Chapter 3

Existence of Some Generalized Mixed Equilibrium Problems in Hilbert Spaces

3.1 Existence of Some Generalized Mixed Equilibrium Problems in Hilbert Spaces

We first prove the existence of (3.1.2). To do this, the following condition is also assumed:

(H') For fixed $r > 0$ and $x \in C$ there exists a bounded set $K \subset C$ and $a \in K$ such that for all $z \in C/K$,

$$-F(a,z) + G(z,a) + \frac{1}{r}\langle a - z, z - x\rangle + \varphi(y) < \varphi(x) \quad for\ all\ y \in C. \tag{3.1.1}$$

Theorem 3.1.1. *Let C be a nonempty closed convex subset of a real Hilbert space H.Let $\varphi : C \to \mathbb{R}$ be convex and lower semicontinuous. Let $F, G : C \times C \to \mathbb{R}$ be two bifunctions which satisfy conditions (F1)-(F4), (G1)-(G3) and (H'). Let $r > 0$ and $x \in C$. Then, there exists $z \in C$ such that*

$$F(z,y) + G(z,y) + \frac{1}{r}\langle y - z, z - x\rangle + \varphi(y) < \varphi(x) \quad for\ all\ y \in C. \tag{3.1.2}$$

Further, if

$T_r^{\varphi}(x) = \{z \in C : F(z,y) + G(z,y) + \frac{1}{r}\langle y - z, z - x\rangle + \varphi(y) < \varphi(x)\ for\ all\ y \in C\}$,

then the follow hold:

(i) T_r^{φ} is single-valued and T_r^{φ} is firmly nonexpansive,

(ii) Γ' is closed and convex and $\Gamma' = Fix(T_r^{\varphi})$.

Proof. Let $G' = G(x,y)+\varphi(y)-\varphi(x)$ $for\ all\ y \in C$, We show that $G' : C\times C \to R$ is bifuntion which satisfies conditions $(G'1)$-$(G'3)$. It easy to show that G' satisfies $(G'1)$. For each $y \in C$,we shall show that $G'(\cdot, y)$ is weakly upper semicontinuous. To do this, suppose $x_n \rightharpoonup x$. By $(G2)$ and lower semicontinuity of φ we have, for any $n \geq 1$,

$$\begin{aligned} limsup_{n\to\infty} G'(x_n, y) &= limsup_{n\to\infty}[G(x_n, y) + \varphi(y) - \varphi(x_n)] \\ &\leq limsup_{n\to\infty} G(x_n, y) + limsup_{n\to\infty}[\varphi(y) - \varphi(x_n)] \\ &\leq G(x, y) + \varphi(y) + limsup_{n\to\infty}[-\varphi(x_n)] \\ &= G(x, y) + \varphi(y) - liminf_{n\to\infty}[\varphi(x_n)], \\ &\leq G(x, y) + \varphi(y) - \varphi(x) \\ &= G'(x, y). \end{aligned} \tag{3.1.3}$$

This implies that G' is weakly upper semicontinuous in the first variable. Moreover, we have that

$$\begin{aligned} G'(x, y) + G'(y, x) &= G(x, y) + \varphi(y) - \varphi(x) + G(y, x) + \varphi(x) - \varphi(y) \\ &= G(x, y) + G(y, x) \\ &\leq 0. \end{aligned} \tag{3.1.4}$$

Hence G' satisfies $(G'2)$. For each $x \in C, G'(x, \cdot)$ is convex. Let $y_1, y_2 \in C$ and $\lambda \in [0, 1]$. By $(G'3)$ and convexity of φ, we have

$$\begin{aligned} \lambda G'(x, y_1) + (1-\lambda)G'(x, y_2) &= \lambda G(x, y_1) + (1-\lambda)G(x, y_2) + \lambda\varphi(y_1) \\ &\quad +(1-\lambda)\varphi(y_2) - \varphi(x) \\ &\geq G(x, \lambda y_1 + (1-\lambda)y_2) + \varphi(\lambda y_1 + (1-\lambda)y_2) - \varphi(x) \\ &= G'(x, \lambda y_1 + (1-\lambda)y_2). \end{aligned} \tag{3.1.5}$$

Therefore G' satisfies $(G'3)$. By Lemma 2.4.7 there exists $z \in C$ such that

$$F(z, y) + G(z, y) + \frac{1}{r}\langle y - z, z - x\rangle + \varphi(y) < \varphi(x) \quad for\ all\ y \in C. \tag{3.1.6}$$

Further, if

$T_r^{\varphi}(x) = \{z \in C : F(z,y) + G(z,y) + \frac{1}{r}\langle y - z, z - x\rangle + \varphi(y) < \varphi(x) \ for \ all \ y \in C\}$,

then the follow hold:

(i) T_r^{φ} is single-valued and T_r^{φ} is firmly nonexpansive,

(ii) Γ' is closed and convex and $\Gamma' = Fix(T_r^{\varphi})$. □

Theorem 3.1.2. *Let C be a nonempty closed convex subset of a real Hilbert space H.Let $F, G : C \times C \to R$ be two bifunctions which satisfy conditions (F2),(G2)and $\varphi : C \to R \cup \{+\infty\}$. Let A α-inverse-strongly monotone mapping of C into H. Then $Au = Av$ for all $u, v \in \Gamma'$.*

Proof. Let $u, v \in \Gamma'$.We then get

$$F(u,y) + G(u,y) + \langle Au, y - u\rangle + \varphi(y) \geq \varphi(u) \ \ for \ all \ y \in C. \tag{3.1.7}$$

and

$$F(v,y) + G(v,y) + \langle Av, y - v\rangle + \varphi(y) \geq \varphi(v) \ \ for \ all \ y \in C. \tag{3.1.8}$$

By letting y=v in (3.1.7) and y=u in (3.1.8) we get

$$F(u,v) + G(u,v) + \langle Au, v - u\rangle + \varphi(v) \geq \varphi(x) \ \ for \ all \ y \in C. \tag{3.1.9}$$

and

$$F(v,u) + G(v,u) + \langle Av, u - v\rangle + \varphi(u) \geq \varphi(v) \ \ for \ all \ y \in C. \tag{3.1.10}$$

By (3.1.9), (3.1.10) and the conditions (F2) and (G2), we have

$$\begin{aligned} \langle Av - Au, u - v\rangle &\geq F(u,v) + F(v,u) + G(u,v) + G(v,u) + \langle Au, v - u\rangle + \langle Av, u - v\rangle \\ &\geq 0. \end{aligned}$$

From A is α-inverse-strongly monotone mapping,

$$0 \leq \alpha\|Au - Av\|^2 \leq \langle Au - Av, u - v\rangle \leq 0.$$

That is $Au = Av$. □

Remark 3.1.3. if $\varphi = 0$ in Theorem 3.1.2 , we obtain that $Au = Av \;\; for\ all\ u, v \in \Gamma$.

Theorem 3.1.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\varphi : C \to H$ be an $\alpha-$inverse strongly monotone mapping and $A : C \to R$ be convex and lower semicontinous. Let $F, G : C \times C \to R$ be two bifunctions which satisfy conditions (F1)-(F4),(G1)-(G3) and (H'). Let $f : C \to H$ be a $\rho-$contraction and $r > 0$ be a constant with $r < 2\delta$. Suppose $\Gamma' \neq \emptyset$ For given $x_0 \in C$ and $u \in C$ arbitrarily, let the sequence $\{x_n\}$ be generated iteratively by*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) P_C[\alpha_n u + (1 - \alpha_n) T_r^{\varphi}(x_n - rAx_n)], \tag{3.1.8}$$

for all $n \geq 0$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0,1] satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\Sigma_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then the sequence $\{x_n\}$ converges strongly to $x^* \in \Gamma'$ which is the unique solution of the following variational inequality: $\langle (I - f)x^*, x - x^* \rangle \geq 0, \;\; x \in \Gamma'$.

Proof. By Theorem 3.1.1, we have then $\Gamma' = Fix(T_r^{\varphi})$ and Theorem 2.5.6 assumes that the sequence generated by 3.1.8 converges strongly to $x^* \in \Gamma'$ which is the unique solution of the following variational inequality:$\langle (I - f)x^*, x - x^* \rangle \geq 0, \;\; x \in \Gamma'$. □

Theorem 3.1.5. *Let C be a nonempty closed convex subset of a real Hilbert space H. Suppose $\Gamma' \neq \emptyset$. For given $u, x_0 \in C$, let the sequence $\{x_n\}$ be generated iteratively by*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) T_r^{\varphi} x_n, \;\; n \geq 0, \tag{3.1.9}$$

where $\{\alpha_n\}$ is sequences in [0,1] satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$,

(ii) $\Sigma_{n=1}^{\infty} \alpha_n = +\infty$,

(iii) $\Sigma_{n=1}^{\infty} \mid \alpha_{n+1} - \alpha_n \mid < +\infty$.

Then the sequence $\{x_n\}$ converges strongly to $x^* \in \Gamma'$.

Proof. By Theorem 3.1.1, we have then $\Gamma' = Fix(T_r^{\varphi})$ Since T_r^{φ} is a nonexpansive, we have by Theorem 2.5.7 that $x_n \to x^* \in \Gamma'$. $\square$