Chapter 3
Main Results

This chapter is divided into two sections. We first characterize CI-graphs of left groups. Another section is to introduce about CI-graphs of right groups which the connection set is a subset of $G \times \{r_i\}$ where $\{r_i\}$ is a singleton subset of the n-element right zero semigroup R_n.

3.1 CI-graphs of left groups

We start with the lemma that will be used in Theorem 3.1.2. The condition for two Cayley digraphs of an arbitrary left group which can be isomorphic will be given.

Lemma 3.1.1. *Let $S = G \times L_n$ be a left group and $A, B \subseteq S$. Then* $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$ *if and only if* $\mathrm{Cay}(G, p_1(A)) \cong \mathrm{Cay}(G, p_1(B))$.

Proof. ($\Longrightarrow$) Let $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$ and $i \in \{1, 2, ..., n\}$. By Lemma 2.2.2, we have $\dot{\bigcup}_{i=1}^{n}\mathrm{Cay}(G \times \{l_i\}, p_1(A) \times \{l_i\}) \cong \dot{\bigcup}_{i=1}^{n}\mathrm{Cay}(G \times \{l_i\}, p_1(B) \times \{l_i\})$ and $\mathrm{Cay}(G, p_1(A)) \cong \mathrm{Cay}(G \times \{l_i\}, p_1(A) \times \{l_i\}) \cong \mathrm{Cay}(G \times \{l_i\}, p_1(B) \times \{l_i\}) \cong \mathrm{Cay}(G, p_1(B))$ as required.

($\Longleftarrow$) Let $\mathrm{Cay}(G, p_1(A)) \cong \mathrm{Cay}(G, p_1(B))$. Then $\mathrm{Cay}(G \times \{l_i\}, p_1(A) \times \{l_i\}) \cong \mathrm{Cay}(G \times \{l_i\}, p_1(B) \times \{l_i\})$ for all $i \in \{1, 2, ..., n\}$ by Lemma 2.2.2 (1). Therefore $\dot{\bigcup}_{i=1}^{n}\mathrm{Cay}(G\times\{l_i\}, p_1(A)\times\{l_i\}) \cong \dot{\bigcup}_{i=1}^{n}\mathrm{Cay}(G\times\{l_i\}, p_1(B)\times\{l_i\})$. Thus we get $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$ by Lemma 2.2.2 (2). □

The next result characterizes the CI-graphs of left groups.

Theorem 3.1.2. *Let $S = G \times L_n$ be a left group and $A \subseteq S$. Then* $\mathrm{Cay}(S, A)$ *is a CI-graph if and only if $n = 1$ and* $\mathrm{Cay}(G, p_1(A))$ *is a CI-graph.*

Proof. ($\Longrightarrow$) Let $\emptyset \neq A \subseteq G \times L_n$ and let $\mathrm{Cay}(S, A)$ be a CI-graph and $n \neq 1$. We start the proof by choosing an element $(g, l_i) \in A$ to consider. Since $n \neq 1$, so $n \geq 2$. Then there exists $k \in \{1, 2, ..., n\}$ such that $k \neq i$ and $l_k \in L_n$. We will consider the following two cases:

case 1: if there exists $(g, l_k) \in A$, consider $B = A \setminus \{(g, l_k)\}$. We will see that $p_1(A) = p_1(B)$ and $\mathrm{Cay}(G, p_1(A)) \cong \mathrm{Cay}(G, p_1(B))$. Thus we have $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$ by Lemma 3.1.1, but $|A| \neq |B|$. So it is easy to see that there is no any functions $f \in \mathrm{Aut}(S)$ such that $f(A) = B$ which satisfy the definition of CI-graph.

case 2: if $(g, l_k) \notin A$, consider $B = A \cup \{(g, l_k)\}$. Similarly to the case 1, $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$, but we can't find any functions $f \in \mathrm{Aut}(S)$ such that $f(A) = B$ since $|A| \neq |B|$. It contradicts the assumption by these two cases. Therefore $n = 1$.

Next, we will show that $\mathrm{Cay}(G, p_1(A))$ is a CI-graph. Suppose that $\mathrm{Cay}(G, p_1(A)) \cong \mathrm{Cay}(G, X)$. Take $B = X \times \{l_1\}$, then $p_1(B) = X$. By Lemma 3.1.1, we get $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$. Since $\mathrm{Cay}(S, A)$ is a CI-graph, there exists $\alpha \in \mathrm{Aut}(S)$ such that $\alpha(A) = B$. Define $f : G \to G$ by $g \mapsto p_1(\alpha(g, l_1))$. Since $\alpha \in \mathrm{Aut}(G \times L_1)$, we have f is bijective. Therefore f is a group homomorphism since $f(g_1)f(g_2) = p_1(\alpha(g_1, l_1))p_1(\alpha(g_2, l_1)) = p_1(\alpha(g_1, l_1)\alpha(g_2, l_1)) = p_1(\alpha(g_1g_2, l_1)) = f(g_1g_2)$ for $g_1, g_2 \in G$. Moreover, $f(p_1(A)) = p_1(\alpha(A)) = p_1(B) = X$. Hence $f \in \mathrm{Aut}(G)$ and $f(p_1(A)) = p_1(B) = X$. Thus $\mathrm{Cay}(G, p_1(A))$ is a CI-graph.

($\Longleftarrow$) Let $\mathrm{Cay}(G, p_1(A))$ be a CI-graph. Let $n = 1$. Suppose that $\mathrm{Cay}(G \times L_1, A) \cong \mathrm{Cay}(G \times L_1, B)$. So, by Lemma 3.1.1, we have $\mathrm{Cay}(G, p_1(A)) \cong \mathrm{Cay}(G, p_1(B))$. Since $\mathrm{Cay}(G, p_1(A))$ is a CI-graph, there exists $\alpha \in \mathrm{Aut}(G)$ such that $\alpha(p_1(A)) = p_1(B)$. Then we define $\beta : G \times \{l_1\} \to G \times \{l_1\}$ by $\beta(g, l_1) = (\alpha(g), l_1)$. Since $\alpha \in \mathrm{Aut}(G)$, it is easy to see that β is also bijective. Therefore β is a group homomorphism since $\beta(g_1, l_1)\beta(g_2, l_1) = (\alpha(g_1), l_1)(\alpha(g_2), l_1) = (\alpha(g_1)\alpha(g_2), l_1) = (\alpha(g_1g_2), l_1) = \beta(g_1g_2, l_1) = \beta((g_1, l_1)(g_2, l_1))$ for $(g_1, l_1), (g_2, l_1) \in G \times \{l_1\}$. In addition, $\beta(A) = \beta(p_1(A) \times \{l_1\}) = \alpha(p_1(A)) \times \{l_1\} = p_1(B) \times \{l_1\} =$

B. Hence $\mathrm{Cay}(S, A)$ is a CI-graph. □

The next example shows that if $n \geq 2$, then $\mathrm{Cay}(S, A)$ is not a CI-graph.

Example 3.1.3. Let $S = \mathbb{Z}_5 \times L_2$. Consider $A = \{(\bar{1}, l_1), (\bar{1}, l_2)\}$ and $B = \{(\bar{1}, l_1)\}$.

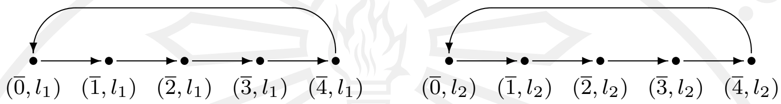

Figure 3: $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$

By the definition of a Cayley digraph, we have $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$, see Figure 3. Since $|A| \neq |B|$, then we can't find any automorphisms f in S such that $f(A) = B$.

3.2 CI-graphs of right groups

Firstly, we introduce one of our main theorems about being CI-graphs of any right groups with a one-element connection set. Theorem 2.1.1 will be helpful in the proof.

Theorem 3.2.1. *Let $S = G \times R_n$ be a right group where G is a cyclic group and R_n is an n-element right zero semigroup. Let $(a, r_i) \in S$ where $i \in \{1, 2, ..., n\}$. Then $\mathrm{Cay}(S, \{(a, r_i)\})$ is a CI-graph.*

Proof. Suppose that $\mathrm{Cay}(S, \{(a, r_i)\}) \cong \mathrm{Cay}(S, \{(b, r_j)\})$ where $(b, r_j) \in S$ for some $j \in \{1, 2, ..., n\}$. By Theorem 2.1.1, we know that $\mathrm{Cay}(G, \{a\})$ is a CI-graph. So for all $b \in G$ such that $\mathrm{Cay}(G, \{b\}) \cong \mathrm{Cay}(G, \{a\})$, there exists $\alpha \in \mathrm{Aut}(G)$ such that $\alpha(a) = b$. Then we define $t : S \to S$ by

$$t(g, r) = \begin{cases} (\alpha(g), r_j) & \text{, if } \quad r = r_i \\ (\alpha(g), r_i) & \text{, if } \quad r = r_j \\ (\alpha(g), r) & \text{, otherwise.} \end{cases}$$

It is obvious that t is bijective. Let $(g,r),(g',r') \in S$. Since S is a right group, there are only 3 cases to be considered depend on r'.

case 1: $r' = r_i$. Then $t((g,r)(g',r_i)) = t(gg',r_i) = (\alpha(gg'),r_j)$ and $t(g,r)t(g',r_i) = (p_1(t(g,r))\alpha(g'),r_j) = (\alpha(g)\alpha(g'),r_j) = (\alpha(gg'),r_j)$.

case 2: $r' = r_j$. Then $t((g,r)(g',r_j)) = t(gg',r_j) = (\alpha(gg'),r_i)$ and $t(g,r)t(g',r_j) = (p_1(t(g,r))\alpha(g'),r_i) = (\alpha(g)\alpha(g'),r_i) = (\alpha(gg'),r_i)$.

case 3: $r' \neq r_i \neq r_j$. Then $t((g,r)(g',r')) = t(gg',r') = (\alpha(gg'),r')$ and $t(g,r)t(g',r') = (p_1(t(g,r))\alpha(g'),r') = (\alpha(g)\alpha(g'),r') = (\alpha(gg'),r')$.

Thus we have t is a semigroup homomorphism. Since $t \in \mathrm{Aut}(S)$ and $t(a,r_i) = (\alpha(a),r_j) = (b,r_j)$, $\mathrm{Cay}(S,\{(a,r_i)\})$ is a CI-graph. $\square$

The following lemma gives the conditions when any two Cayley digraphs of an arbitrary right group which each of its connection set is a subset of the cartesian product of a group G and a singleton subset of the n-element right zero semigroup R_n. Throughout the proof, N_0^H denotes the number of vertices u in a digraph H such that $\overrightarrow{d}(u) = 0$.

Lemma 3.2.2. *Let $S = G \times R_n$ be a right group. Let $A \subseteq G \times \{r_i\}$ where $i \in \{1,2,...,n\}$. Then $\mathrm{Cay}(S,A) \cong \mathrm{Cay}(S,B)$ if and only if the following conditions hold:*

(1) *$B \subseteq G \times \{r_j\}$ for some $j \in \{1,2,...,n\}$,*

(2) *there exists a graph isomorphism $f : \mathrm{Cay}(G \times \{r_i\}, A) \to \mathrm{Cay}(G \times \{r_j\}, B)$ such that $((g,r_k),(g',r_i)) \in E(\mathrm{Cay}(S,A))$ if and only if $(f(g,r_k), f(g',r_i)) \in E(\mathrm{Cay}(S,B))$ for any $k \in \{1,2,...,n\}$.*

Proof. ($\Longrightarrow$) Let $\mathrm{Cay}(S,A) \cong \mathrm{Cay}(S,B)$.

1. Suppose that $B \nsubseteq G \times \{r_j\}$ for all $j \in \{1,2,...,n\}$. Then $|\{j|B \cap (G \times \{r_j\}) = \emptyset\}| \neq |\{j|A \cap (G \times \{r_j\}) = \emptyset\}|$. By Lemma 2.3.5, $N_0^{\mathrm{Cay}(S,B)} = |\{j|B \cap (G \times \{r_j\}) = \emptyset\}||G|$ and $N_0^{\mathrm{Cay}(S,A)} = |\{j|A \cap (G \times \{r_j\}) = \emptyset\}||G|$. Therefore $N_0^{\mathrm{Cay}(S,A)} \neq N_0^{\mathrm{Cay}(S,B)}$, which contradicts $\mathrm{Cay}(S,A) \cong \mathrm{Cay}(S,B)$. Then $B \subseteq G \times \{r_j\}$ for some $j \in \{1,2,...,n\}$.

2. Since $\mathrm{Cay}(S,A) \cong \mathrm{Cay}(S,B)$, there exists a graph isomorphism $s : \mathrm{Cay}(S,A) \to \mathrm{Cay}(S,B)$. Next, we can define $t : \mathrm{Cay}(G \times \{r_i\}, A) \to \mathrm{Cay}(G \times \{r_j\}, B)$ as the restriction of s to $G \times \{r_i\}$, i.e. $t = s_{|G\times\{r_i\}}$ by Lemma 2.3.5. It is obvious that t is also a graph isomorphism by the definition of s. Therefore $\mathrm{Cay}(G \times \{r_i\}, A) \cong \mathrm{Cay}(G \times \{r_j\}, B)$. The statement $((g, r_k), (g', r_i)) \in E(\mathrm{Cay}(S,A))$ if and only if $(t(g, r_k), t(g', r_i)) \in E(\mathrm{Cay}(S,B))$ for any $k \in \{1, 2, ..., n\}$ is also true by the assumption.

($\Longleftarrow$) We define $\varphi : \mathrm{Cay}(S,A) \to \mathrm{Cay}(S,B)$ by

$$\varphi(g,r) = \begin{cases} (p_1 f(g,r_i), r_j) & \text{, if} \quad r = r_i \\ (p_1 f(g,r_i), r_i) & \text{, if} \quad r = r_j \\ (p_1 f(g,r_i), r) & \text{, otherwise.} \end{cases}$$

By the assumption, it is obviously concluded that φ is a graph isomorphism from $\mathrm{Cay}(S,A)$ to $\mathrm{Cay}(S,B)$. Therefore $\mathrm{Cay}(S,A) \cong \mathrm{Cay}(S,B)$. □

The following lemma is similar to Lemma 3.1.1. We give the condition for two Cayley digraphs of a right group can be isomorphic. The connection set which will be considered is a subset of the cartesian product of a group G and a one-element subset of the right zero semigroup R_n.

Lemma 3.2.3. *Let $S = G \times R_n$ be a right group, $A \subseteq G \times \{r_i\}$ where $i \in \{1, 2, ..., n\}$, and $B \subseteq S$. Then $\mathrm{Cay}(S,A) \cong \mathrm{Cay}(S,B)$ if and only if $\mathrm{Cay}(G, p_1(A)) \cong \mathrm{Cay}(G, p_1(B))$.*

Proof. Let $i \in \{1, 2, ..., n\}$ and $A \subseteq G \times \{r_i\}$.

($\Longrightarrow$) Let $\mathrm{Cay}(S,A) \cong \mathrm{Cay}(S,B)$. By Lemma 3.2.2, there exists $j \in \{1, 2, ..., n\}$ such that $B \subseteq G \times \{r_j\}$ and $\mathrm{Cay}(G \times \{r_i\}, A) \cong \mathrm{Cay}(G \times \{r_j\}, B)$. Therefore, by Lemma 2.3.2, we have $\mathrm{Cay}(G, p_1(A)) \cong \mathrm{Cay}(G, p_1(B))$.

($\Longleftarrow$) Let $\mathrm{Cay}(G, p_1(A)) \cong \mathrm{Cay}(G, p_1(B))$. Then there exists $\varphi : \mathrm{Cay}(G, p_1(A)) \to \mathrm{Cay}(G, p_1(B))$ which is a digraph isomorphism. We define

$f : \mathrm{Cay}(S, A) \to \mathrm{Cay}(S, B)$ by

$$f(g, r) = \begin{cases} (\varphi(g), r_j) & ,\text{if} \quad r = r_i \\ (\varphi(g), r_i) & ,\text{if} \quad r = r_j \\ (\varphi(g), r) & ,\text{otherwise.} \end{cases}$$

It is obvious that f is bijective. Let $(g, r_a), (g', r_b) \in \mathrm{Cay}(S, A)$ and $((g, r_a), (g', r_b)) \in E(\mathrm{Cay}(S, A))$. There exists $(a, r_i) \in A$ such that $(g', r_b) = (g, r_a)(a, r_i)$. Then $g' = ga$ and $r_b = r_i$. Hence $(g, g') \in E(\mathrm{Cay}(G, p_1(A)))$ and $f(g', r_b) = f(g', r_i) = (\varphi(g'), r_j)$. Thus we have $(\varphi(g), \varphi(g')) \in E(\mathrm{Cay}(G, p_1(B)))$ by the assumption. Then there exists $b \in p_1(B)$ such that $\varphi(g') = \varphi(g)b$. Since $f(g', r_b) = (\varphi(g'), r_j) = (\varphi(g)b, r_j) = (\varphi(g), r_a)(b, r_j) = f(g, r_a)(b, r_j)$, $(f(g, r_a), f(g', r_b)) \in E(\mathrm{Cay}(S, B))$ where $(b, r_j) \in B$. Thus we have f preserves arcs, and then f^{-1} preserves arcs can prove in the same way. Therefore $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$. □

Here we come to our main theorem of the right group. The preceding lemma will be used in the proof.

Theorem 3.2.4. *Let $S = G \times R_n$ be a right group and $A \subseteq G \times \{r_i\}$ where $i \in \{1, 2, ..., n\}$. Then $\mathrm{Cay}(S, A)$ is a CI-graph if and only if $\mathrm{Cay}(G, p_1(A))$ is a CI-graph.*

Proof. Let $i \in \{1, 2, ..., n\}$.

($\Longrightarrow$) Let $\mathrm{Cay}(S, A)$ be a CI-graph. Suppose that $\mathrm{Cay}(G, p_1(A)) \cong \mathrm{Cay}(G, B)$. Take $X = B \times \{r_j\}$ for some $j \in \{1, 2, ..., n\}$. By Lemma 3.2.3, we get $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, X)$. So there exists $f \in \mathrm{Aut}(S)$ such that $f(A) = X$. Define $\varphi : G \to G$ by $g \mapsto p_1(f(g, r_i))$. Clearly, φ is bijective. Then φ is also a group homomorphism since $\varphi(g_1)\varphi(g_2) = p_1(f(g_1, r_i))p_1(f(g_2, r_i)) = p_1(f(g_1, r_i)f(g_2, r_i)) = p_1 f(g_1 g_2, r_i) = \varphi(g_1 g_2)$. Let $t \in \varphi(p_1(A))$, i.e. $t = p_1(f(x, r_i))$ for some $(x, r_i) \in A$. Then $t \in p_1(f(A)) = p_1(X) = B$. Conversely, let $t \in B = p_1(X)$, i.e. $t = p_1(t, r_j)$. Since $f(A) = X$, there exists $(h, r_i) \in A$ such that $f(h, r_i) = (t, r_j)$ and thus

$t = p_1(f(h, r_i)) \in \varphi(p_1(A))$. Hence $\varphi(p_1(A)) = B$. Therefore $\mathrm{Cay}(G, p_1(A))$ is a CI-graph.

($\Longleftarrow$) Let $\mathrm{Cay}(G, p_1(A))$ be a CI-graph. Suppose that $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$. By Lemma 3.2.3, we have $\mathrm{Cay}(G, p_1(A)) \cong \mathrm{Cay}(G, p_1(B))$ where $B \subseteq G \times \{r_j\}$ for some $j \in \{1, 2, ..., n\}$. Then there exists $f \in \mathrm{Aut}(G)$ such that $f(p_1(A)) = p_1(B)$. Define $\varphi : S \to S$ by

$$\varphi(g, r) = \begin{cases} (f(g), r_j) & , \text{if} \quad r = r_i \\ (f(g), r_i) & , \text{if} \quad r = r_j \\ (f(g), r) & , \text{otherwise.} \end{cases}$$

It is easy to check that φ is bijective. About to prove that φ is a semigroup homomorphism is similar to Theorem 3.2.1. Next, we will prove that $\varphi(A) = B$. Let $t \in \varphi(A) = \varphi(p_1(A) \times \{r_i\})$. Then $t = \varphi(x, r_i)$ for some $x \in p_1(A)$. So $t = (f(x), r_j) \in B$. Therefore $\varphi(A) \subseteq B$. Conversely, let $t \in B$. Suppose that $t = (g, r_j)$ for some $g \in G$. Since $f(p_1(A)) = p_1(B)$, there exists $h \in p_1(A)$, i.e. $(h, r_i) \in A$ such that $f(h) = g$. Hence $t = (f(h), r_j) = \varphi(h, r_i) \in \varphi(A)$. Therefore $B \subseteq \varphi(A)$. So we can conclude that $\mathrm{Cay}(S, A)$ is a CI-graph. $\square$

We now show another example which can be concluded by Theorem 3.2.4.

Example 3.2.5. Let $G = \mathbb{Z}_9$ and $S = \mathbb{Z}_9 \times R_n$. Consider $A = \{\overline{1}, \overline{4}, \overline{6}, \overline{7}\}$ and $B = \{\overline{1}, \overline{3}, \overline{4}, \overline{7}\}$.

Define $\beta : \mathrm{Cay}(G, A) \to \mathrm{Cay}(G, B)$ by $0 \mapsto 6, 1 \mapsto 1, 2 \mapsto 2, 3 \mapsto 3, 4 \mapsto 7, 5 \mapsto 8, 6 \mapsto 0, 7 \mapsto 4$ and $8 \mapsto 5$. We have $\mathrm{Cay}(G, A) \cong \mathrm{Cay}(G, B)$, but there is no Cayley isomorphisms mapping A to B, that is, $\mathrm{Cay}(G, A)$ is not a CI-graph. Therefore, by Theorem 3.2.4, we can conclude that $\mathrm{Cay}(S, A \times \{r_i\})$ is not a CI-graph for all $i \in \{1, 2, ..., n\}$.