# Chapter 2

# Preliminaries

In this chapter, we collect information that need for an understanding of the research work.

## 2.1 Basic definitions and results

### 2.1.1 Semigroups

A semigroup $S$ is said to be a *left* (*right*) *zero semigroup* if $xy = x$ ($xy = y$) for all $x, y \in S$.

Let $G$ be a group, $L_n$, for $n \in \mathbb{N}$, the $n$-element left zero semigroup, and set $S = G \times L_n$. Define the multiplication on $S$ componentwise by $(g, l)(g', l') = (gg', l)$ for $g, g' \in G$ and $l, l' \in L_n$. We call the semigroup $S$ a *left zero union of groups* (*LZUG*) over $G$.

Correspondingly, if $R_n$ for $n \in \mathbb{N}$ is the $n$-element right zero semigroup, we set $S = G \times R_n$ and define the multiplication on $S$ componentwise by $(g, r)(g', r') = (gg', r')$ for $g, g' \in G$ and $r, r' \in R_n$. We call this semigroup a *right zero union of groups* (*LZUG*) over $G$.

Note that *LZUG* over $G$ and *RZUG* over $G$ are exactly the left and the right groups over $G$, where a semigroup $S$ is called a *left* (*right*) *group*, if it is uniquely left (right) solvable, i.e. for all $r, t \in S$ there exists a unique $s \in S$ such that $rs = t$ ($sr = t$).

### 2.1.2 Digraphs

A *directed graph* or *digraph* $D$ is a finite nonempty set $V$ together with a set $E$ of ordered pairs of elements of $V$. Each element of $V$ is referred to as a *vertex* and $V$ itself as the *vertex set* of $D$; the members of the *arc set* $E$ are called *arcs.* We write $D = (V, E)$. By an *element of a digraph*, we shall mean a vertex or an arc. The number of elements in the vertex set is called the *order* of $D$. A digraph $D_1 = (V_1, E_1)$ is called a *subdigraph* of a digraph $D = (V, E)$ if $V_1 \subseteq V$ and $E_1 \subseteq E$. A subdigraph $D_1 = (V_1, E_1)$ is called a *strong subdigraph* of a digraph $D = (V, E)$ if it is the maximal subdigraph of $D$ with the vertex set $V_1$

Also, recall that if $(u, v)$ is an arc of a digraph, then $u$ is said to be *adjacent to* $v$ and $v$ is *adjacent from* $u$. The vertices $u$ and $v$ are also said to be *incident with* the arc $(u, v)$. The *indegree* $\overrightarrow{d}(v)$ of a vertex $v$ of a digraph $D$ is the number of vertices of $D$ that end in $v$. The *outdegree* $\overleftarrow{d}(v)$ of $v$ is the number of arcs of $D$ start from $v$.

Now Let $D$ be a digraph. A sequence

$$W : (u = u_0, u_1, ..., u_k = v)$$

of vertices of $D$ such that $u_i$ is adjacent to $u_{i+1}$ for all $i$ $(1 \leq i \leq k-1)$ is called a (*directed*) $u - v$ *walk* in $D$. Each arc $(u_i, u_{i+1}), 1 \leq i \leq k-1$, is said to be lie on or belong to $W$. The number of occurrences of arcs on a walk is the *length* of the walk. So the length of the walk $W : (u = u_0, u_1, ..., u_k = v)$ is $k$. A walk in which no arc is repeated is a (*directed*) *trail*; while a walk in which no vertex is repeated is a (*directed*) *path.* A $u - v$ walk is *closed* if $u = v$. A closed trail of length at least 2 is a (*directed*) *circuit*; a closed walk of length at least 2 in which no vertex is repeated except for the initial and terminal vertices is a (*directed*) *cycle.*

The digraph $D$ is said to be *connected* if, for each pair of vertices $u, v$ of $D$, there exists a $u - v$ (directed) path. A maximal connected subdigraph of a digraph $D$ is called a *component* of $D$

Let $D_1 = (V_1, E_1)$ and $D_2 = (V_2, E_2)$ be digraphs. A mapping $\varphi : V_1 \to V_2$ is called a *digraph homomorphism* if $u, v \in E_1$ implies $((\varphi(u)), (\varphi(v))) \in E_2$, i.e. $\varphi$ preserves arcs. We write $\varphi : D_1 \to D_2$. A digraph homomorphism $\varphi : D \to D$ is called a *digraph endomorphism.* If $\varphi : D_1 \to D_2$ is a bijective digraph homomorphism and $\varphi^{-1}$ is also a digraph homomorphism, then $\varphi$ is called a *digraph* *isomorphism*, we write $D_1 \cong D_2$ and say that $D_1$ and $D_2$ are isomorphic. A digraph isomorphism $\varphi : D \to D$ is called a *digraph automorphism*.

A digraph $D$ is called a *semigroup* (*group*) *digraph* or *digraph of a semigroup* (*group*) if there exists a semigroup (group) $S$ and a connection set $A \subseteq S$ such that $D$ is isomorphic to the Cayley digraph $\mathrm{Cay}(S, A)$.

### 2.1.3 Basic theorems

Now we show an interesting basic theorem which can describe a form of CI-graphs.

Note that the *cyclic group of order n* is the group $G = \{e, a, a^2, ..., a^{n-1}\}$, $a^n = e$ where $n \geq 1$ and $e$ is the identity element of $G$. The element $a$ is called a *generator* of $G$. Our insistence that $|G| = n$ means that $1, a, a^2, ..., a^{n-1}$ are distinct elements of $G$.

**Theorem 2.1.1.** [11] *A cyclic group $G$ is a 2-DCI-group, that is, all Cayley digraphs of $G$ of valency at most 2 are CI-graphs.*

S. Panma characterizes digraphs which are Cayley digraphs of left and right groups in [16]. Hence we shall introduce these useful results to describe the structures of the Cayley digraphs of both groups in the next following two sections.

In this thesis, $p_i$ denotes the projection map on the $i^{th}$ coordinate of an ordered pair.

## 2.2 Cayley digraphs of left groups

**Theorem 2.2.1.** [16] *Let $(V, E)$ be a digraph. Then $(V, E)$ is a Cayley digraph of left groups if and only if the following conditions hold:*

(1) *$(V, E)$ is the disjoint union of $n$ isomorphic subdigraphs $(V_1, E_1), (V_2, E_2), ...,$ $(V_n, E_n)$ for some $n \in \mathbb{N}$,*

(2) *there exists a group $G$ such that $(V_i, E_i)$, $i \in \{1, 2, ..., n\}$, are strong subdigraph Cayley digraphs of $G$,*

(3) *there exists a digraph isomorphism $\varphi_i : (V_i, E_i) \to \mathrm{Cay}(G, A_i)$, for some $A_i \subseteq G$, and $A_j = A_k$ for all $j, k \in \{1, 2, ..., n\}$,*

(4) *for $u, v \in V_i$, $(u, v) \in E$ if and only if $\varphi_i(v) = \varphi_i(u)a$ for some $a \in A_i$.*

So Theorem 2.2.1 is helpful for us to state a new lemma which will be easier to used in the proof of the main results about left groups.

Let $(V_1, E_1), (V_2, E_2), ..., (V_n, E_n)$ be digraphs and $V_i \cap V_j = \emptyset$ for all $i \neq j$. The *disjoint union* of $(V_1, E_1), (V_2, E_2), ..., (V_n, E_n)$ is defined as $\dot{\bigcup}_{i=1}^{n}(V_i, E_i) := (V_1 \cup V_2 \cup ... \cup V_n, E_1 \cup E_2 \cup ... \cup E_n)$.

**Lemma 2.2.2.** *Let $S = G \times L_n$ be a left group and $A \subseteq S$. Then the following conditions hold:*

(1) *for each $i \in \{1, 2, ..., n\}$,* $\mathrm{Cay}(G \times \{l_i\}, p_1(A) \times \{l_i\}) \cong \mathrm{Cay}(G, p_1(A))$

(2) $\mathrm{Cay}(S, A) = \dot{\bigcup}_{i=1}^{n} \mathrm{Cay}(G \times \{l_i\}, p_1(A) \times \{l_i\})$.

**Example 2.2.3.** Let $S = \mathbb{Z}_5 \times L_2$. Consider $A = \{(\bar{1}, l_2)\}$.

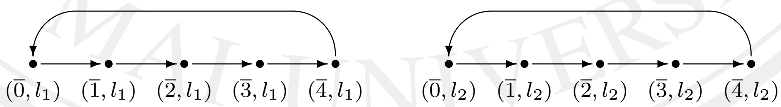


Figure 1: $\mathrm{Cay}(\mathbb{Z}_5 \times L_2, \{(\bar{1}, l_2)\})$

From Figure 1., we have

(1) $\mathrm{Cay}(S, A)$ is the disjoint union of two isomorphic subdigraphs $(\mathbb{Z}_5 \times \{l_1\}, E_1)$ and $(\mathbb{Z}_5 \times \{l_2\}, E_2)$.

(2) For each $i \in \{1, 2\}$, $\mathbb{Z}_5 \times \{l_i\}$ contains a strong subdigraph Cayley digraph of $\mathbb{Z}_5$: $(\mathbb{Z}_5 \times \{l_i\}, E_i) \cong \mathrm{Cay}(\mathbb{Z}_5 \times \{l_i\}, \{(\bar{1}, l_1)\}) \cong \mathrm{Cay}(\mathbb{Z}_5, \{\bar{1}\})$.

(3) From (2), we have $A_1 = A_2 = \{\bar{1}\}$ and $\varphi_i : (\mathbb{Z}_5 \times \{l_i\}, E_i) \to \mathrm{Cay}(\mathbb{Z}_5, A_i)$ is a digraph isomorphism for all $i \in \{1, 2\}$.

(4) We see that $((g, l_i), (g', l_i))$ is an arc in $\mathrm{Cay}(S, A)$ if and only if $g' = ga$ where $a = \bar{1}$.

## 2.3 Cayley digraphs of right groups

**Theorem 2.3.1.** [16] *Let $(V, E)$ be a digraph. Then $(V, E)$ is a Cayley digraph of right groups if and only if the following conditions hold:*

(1) *there exists a group $G$ and $m \in \mathbb{N}$ such that $(V, E)$ contains $m$ disjoint strong subdigraphs $(V_1, E_1), (V_2, E_2), ..., (V_m, E_m)$ which are Cayley digraphs of $G$ and $V_i = \bigcup_{\alpha=1}^{m} V_{i\alpha}$,*

(2) *for each $\alpha \in \{1, 2, ..., m\}$, there exists a digraph isomorphism $\varphi_\alpha : (V_\alpha, E_\alpha) \to \mathrm{Cay}(G, A_\alpha)$, for some $A_\alpha \subseteq G$,*

(3) *for each $\alpha, \beta \in \{1, 2, ..., m\}$, and for each $u \in V_\alpha, v \in V_\beta, (u, v) \in E$ if and only if $\varphi_\beta(v) = \varphi_\alpha(u)a$ for some $a \in A_\beta$.*

From Theorem 2.3.1, we can state the following lemma which will be used in the next chapter.

**Lemma 2.3.2.** *Let $S = G \times R_n$ be a right group and $A \subseteq S$. If $A \subseteq G \times \{r_i\}$ where $i \in \{1, 2, ..., n\}$, then $\mathrm{Cay}(G \times \{r_i\}, A) \cong \mathrm{Cay}(G, p_1(A))$.*

**Example 2.3.3.** Let $S = \mathbb{Z}_5 \times R_3$. Consider $A = \{(\bar{1}, r_2)\}$.

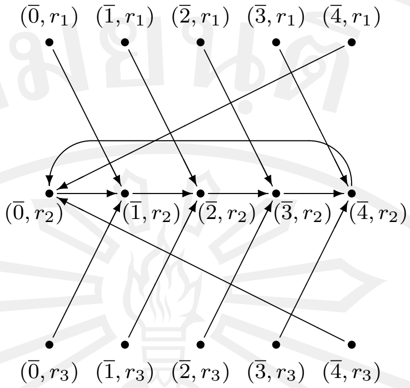


Figure 2: $\mathrm{Cay}(\mathbb{Z}_5 \times R_3, \{(\overline{1}, r_2)\})$

From Figure 2., we have

(1) $(\mathbb{Z}_5 \times R_3, E)$ contains three strong subdigraph Cayley digraphs of $\mathbb{Z}_5$;
$(\mathbb{Z}_5 \times \{r_1\}, E_1) \cong \mathrm{Cay}(\mathbb{Z}_5 \times \{r_1\}, A_1) \cong \mathrm{Cay}(\mathbb{Z}_5, \{\ \})$,
$(\mathbb{Z}_5 \times \{r_2\}, E_2) \cong \mathrm{Cay}(\mathbb{Z}_5 \times \{r_2\}, A_2) \cong \mathrm{Cay}(\mathbb{Z}_5, \{\overline{1}\})$,
$(\mathbb{Z}_5 \times \{r_3\}, E_3) \cong \mathrm{Cay}(\mathbb{Z}_5 \times \{r_3\}, A_3) \cong \mathrm{Cay}(\mathbb{Z}_5, \{\ \})$.

(2) From (2), we have $A_1 = \{\ \}, A_2 = \{\overline{1}\}, A_3 = \{\ \}$ and $\varphi_\alpha : (\mathbb{Z}_5 \times \{r_\alpha\}, E_\alpha) \to \mathrm{Cay}(\mathbb{Z}_5, A_\alpha)$ is a digraph isomorphism for all $\alpha \in \{1, 2, 3\}$.

(3) For each $\alpha, \beta \in \{1, 2, 3\}$, and for each $u \in V_\alpha, v \in V_\beta, (u, v) \in E$ if and only if $\varphi_\beta(v) = \varphi_\alpha(u)a$ for some $a \in A_\beta$. For example, we have $((\overline{3}, r_3), (\overline{4}, r_2))$ is an arc in $\mathrm{Cay}(S, A)$ since $\overline{4} = \overline{3} + \overline{1}$ and $\overline{1} \in A_2$.

Next, we show the condition when any two Cayley digraphs of a given right group with a one-element connection set are isomorphic.

**Lemma 2.3.4.** [12] *Let $S = G \times R_n$ be a right group, and $(g, r), (g', r') \in S$ where $g, g' \in G$ and $r, r' \in R_n$. Then $\mathrm{Cay}(S, \{(g, r)\}) \cong \mathrm{Cay}(S, \{(g', r')\})$ if and only if $|g| = |g'|$.*

**Lemma 2.3.5.** *Let $S = G \times R_n$ be a right group and $A \subseteq S$. Let $i \in \{1, 2, ..., n\}$. Then $A \cap (G \times \{r_i\}) = \emptyset$ if and only if $\overrightarrow{d}(u) = 0$ for all $u \in (G \times \{r_i\})$.*

*Proof.* Let $i \in \{1, 2, ..., n\}$.

($\Longrightarrow$) Assume that $A \cap (G \times \{r_i\}) = \emptyset$. Suppose that there exists $u \in (G \times \{r_i\})$ such that $\overrightarrow{d}(u) \neq 0$. Hence there exists an element $a \in A$ such that $xa = u$ for some $x \in S$. Since $S$ is a right group, we have $a \in (G \times \{r_i\})$. Then $a \in A \cap (G \times \{r_i\})$, contrary to $A \cap (G \times \{r_i\}) = \emptyset$. Therefore $\overrightarrow{d}(u) = 0$ for all $u \in (G \times \{r_i\})$.

($\Longleftarrow$) Let $u, v \in (G \times \{r_i\})$ and $\overrightarrow{d}(u) = 0, \overrightarrow{d}(v) = 0$. Suppose that $A \cap (G \times \{r_i\}) \neq \emptyset$. So there exists an element $a \in A \cap (G \times \{r_i\})$ such that $(u, v)$ is an arc in $\mathrm{Cay}(S, A)$, and then $\overrightarrow{d}(v) \neq 0$, a contradiction. Hence $A \cap (G \times \{r_i\}) = \emptyset$. $\square$