Hence $\alpha \nleq \beta$ which is a contradiction. Then $x\alpha \in Y$. So $x\alpha\gamma = y$. Thus $x\beta\gamma = y = x\alpha\gamma$.

Therefore $\alpha\gamma$ and $\beta\gamma$ satisfy (i)-(iii) of Theorem 3.1.1, so we conclude that $\alpha\gamma \leq \beta\gamma$. Hence γ is a right compatible element. ■

Lemma 3.3.12 *Let $\gamma \in S(X,Y)$. If $\gamma|_Y$ is a constant map and $\gamma : X\setminus Y \to Y\setminus Y\gamma$ is injective, then γ is a right compatible element.*

Proof. Assume that $\gamma|_Y$ is a constant map with image $\{y\}$ and $\gamma : X\setminus Y \to Y\setminus Y\gamma$ is injective. Let $\alpha, \beta \in S(X,Y)$ be such that $\alpha \leq \beta$. We show that $\alpha\gamma \leq \beta\gamma$.

(i) Since $X\alpha \subseteq X\beta$ and $Y\alpha \subseteq Y\beta$, we have $X\alpha\gamma \subseteq X\beta\gamma$ and $Y\alpha\gamma \subseteq Y\beta\gamma$.

(ii) By Lemma 3.3.10(ii), we get $\pi_{\beta\gamma}$ refines $\pi_{\alpha\gamma}$ and $\pi_{\beta\gamma}(Y)$ refines $\pi_{\alpha\gamma}(Y)$.

(iii) Let $x\beta\gamma \in X\alpha\gamma$. Then $x\beta\gamma = y'\gamma$ for some $y' \in X\alpha$.

Case $x\beta \in X\setminus Y$: Since $\gamma : X\setminus Y \to Y\setminus Y\gamma$, we get $x\beta\gamma = g$ for some $g \in Y\setminus Y\gamma$. So $x\beta\gamma = g = y'\gamma$. This implies $y' \in X\setminus Y$. Since $\gamma : X\setminus Y \to Y\setminus Y\gamma$ is injective, we have $x\beta = y'$. Since $x\beta = y' \in X\alpha$, we have $x\beta = x\alpha$. Thus $x\beta\gamma = x\alpha\gamma$.

Case $x\beta \in Y$: Then $x\beta\gamma = y$. If $x \in Y$, then $x\alpha \in Y\alpha \subseteq Y$. Then $x\alpha\gamma = y$. Hence $x\beta\gamma = y = x\alpha\gamma$. If $x \in X\setminus Y$, we know that $x\beta \in Y$ so $x\beta = g$ for some $g \in Y$. Since $g \in X\beta \cap Y$, we get $x \in g\beta^{-1} \in \pi_\beta(Y)$. Suppose that $x\alpha \in X\setminus Y$. Then $x\alpha = h$ for some $h \in X\setminus Y$. Then $h\alpha^{-1}$ contains x. Since α is a function, there is no element in $\pi_\alpha(Y)$ which contain x. Thus $g\beta^{-1} \nsubseteq A$ for all $A \in \pi_\alpha(Y)$. Hence $\alpha \nleq \beta$ which is a contradiction. Then $x\alpha \in Y$. So $x\alpha\gamma = y$. Thus $x\beta\gamma = y = x\alpha\gamma$.

Therefore $\alpha\gamma$ and $\beta\gamma$ satisfy (i)-(iii) of Theorem 3.1.1, so we conclude that $\alpha\gamma \leq \beta\gamma$. Hence γ is a right compatible element. ■

Lemma 3.3.13 *Let $\gamma \in S(X,Y)$ be such that γ is a right compatible element, $|Y| \geq 2$ and $|X\setminus Y| \leq 1$. If γ is not a constant map, then $(X\setminus Y)\gamma \subseteq X\setminus Y$ or $(X\setminus Y)\gamma \subseteq Y\setminus Y\gamma$.*

Proof. Assume that γ is not a constant map. Then there exist $a, b \in X$ such that $a \neq b$ and $a\gamma \neq b\gamma$. If $|X\setminus Y| = 0$, then $X = Y$. So $(X\setminus Y)\gamma \subseteq X\setminus Y$ or $(X\setminus Y)\gamma \subseteq Y\setminus Y\gamma$ holds. Now, consider the case $|X\setminus Y| = 1$ and suppose that $(X\setminus Y)\gamma \nsubseteq X\setminus Y$ and $(X\setminus Y)\gamma \nsubseteq Y\setminus Y\gamma$. Let $X\setminus Y = \{c\}$. Then $c\gamma \in Y$ and ($c\gamma \in Y\gamma$ or $c\gamma \in X\setminus Y$). This implies $c\gamma \in Y\gamma$. So $c\gamma = y\gamma$ for some $y \in Y$.

Case $a = y$ or $a = c$: Then $c\gamma = y\gamma = a\gamma$. If $b = c$, then $b\gamma = c\gamma = a\gamma$ which is a contradiction, thus $b \neq c$. This implies $b \in Y$. If $b = y$, then $b\gamma = y\gamma = a\gamma$ which is also a contradiction, thus $b \neq y$. Since $b, y \in Y$ and $b \neq y$ and $c \in X \setminus Y$, we obtain b, c, y are all distinct. Let $\alpha \in S(X, Y)$ be such that

$$x\alpha = \begin{cases} b & \text{if } x \in Y, \\ c & \text{if } x = c. \end{cases}$$

Let $\beta \in S(X, Y)$ be such that

$$x\beta = \begin{cases} y & \text{if } x = y, \\ b & \text{if } x \in Y \setminus \{y\}, \\ c & \text{if } x = c. \end{cases}$$

Then $\alpha \leq \beta$. Thus

$$x\alpha\gamma = \begin{cases} b\gamma & \text{if } x \in Y, \\ a\gamma & \text{if } x = c, \end{cases}$$

and

$$x\beta\gamma = \begin{cases} b\gamma & \text{if } x \in Y \setminus \{y\}, \\ a\gamma & \text{if } x \in \{y, c\}. \end{cases}$$

From $\pi_{\beta\gamma} = \{\{y, c\}, Y \setminus \{y\}\}$ and $\pi_{\alpha\gamma} = \{\{c\}, Y\}$, we see that $\pi_{\beta\gamma}$ does not refine $\pi_{\alpha\gamma}$. Then $\alpha\gamma \not\leq \beta\gamma$.

Case $a \neq y$ and $a \neq c$ and $a\gamma = y\gamma$: Since $a \neq c$, we get $a \in Y$. If $b = c$, then $b\gamma = c\gamma = y\gamma = a\gamma$ which is a contradiction, thus $b \neq c$. This implies $b \in Y$. If $b = y$, then $b\gamma = y\gamma = a\gamma$ which is also a contradiction, thus $b \neq y$. Now, we have a, b, y are all distinct. Since $a, b, y \in Y$ and $c \in X \setminus Y$, we obtain a, b, c, y are all distinct. Let $\alpha \in S(X, Y)$ be such that

$$x\alpha = \begin{cases} a & \text{if } x \in Y \setminus \{y\}, \\ b & \text{if } x \in \{y, c\}. \end{cases}$$

Let $\beta \in S(X,Y)$ be such that

$$x\beta = \begin{cases} a & \text{if } x \in Y \setminus \{y\}, \\ b & \text{if } x = y, \\ y & \text{if } x = c. \end{cases}$$

Then $\alpha \leq \beta$. Thus

$$x\alpha\gamma = \begin{cases} a\gamma & \text{if } x \in Y \setminus \{y\}, \\ b\gamma & \text{if } x \in \{y, c\}, \end{cases}$$

and

$$x\beta\gamma = \begin{cases} a\gamma & \text{if } x \in X \setminus \{y\}, \\ b\gamma & \text{if } x = y. \end{cases}$$

From $\pi_{\beta\gamma} = \{\{y\}, X \setminus \{y\}\}$ and $\pi_{\alpha\gamma} = \{\{y,c\}, Y \setminus \{y\}\}$, we see that $\pi_{\beta\gamma}$ does not refine $\pi_{\alpha\gamma}$. Then $\alpha\gamma \not\leq \beta\gamma$.

Case $a \neq y$ and $a \neq c$ and $a\gamma \neq y\gamma$: Since $a \neq c$, we get $a \in Y$. Since $a, y \in Y$ and $a \neq y$ and $c \in X \setminus Y$, we obtain a, c, y are all distinct. Let $\alpha \in S(X,Y)$ be such that

$$x\alpha = \begin{cases} a & \text{if } x \in Y, \\ c & \text{if } x = c. \end{cases}$$

Let $\beta \in S(X,Y)$ be such that

$$x\beta = \begin{cases} a & \text{if } x \in Y \setminus \{y\}, \\ y & \text{if } x = y, \\ c & \text{if } x = c. \end{cases}$$

Then $\alpha \leq \beta$. Thus

$$x\alpha\gamma = \begin{cases} a\gamma & \text{if } x \in Y, \\ y\gamma & \text{if } x = c, \end{cases}$$

and

$$x\beta\gamma = \begin{cases} a\gamma & \text{if } x \in Y \setminus \{y\}, \\ y\gamma & \text{if } x \in \{y, c\}. \end{cases}$$

From $\pi_{\beta\gamma} = \{\{y, c\}, Y \setminus \{y\}\}$ and $\pi_{\alpha\gamma} = \{\{c\}, Y\}$, we see that $\pi_{\beta\gamma}$ does not refine $\pi_{\alpha\gamma}$. Then $\alpha\gamma \nleq \beta\gamma$.

In any cases, we have γ is not a right compatible element which is a contradiction. Therefore $(X \setminus Y)\gamma \subseteq X \setminus Y$ or $(X \setminus Y)\gamma \subseteq Y \setminus Y\gamma$. ■

Theorem 3.3.14 *Let X be a nonempty set and $Y \subseteq X$ such that $|Y| \geq 2$ and $|X \setminus Y| \leq 1$. Then $\gamma \in S(X, Y)$ is a right compatible element if and only if one of the following statements holds.*

(i) *γ is a constant map.*

(ii) *[γ is injective or $\gamma|_Y$ is a constant map] and*
[$(X \setminus Y)\gamma \subseteq X \setminus Y$ or $(X \setminus Y)\gamma \subseteq Y \setminus Y\gamma$].

Proof. Assume that γ is a right compatible element and γ is not a constant map. By Lemma 3.3.13, we have $(X \setminus Y)\gamma \subseteq X \setminus Y$ or $(X \setminus Y)\gamma \subseteq Y \setminus Y\gamma$. We show that γ is injective or $\gamma|_Y$ is a constant map by supposing that this is false. So γ is not injective and $\gamma|_Y$ is not a constant map. Then there exist $c, d \in X$ such that $c \neq d$ and $c\gamma = d\gamma$. Also, there exist $a, b \in Y$ such that $a \neq b$ and $a\gamma \neq b\gamma$. If $|X \setminus Y| = 0$, then $c, d \in Y$. Next, we consider in case $|X \setminus Y| = 1$, if $c \in X \setminus Y$, then $d \in Y$ since $|X \setminus Y| = 1$. So $c\gamma \in (X \setminus Y)\gamma \subseteq X \setminus Y$ or $c\gamma \in (X \setminus Y)\gamma \subseteq Y \setminus Y\gamma$. Thus $d\gamma \in Y\gamma$ but $c\gamma \notin Y\gamma$, and so $c\gamma \neq d\gamma$ which is a contradiction. Similarly, if $d \in X \setminus Y$, then it will lead to a contradiction. Hence c and d are both belong to Y. So, we consider the following two cases.

Case $c, d \in Y$ and $c\gamma \neq a\gamma$: Then $c \neq a$. If $a = d$, then $a\gamma = d\gamma = c\gamma$ which is a contradiction. Then $a \neq d$. Hence a, c, d are all distinct. Let $\alpha \in S(X, Y)$ be such that

$$x\alpha = \begin{cases} a & \text{if } x \in \{c, d\}, \\ c & \text{if } x \in X \setminus \{c, d\}. \end{cases}$$

Let $\beta \in S(X,Y)$ be such that

$$x\beta = \begin{cases} a & \text{if } x = d, \\ d & \text{if } x = c, \\ c & \text{if } x \in X \setminus \{c,d\}. \end{cases}$$

Then $\alpha \leq \beta$. Thus

$$x\alpha\gamma = \begin{cases} a\gamma & \text{if } x \in \{c,d\}, \\ c\gamma & \text{if } x \in X \setminus \{c,d\}, \end{cases}$$

and

$$x\beta\gamma = \begin{cases} a\gamma & \text{if } x = d, \\ c\gamma & \text{if } x \in X \setminus \{d\}. \end{cases}$$

From $\pi_{\beta\gamma} = \{\{d\}, X \setminus \{d\}\}$ and $\pi_{\alpha\gamma} = \{\{c,d\}, X \setminus \{c,d\}\}$, we see that $\pi_{\beta\gamma}$ does not refine $\pi_{\alpha\gamma}$. Then $\alpha\gamma \nleq \beta\gamma$ which is a contradiction.

Case $c, d \in Y$ and $c\gamma = a\gamma$: Since $a\gamma \neq b\gamma$, we have $c\gamma \neq b\gamma$, this implies $c \neq b$. If $b = d$, then $b\gamma = d\gamma = c\gamma = a\gamma$ which is a contradiction. Thus $b \neq d$, and hence b, c, d are all distinct. Let $\alpha \in S(X,Y)$ be such that

$$x\alpha = \begin{cases} b & \text{if } x \in \{c,d\}, \\ c & \text{if } x \in X \setminus \{c,d\}. \end{cases}$$

Let $\beta \in S(X,Y)$ be such that

$$x\beta = \begin{cases} d & \text{if } x = c, \\ b & \text{if } x = d, \\ c & \text{if } x \in X \setminus \{c,d\}. \end{cases}$$

Then $\alpha \leq \beta$. Thus

$$x\alpha\gamma = \begin{cases} b\gamma & \text{if } x \in \{c,d\}, \\ a\gamma & \text{if } x \in X \setminus \{c,d\}, \end{cases}$$

and

$$x\beta\gamma = \begin{cases} b\gamma & \text{if } x = d, \\ a\gamma & \text{if } x \in X \setminus \{d\}. \end{cases}$$

From $\pi_{\beta\gamma} = \{\{d\}, X \setminus \{d\}\}$ and $\pi_{\alpha\gamma} = \{\{c,d\}, X \setminus \{c,d\}\}$, we see that $\pi_{\beta\gamma}$ does not refine $\pi_{\alpha\gamma}$. Then $\alpha\gamma \not\leq \beta\gamma$ which is a contradiction.

Therefore γ is injective or $\gamma|_Y$ is a constant map.

The converse is true by Lemma 3.3.4, Lemma 3.3.6, Lemma 3.3.9, Lemma 3.3.11 and Lemma 3.3.12. ∎

Example 3.3.15 Let $X = \{1,2,3,4,5,6\}, Y = \{1,2,3,4,5\}$, and define $\alpha \in S(X,Y)$ by

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 2 & 2 & 2 & 2 & 3 \end{pmatrix}$$

We see that $\alpha|_Y$ is a constant map and $(X \setminus Y)\alpha = \{3\} \subseteq \{1,3,4,5\} = Y \setminus Y\alpha$. Thus α is a right compatible element with $\leq$ on $S(X,Y)$ by Theorem 3.3.14. ∎

Lemma 3.3.16 *Let $\gamma \in S(X,Y)$ be such that γ is a right compatible element, $|Y| \geq 2$ and $|X \setminus Y| > 1$. If γ is not a constant map, then $(X \setminus Y)\gamma \subseteq X \setminus Y$ or $(X \setminus Y)\gamma \subseteq Y \setminus Y\gamma$.*

Proof. Assume that γ is not a constant map. Then there exist $a, b \in X$ such that $a \neq b$ and $a\gamma \neq b\gamma$. Suppose that $(X \setminus Y)\gamma \nsubseteq X \setminus Y$ and $(X \setminus Y)\gamma \nsubseteq Y \setminus Y\gamma$. Then there exists $c \in (X \setminus Y)\gamma$ such that $c \notin X \setminus Y$, and so $c \in Y$. Since $c \in (X \setminus Y)\gamma$, there exists $c' \in X \setminus Y$ such that $c = c'\gamma$. Since $(X \setminus Y)\gamma \nsubseteq Y \setminus Y\gamma$, there exists $d \in (X \setminus Y)\gamma$ such that $d \notin Y \setminus Y\gamma$. So $d \in X \setminus Y$ or $d \in Y\gamma$. Since $d \in (X \setminus Y)\gamma$, there exists $d' \in X \setminus Y$ such that $d = d'\gamma$. Since $|X \setminus Y| > 1$ and $c' \in X \setminus Y$, we have $X \setminus (Y \cup \{c'\}) \neq \emptyset$.

Case $d \in X \setminus Y$: Since $c'\gamma = c \in Y$ and $d'\gamma = d \in X \setminus Y$, we have $c'\gamma \neq d'\gamma$. This implies $c' \neq d'$. Since $c', d' \in X \setminus Y$ and $c' \neq d'$ and $c \in Y$, we obtain c, c', d' are all distinct. Let $\alpha \in S(X,Y)$ be such that

$$x\alpha = \begin{cases} c & \text{if } x \in Y, \\ d' & \text{if } x \in X \setminus Y. \end{cases}$$

Let $\beta \in S(X,Y)$ be such that

$$x\beta = \begin{cases} c & \text{if } x \in Y, \\ c' & \text{if } x = c', \\ d' & \text{if } x \in X \setminus (Y \cup \{c'\}). \end{cases}$$

Then $\alpha \leq \beta$. Thus

$$x\alpha\gamma = \begin{cases} c\gamma & \text{if } x \in Y, \\ d & \text{if } x \in X \setminus Y, \end{cases}$$

and

$$x\beta\gamma = \begin{cases} c\gamma & \text{if } x \in Y, \\ c & \text{if } x = c', \\ d & \text{if } x \in X \setminus (Y \cup \{c'\}). \end{cases}$$

Since $c, c\gamma \in Y$ and $d \in X \setminus Y$, we have $c \neq d$ and $c\gamma \neq d$. From $\pi_{\beta\gamma}(Y) = \{Y, \{c'\}\}$ and $\pi_{\alpha\gamma}(Y) = \{Y\}$, we see that $\pi_{\beta\gamma}(Y)$ does not refine $\pi_{\alpha\gamma}(Y)$. Then $\alpha\gamma \nleq \beta\gamma$.

Case $d \in Y\gamma$ and $c \neq d$: Then $d = y\gamma$ for some $y \in Y$. If $c' = d'$, then $c = c'\gamma = d'\gamma = d$ which is a contradiction. Then $c' \neq d'$. Since $c', d' \in X \setminus Y$ and $c' \neq d'$ and $y \in Y$, we obtain y, c', d' are all distinct. Let $\alpha \in S(X,Y)$ be such that

$$x\alpha = \begin{cases} y & \text{if } x \in Y, \\ c' & \text{if } x \in X \setminus Y. \end{cases}$$

Let $\beta \in S(X,Y)$ be such that

$$x\beta = \begin{cases} y & \text{if } x \in Y, \\ c' & \text{if } x = c', \\ d' & \text{if } x \in X \setminus (Y \cup \{c'\}). \end{cases}$$

Then $\alpha \leq \beta$. Thus

$$x\alpha\gamma = \begin{cases} d & \text{if } x \in Y, \\ c & \text{if } x \in X \setminus Y, \end{cases}$$

and

$$x\beta\gamma = \begin{cases} d & \text{if } x \in X \setminus \{c'\}, \\ c & \text{if } x = c'. \end{cases}$$

From $\pi_{\beta\gamma} = \{\{c'\}, X \setminus \{c'\}\}$ and $\pi_{\alpha\gamma} = \{Y, X \setminus Y\}$, we see that $\pi_{\beta\gamma}$ does not refine $\pi_{\alpha\gamma}$. Then $\alpha\gamma \nleq \beta\gamma$.

Case $d \in Y\gamma$ and $c = d \neq a\gamma$: Then $d = y\gamma$ for some $y \in Y$. Since $y\gamma = d \neq a\gamma$, we have $y \neq a$. Also, since $c'\gamma = c \neq a\gamma$, we get $c' \neq a$. From $y \in Y$ and $c' \in X \setminus Y$, we have $y \neq c'$. Hence a, y, c' are all distinct. Let $\alpha \in S(X,Y)$ be such that

$$x\alpha = \begin{cases} y & \text{if } x \in Y, \\ a & \text{if } x \in X \setminus Y. \end{cases}$$

Let $\beta \in S(X,Y)$ be such that

$$x\beta = \begin{cases} y & \text{if } x \in Y, \\ c' & \text{if } x = c', \\ a & \text{if } x \in X \setminus (Y \cup \{c'\}). \end{cases}$$

Then $\alpha \leq \beta$. Thus

$$x\alpha\gamma = \begin{cases} c & \text{if } x \in Y, \\ a\gamma & \text{if } x \in X \setminus Y, \end{cases}$$

and

$$x\beta\gamma = \begin{cases} c & \text{if } x \in Y \cup \{c'\}, \\ a\gamma & \text{if } x \in X \setminus (Y \cup \{c'\}). \end{cases}$$

From $\pi_{\beta\gamma} = \{Y \cup \{c'\}, X \setminus (Y \cup \{c'\})\}$ and $\pi_{\alpha\gamma} = \{Y, X \setminus Y\}$, we see that $\pi_{\beta\gamma}$ does not refine $\pi_{\alpha\gamma}$. Then $\alpha\gamma \nleq \beta\gamma$.

Case $d \in Y\gamma$ and $c = d = a\gamma$: Then $d = y\gamma$ for some $y \in Y$. Since $y\gamma = d = a\gamma \neq b\gamma$, we have $y \neq b$. Also, since $c'\gamma = c = a\gamma \neq b\gamma$, we get $c' \neq b$. From $y \in Y$ and $c' \in X \setminus Y$, we have $y \neq c'$. Hence b, y, c' are all distinct. Let $\alpha \in S(X,Y)$ be such that

$$x\alpha = \begin{cases} y & \text{if } x \in Y, \\ b & \text{if } x \in X \setminus Y. \end{cases}$$

Let $\beta \in S(X,Y)$ be such that

$$x\beta = \begin{cases} y & \text{if } x \in Y, \\ c' & \text{if } x = c', \\ b & \text{if } x \in X \setminus (Y \cup \{c'\}). \end{cases}$$

Then $\alpha \leq \beta$. Thus

$$x\alpha\gamma = \begin{cases} a\gamma & \text{if } x \in Y, \\ b\gamma & \text{if } x \in X \setminus Y, \end{cases}$$

and

$$x\beta\gamma = \begin{cases} a\gamma & \text{if } x \in Y \cup \{c'\}, \\ b\gamma & \text{if } x \in X \setminus (Y \cup \{c'\}). \end{cases}$$

From $\pi_{\beta\gamma} = \{Y \cup \{c'\}, X \setminus (Y \cup \{c'\})\}$ and $\pi_{\alpha\gamma} = \{Y, X \setminus Y\}$, we see that $\pi_{\beta\gamma}$ does not refine $\pi_{\alpha\gamma}$. Then $\alpha\gamma \not\leq \beta\gamma$.

In any cases, we have γ is not a right compatible element which is a contradiction. Hence $(X \setminus Y)\gamma \subseteq X \setminus Y$ or $(X \setminus Y)\gamma \subseteq Y \setminus Y\gamma$. ■

Theorem 3.3.17 *Let X be a nonempty set and $Y \subseteq X$ such that $|Y| \geq 2$ and $|X \setminus Y| > 1$. Then $\gamma \in S(X,Y)$ is a right compatible element if and only if one of the following statements holds.*

(i) *γ is a constant map.*

(ii) *[(γ is injective) or ($\gamma|_Y$ is a constant map and $\gamma|_{X\setminus Y}$ is injective)] and [$(X \setminus Y)\gamma \subseteq X \setminus Y$ or $(X \setminus Y)\gamma \subseteq Y \setminus Y\gamma$].*

Proof. Assume that γ is a right compatible element and γ is not a constant map. By Lemma 3.3.16, we have $(X \setminus Y)\gamma \subseteq X \setminus Y$ or $(X \setminus Y)\gamma \subseteq Y \setminus Y\gamma$. Assume that γ is not injective. We show that $\gamma|_Y$ is a constant map by supposing that this is false. That is $\gamma|_Y$ is not a constant map. Then there exist $a, b \in Y$ such that $a \neq b$ and

$a\gamma \neq b\gamma$. Since γ is not injective, there exist $c, d \in X$ such that $c \neq d$ and $c\gamma = d\gamma$.

Case $c \in Y$ and $d \in X \setminus Y$: Since $d\gamma \in (X \setminus Y)\gamma$ and $(X \setminus Y)\gamma \subseteq X \setminus Y$ or $(X \setminus Y)\gamma \subseteq Y \setminus Y\gamma$, we obtain $d\gamma \notin Y\gamma$. But $c\gamma \in Y\gamma$. Then $c\gamma \neq d\gamma$ which is a contradiction.

Case $c, d \in Y$ and $c\gamma \neq a\gamma$: Then $a \neq c$. Since $d\gamma = c\gamma \neq a\gamma$, we have $a \neq d$. Hence a, c, d are all distinct. Let $\alpha \in S(X, Y)$ be such that

$$x\alpha = \begin{cases} a & \text{if } x \in \{c, d\}, \\ c & \text{if } x \in X \setminus \{c, d\}. \end{cases}$$

Let $\beta \in S(X, Y)$ be such that

$$x\beta = \begin{cases} a & \text{if } x = c, \\ d & \text{if } x = d, \\ c & \text{if } x \in X \setminus \{c, d\}. \end{cases}$$

Then $\alpha \leq \beta$. Thus

$$x\alpha\gamma = \begin{cases} a\gamma & \text{if } x \in \{c, d\}, \\ c\gamma & \text{if } x \in X \setminus \{c, d\}, \end{cases}$$

and

$$x\beta\gamma = \begin{cases} a\gamma & \text{if } x = c, \\ c\gamma & \text{if } x \in X \setminus \{c\}. \end{cases}$$

From $\pi_{\beta\gamma} = \{\{c\}, X \setminus \{c\}\}$ and $\pi_{\alpha\gamma} = \{\{c, d\}, X \setminus \{c, d\}\}$, we see that $\pi_{\beta\gamma}$ does not refine $\pi_{\alpha\gamma}$. Then $\alpha\gamma \nleq \beta\gamma$ which contradicts γ is a right compatible element.

Case $c, d \in Y$ and $c\gamma = a\gamma$: Since $c\gamma = a\gamma \neq b\gamma$, we get $b \neq c$. Since $d\gamma = c\gamma = a\gamma \neq b\gamma$, we have $b \neq d$. Hence b, c, d are all distinct. Let $\alpha \in S(X, Y)$ be such that

$$x\alpha = \begin{cases} b & \text{if } x \in \{c, d\}, \\ c & \text{if } x \in X \setminus \{c, d\}. \end{cases}$$

Let $\beta \in S(X,Y)$ be such that

$$x\beta = \begin{cases} b & \text{if } x = c, \\ d & \text{if } x = d, \\ c & \text{if } x \in X \setminus \{c,d\}. \end{cases}$$

Then $\alpha \leq \beta$. Thus

$$x\alpha\gamma = \begin{cases} b\gamma & \text{if } x \in \{c,d\}, \\ a\gamma & \text{if } x \in X \setminus \{c,d\}, \end{cases}$$

and

$$x\beta\gamma = \begin{cases} b\gamma & \text{if } x = c, \\ a\gamma & \text{if } x \in X \setminus \{c\}. \end{cases}$$

From $\pi_{\beta\gamma} = \{\{c\}, X \setminus \{c\}\}$ and $\pi_{\alpha\gamma} = \{\{c,d\}, X \setminus \{c,d\}\}$, we see that $\pi_{\beta\gamma}$ does not refine $\pi_{\alpha\gamma}$. Then $\alpha\gamma \not\leq \beta\gamma$ which contradicts γ is a right compatible element.

Case $c, d \in X \setminus Y$: We have a, b, c, d are all distinct. Let $\alpha \in S(X,Y)$ be such that

$$x\alpha = \begin{cases} c & \text{if } x = c, \\ a & \text{if } x \in X \setminus \{c\}. \end{cases}$$

Let $\beta \in S(X,Y)$ be such that

$$x\beta = \begin{cases} a & \text{if } x \in Y, \\ c & \text{if } x = c, \\ d & \text{if } x \in X \setminus (Y \cup \{c\}). \end{cases}$$

Then $\alpha \leq \beta$. Thus

$$x\alpha\gamma = \begin{cases} c\gamma & \text{if } x = c, \\ a\gamma & \text{if } x \in X \setminus \{c\}, \end{cases}$$

and

$$x\beta\gamma = \begin{cases} a\gamma & \text{if } x \in Y, \\ c\gamma & \text{if } x \in X \setminus Y. \end{cases}$$

Since $c\gamma \in (X\setminus Y)\gamma$ and $(X\setminus Y)\gamma \subseteq X\setminus Y$ or $(X\setminus Y)\gamma \subseteq Y\setminus Y\gamma$, we have $c\gamma \notin Y\gamma$. But $a\gamma \in Y\gamma$, we get $c\gamma \neq a\gamma$. From $\pi_{\beta\gamma} = \{Y, X\setminus Y\}$ and $\pi_{\alpha\gamma} = \{\{c\}, X\setminus\{c\}\}$, we see that $\pi_{\beta\gamma}$ does not refine $\pi_{\alpha\gamma}$. Then $\alpha\gamma \nleq \beta\gamma$ which contradicts γ is a right compatible element.

In any cases, it is a contradiction. Hence $\gamma|_Y$ is a constant map.

Next, we show that $\gamma|_{X\setminus Y}$ is injective. We suppose that this is false. Then there exist $g, h \in X\setminus Y$ such that $g \neq h$ and $g\gamma = h\gamma$. From $\gamma|Y$ is a constant map. Then $Y\gamma = \{y\}$ for some $y \in Y$.

Case $g\gamma = y$: If $x\gamma = y$ for all $x \in X\setminus Y$, then γ is a constant map which is a contradiction. Then there exists $k \in (X\setminus Y)\setminus\{g,h\}$ such that $k\gamma \neq y$. We have k, g, h are all distinct. Since $k, g, h \in X\setminus Y$ and $y \in Y$, we obtain y, k, g, h are all distinct. Let $\alpha \in S(X,Y)$ be such that

$$x\alpha = \begin{cases} y & \text{if } x \in Y, \\ k & \text{if } x \in X\setminus Y. \end{cases}$$

Let $\beta \in S(X,Y)$ be such that

$$x\beta = \begin{cases} y & \text{if } x \in Y, \\ g & \text{if } x = g, \\ h & \text{if } x = h, \\ k & \text{if } x \in X\setminus(Y\cup\{g,h\}). \end{cases}$$

Then $\alpha \leq \beta$. Thus

$$x\alpha\gamma = \begin{cases} y & \text{if } x \in Y, \\ k\gamma & \text{if } x \in X\setminus Y, \end{cases}$$

and

$$x\beta\gamma = \begin{cases} y & \text{if } x \in Y\cup\{g,h\}, \\ k\gamma & \text{if } x \in X\setminus(Y\cup\{g,h\}). \end{cases}$$

From $\pi_{\beta\gamma} = \{Y\cup\{g,h\}, X\setminus(Y\cup\{g,h\})\}$ and $\pi_{\alpha\gamma} = \{Y, X\setminus Y\}$, we see that $\pi_{\beta\gamma}$ does not refine $\pi_{\alpha\gamma}$. Then $\alpha\gamma \nleq \beta\gamma$ which contradicts γ is a right compatible element.

Case $g\gamma \neq y$: Then $g\gamma = x'$ for some $x' \in X \setminus \{y\}$. Since $g, h \in X \setminus Y$ and $g \neq h$ and $y \in Y$, we obtain y, g, h are all distinct. Let $\alpha \in S(X, Y)$ be such that

$$x\alpha = \begin{cases} y & \text{if } x \in Y \cup \{g\}, \\ h & \text{if } x \in X \setminus (Y \cup \{g\}). \end{cases}$$

Let $\beta \in S(X, Y)$ be such that

$$x\beta = \begin{cases} y & \text{if } x \in Y, \\ g & \text{if } x = g, \\ h & \text{if } x \in X \setminus (Y \cup \{g\}). \end{cases}$$

Then $\alpha \leq \beta$. Thus

$$x\alpha\gamma = \begin{cases} y & \text{if } x \in Y \cup \{g\}, \\ x' & \text{if } x \in X \setminus (Y \cup \{g\}), \end{cases}$$

and

$$x\beta\gamma = \begin{cases} y & \text{if } x \in Y, \\ x' & \text{if } x \in X \setminus Y. \end{cases}$$

From $\pi_{\beta\gamma} = \{Y, X \setminus Y\}$ and $\pi_{\alpha\gamma} = \{Y \cup \{g\}, X \setminus (Y \cup \{g\})\}$, we see that $\pi_{\beta\gamma}$ does not refine $\pi_{\alpha\gamma}$. Then $\alpha\gamma \nleq \beta\gamma$ which contradicts γ is a right compatible element.

In both cases, we have γ is not a right compatible element which is a contradiction. Hence $\gamma|_{X \setminus Y}$ is injective.

The converse is true by Lemma 3.3.4, Lemma 3.3.6, Lemma 3.3.9, Lemma 3.3.11 and Lemma 3.3.12. ■

Example 3.3.18 Let X be the set of all natural numbers and Y the set of all positive even integers. Consider

$$\alpha = \begin{pmatrix} 2 & 4 & 6 & 8 & 10 & 12 & \ldots & 1 & 3 & 5 & 7 & 9 & 11 & \ldots \\ 4 & 6 & 8 & 10 & 12 & 14 & \ldots & 3 & 5 & 7 & 9 & 11 & 13 & \ldots \end{pmatrix},$$

We see that α is injective and $(X \setminus Y)\alpha = \{3, 5, 7, 9, \ldots\} \subseteq \{1, 3, 5, 7, \ldots\} = X \setminus Y$. Thus α is a right compatible element with $\leq$ on $S(X, Y)$ by Theorem 3.3.17. ■

3.4 The Numbers of Minimal and Maximal Elements

From now on, we let X be a finite set with $|X| = n$ and Y be a nonempty subset of X with $|Y| = r$. First, we find the number of minimal elements of $S(X, Y)$.

Theorem 3.4.1 *The number of minimal elements of $S(X, Y)$ is r.*

Proof. By Thoerem 3.2.1, $\alpha \in S(X, Y)$ is a minimal element if and only if α is a constant map. Since $|Y| = r$, we obtain that there are r constant maps. Hence the number of minimal elements of $S(X, Y)$ is r. ∎

The following theorem is needed in order to find the number of maximal elements of $S(X, Y)$.

Theorem 3.4.2 *Let $\alpha \in S(X, Y)$. Then α is a maximal element if and only if $Y \subseteq X\alpha$ and $\alpha : X \setminus Y \to X \setminus Y\alpha$ is injective.*

Proof. Assume that α is a maximal element. We show that $Y \subseteq X\alpha$. Suppose that $Y \nsubseteq X\alpha$. Then there is $z \in Y$ such that $z \notin X\alpha$. From X is a finite set, this implies Y is a finite set. Moreover, there exist $a, b \in Y$ such that $a \neq b$ and $a\alpha = b\alpha$. We define $\beta \in S(X, Y)$ by

$$x\beta = \begin{cases} x\alpha & \text{if } x \neq b, \\ z & \text{if } x = b. \end{cases}$$

Since $b\alpha \in X\alpha$ and $b\beta = z \notin X\alpha$, we have $b\alpha \neq b\beta$. Then $\alpha \neq \beta$. We show that $\alpha \leq \beta$. Since $X\beta = X\alpha\dot{\cup}\{z\}$, we have $X\alpha \subseteq X\beta$. Since $Y\beta = Y\alpha\dot{\cup}\{z\}$, we have $Y\alpha \subseteq Y\beta$. Also, $z\beta^{-1} = \{b\} \subseteq (b\alpha)\alpha^{-1}$ and $u\beta^{-1} \subseteq u\alpha^{-1}$ for all $u \in X\beta \setminus \{z\}$. This implies π_β refines π_α and $\pi_\beta(Y)$ refines $\pi_\alpha(Y)$. If $x \in X$ and $x\beta \in X\alpha$, then $x \neq b$, so $x\beta = x\alpha$ by the definition of β. Then $\alpha \leq \beta$ which contradicts the maximality of α. Hence $Y \subseteq X\alpha$.

Next, we show that $\alpha : X \setminus Y \to X \setminus Y\alpha$. Suppose that there is $c \in X \setminus Y$ such that $c\alpha \in Y\alpha$. Since $c\alpha \in Y$ and $X \setminus Y$ is a finite set, there exists $d \in X \setminus Y$

such that $d \notin (X \setminus Y)\alpha$. We define $\gamma \in S(X,Y)$ by

$$x\gamma = \begin{cases} x\alpha & \text{if } x \neq c, \\ d & \text{if } x = c. \end{cases}$$

Since $c\alpha \in Y$ and $c\gamma = d \in X \setminus Y$, we have $c\alpha \neq c\gamma$. Then $\alpha \neq \gamma$. We show that $\alpha \leq \gamma$. Since $X\gamma = X\alpha \cup \{d\}$, we have $X\alpha \subseteq X\gamma$. Since $c \in X \setminus Y$, we have $Y\alpha = Y\gamma$ by the definition of γ. Also, $d\gamma^{-1} = \{c\} \subseteq (c\alpha)\alpha^{-1}$ and $u\gamma^{-1} \subseteq u\alpha^{-1}$ for all $u \in X\gamma \setminus \{d\}$. This implies π_γ refines π_α and $\pi_\gamma(Y)$ refines $\pi_\alpha(Y)$. If $x \in X$ and $x\gamma \in X\alpha$, then $x \neq c$, so $x\gamma = x\alpha$ by the definition of γ. Then $\alpha \leq \gamma$ which contradicts the maximality of α. Hence $\alpha : X \setminus Y \to X \setminus Y\alpha$.

Finally, we show that $\alpha : X \setminus Y \to X \setminus Y\alpha$ is injective. Suppose that there exist $u, v \in X \setminus Y$ such that $u \neq v$ and $u\alpha = v\alpha$. Since $X \setminus Y$ is a finite set, there exists $w \in X \setminus Y$ such that $w \notin (X \setminus Y)\alpha$. We define $\delta \in S(X,Y)$ by

$$x\delta = \begin{cases} x\alpha & \text{if } x \neq v, \\ w & \text{if } x = v. \end{cases}$$

Since $v\alpha \in (X \setminus Y)\alpha$ and $v\delta = w \notin (X \setminus Y)\alpha$, we have $v\alpha \neq v\delta$. Then $\alpha \neq \delta$. By the same proof as given above, we get $\alpha \leq \delta$ which contradicts the maximality of α. Hence $\alpha : X \setminus Y \to X \setminus Y\alpha$ is injective.

The converse is true by Lemma 3.2.4. ■

Theorem 3.4.3 *Let $|X| = n = |Y|$. Then the number of maximal elements of $S(X,Y)$ is $n!$.*

Proof. By Theorem 3.4.2, we have $\alpha \in S(X,Y)$ is a maximal element if and only if $Y \subseteq X\alpha$ and $\alpha : X \setminus Y \to X \setminus Y\alpha$ is injective.

Let α be a maximal element in $S(X,Y)$. Since $X = Y \subseteq X\alpha \subseteq X$, we have $X = Y = X\alpha$. Also, since X is a finite set, we obtain α is injective. Hence the number of maximal elements of $S(X,Y)$ is equal to the number of all injective functions of $S(X,Y)$ which is equal to $n!$. ■

Theorem 3.4.4 *Let $|X| = n > 1$ and $|Y| = 1$. Then the number of maximal elements of $S(X,Y)$ is $(n-1)!$.*

Proof. By Theorem 3.4.2, we have $\alpha \in S(X,Y)$ is a maximal element if and only if $Y \subseteq X\alpha$ and $\alpha : X \setminus Y \to X \setminus Y\alpha$ is injective.

Let α be a maximal element in $S(X,Y)$. Since $\emptyset \neq Y\alpha \subseteq Y$ and $|Y| = 1$, we have $Y = Y\alpha$. Then $\alpha : X\setminus Y \to X\setminus Y$ is injective. Since $X\setminus Y$ is a finite set, we obtain $(X\setminus Y)\alpha = X \setminus Y$. From $|Y\alpha| = |Y| = 1$ and $|(X\setminus Y)\alpha| = |X\setminus Y| = n-1$, it follows that $Y\alpha$ can have 1 choice and $(X\setminus Y)\alpha$ can have $(n-1)!$ choices, thus there are $(n-1)!$ ways to choose $Y\alpha$ and $(X\setminus Y)\alpha$. Hence the number of maximal elements of $S(X,Y)$ is $(n-1)!$. ■

To count the number of maximal elements of $S(X,Y)$ in the case $|X| = n > r = |Y| > 1$, we need the following combinatorics result.

Lemma 3.4.5 *The number of r arrangements of objects chosen from unlimited supplies of k types of objects such that each type will be use at least once is*

$$\sum_{j=1}^{k}(-1)^{j-1}\binom{k}{j-1}(k-(j-1))^r \quad \textit{choices.}$$

Proof. Now let us solve this problem with exponential generating functions. The exponential generating function for this problem is

$$\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)^k$$

$$= (e^x - 1)^k$$

$$= \tbinom{k}{0}(e^x)^k(-1)^0 + \tbinom{k}{1}(e^x)^{k-1}(-1)^1 + \tbinom{k}{2}(e^x)^{k-2}(-1)^2$$

$$+ \dots + \tbinom{k}{k-1}(e^x)^{k-(k-1)}(-1)^{k-1} + \tbinom{k}{k}(e^x)^0(-1)^k.$$

From $e^{nx} = 1 + nx + \frac{n^2x^2}{2!} + \frac{n^3x^3}{3!} + \dots + \frac{n^rx^r}{r!} + \dots$, we have the coefficient of $\frac{x^r}{r!}$ in this generating function

$$= \binom{k}{0}(k)^r - \binom{k}{1}(k-1)^r + \binom{k}{2}(k-2)^r + \ldots + (-1)^{k-1}\binom{k}{k-1}(k-(k-1))^r$$
$$= \sum_{j=1}^{k}(-1)^{j-1}\binom{k}{j-1}(k-(j-1))^r.$$

Hence the number of r arrangements of objects chosen from unlimited supplies of k types of objects such that each type will be use at least once is

$$\sum_{j=1}^{k}(-1)^{j-1}\binom{k}{j-1}(k-(j-1))^r \quad \text{choices.}$$ ■

Theorem 3.4.6 *Let $|X| = n > r$ and $|Y| = r > 1$. Then the number of maximal elements of $S(X,Y)$ is*

$$r!(n-r)! + \sum_{i=1}^{m}\left\{\binom{r}{r-i}\sum_{j=1}^{r-i}(-1)^{j-1}\binom{r-i}{j-1}(r-i-(j-1))^r \cdot P(n-r,i)\cdot P(n-r,n-r-i)\right\},$$

where m is the minimum of $n-r$ and $r-1$.

Proof. By Theorem 3.4.2, we have $\alpha \in S(X,Y)$ is a maximal element if and only if $Y \subseteq X\alpha$ and $\alpha : X \setminus Y \to X \setminus Y\alpha$ is injective. Let α be a maximal element in $S(X,Y)$.

Case $Y\alpha = Y$: Since $\alpha : X \setminus Y \to X \setminus Y$ is injective and $X \setminus Y$ is a finite set, so we have $(X \setminus Y)\alpha = X \setminus Y$. From $|Y\alpha| = |Y| = r$ and $|(X \setminus Y)\alpha| = |X \setminus Y| = n-r$, so $Y\alpha$ can have $r!$ choices and for each choice of $Y\alpha$, $(X \setminus Y)\alpha$ can have $(n-r)!$ choices, thus there are $r!(n-r)!$ ways to choose $Y\alpha$ and $(X \setminus Y)\alpha$. Hence the number of maximal elements in this case is $r!(n-r)!$.

Case $Y\alpha \subsetneq Y$: Let $|Y\alpha| = k$. Then $k \geq 1$ and $k \leq r-1$. Suppose that $r-k > n-r$. Since $(X \setminus Y)\alpha \subseteq X \setminus Y\alpha$, we obtain $Y\alpha$ and $(X \setminus Y)\alpha$ are disjoint sets. Then we have

$$\begin{aligned}|X\alpha| &= |Y\alpha| + |(X \setminus Y)\alpha| \\ &= k + (n-r) \\ &< k + (r-k) = r = |Y|.\end{aligned}$$

This implies $Y \nsubseteq X\alpha$ which is a contradiction. Hence $r-k \leq n-r$, that is $|Y\alpha| = k \geq r-(n-r)$.

Let $|Y\alpha| = r - s$. Since $|Y\alpha| \geq 1$, this implies $r - s \geq 1$, that is $s \leq r - 1$. Also, since $|Y\alpha| \geq r - (n - r)$, we obtain $r - s \geq r - (n - r)$, that is $s \leq n - r$. Let m be the minimum of $n - r$ and $r - 1$. We can write $|Y\alpha|$ in the form

$$|Y\alpha| = r - i \quad \forall i = 1, \ldots, m.$$

Consider for each $i \in \{1, \ldots, m\}$, let t be the number of r arrangements of objects chosen from unlimited supplies of $r - i$ types of objects such that each type will be use at least once. Then $Y\alpha$ can have $\binom{r}{r-i}t$ choices. By Lemma 3.4.5, $Y\alpha$ can have $\binom{r}{r-i}\sum_{j=1}^{r-i}(-1)^{j-1}\binom{r-i}{j-1}(r-i-(j-1))^r$ choices. And for each choice of $Y\alpha$, we need to find the number of ways to choose $(X \setminus Y)\alpha$. Since $\alpha : X \setminus Y \to X \setminus Y\alpha$ is injective and $X \setminus Y$ is a finite set, we have $|X \setminus Y| = n - r = |(X \setminus Y)\alpha|$. Also, we have $|Y \setminus Y\alpha| = r - (r - i) = i$. Since $Y \subseteq X\alpha$, this implies $Y \setminus Y\alpha \subseteq (X \setminus Y)\alpha$. Thus $(X \setminus Y)\alpha$ can have $P(n - r, i) \cdot P(n - r, n - r - i)$ choices. Then there are

$$\binom{r}{r-i}\sum_{j=1}^{r-i}(-1)^{j-1}\binom{r-i}{j-1}(r - i - (j - 1))^r \cdot P(n - r, i) \cdot P(n - r, n - r - i)$$

ways to choose $Y\alpha$ and $(X \setminus Y)\alpha$ such that $|Y\alpha| = r - i$. Hence the number of maximal elements in this case is

$$\sum_{i=1}^{m}\Big\{\binom{r}{r-i}\sum_{j=1}^{r-i}(-1)^{j-1}\binom{r-i}{j-1}(r - i - (j - 1))^r \cdot P(n - r, i) \cdot P(n - r, n - r - i)\Big\}.$$

Therefore the number of maximal elements of $S(X, Y)$ is

$$r!(n-r)! + \sum_{i=1}^{m}\Big\{\binom{r}{r-i}\sum_{j=1}^{r-i}(-1)^{j-1}\binom{r-i}{j-1}(r-i-(j-1))^r \cdot P(n-r, i) \cdot P(n-r, n-r-i)\Big\}$$

■

Example 3.4.7 Let $X = \{1, 2, 3, 4\}, Y = \{1, 2\}$. Then $|X| = 4$ and $|Y| = 2$. By Theorem 3.4.6, we have $m = 1$ and the number of maximal elements of $S(X, Y)$

$$= 2!2! + \sum_{i=1}^{1}\Big\{\binom{2}{2-i}\sum_{j=1}^{2-i}(-1)^{j-1}\binom{2-i}{j-1}(2 - i - (j - 1))^2 \cdot P(2, i) \cdot P(2, 2 - i)\Big\}$$

$$= 4 + \binom{2}{1}\sum_{j=1}^{1}(-1)^0\binom{2-1}{j-1}(2 - 1 - (j - 1))^2 \cdot P(2, 1) \cdot P(2, 2 - 1)$$

$$= 4 + \binom{2}{1}\binom{1}{0}(1)^2 \cdot 2! \cdot 2!$$

$= 4+(2)(1)(2)(2)$

$= 4+8$

$= 12.$

Moreover, by Theorem 3.4.6 the maximal elements of $S(X,Y)$ consist of

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 4 & 2 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 1 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 4 & 1 \end{pmatrix}.$$

■