Chapter 4

Strong Convergence Theorems for a Common Fixed Point of Nonexpansive Mappings, Nonspreading Mappings and Mixed Equilibrium Problems

4.1 Strong Convergence Theorem for Countable Family of Nonexpansive Mappings

Theorem 4.1.1. *Let C be a nonempty closed convex subset of a Hilbert space H and F, G bifunctions from $C \times C$ to $\mathbb{R}$ which satisfies (F1)-(F4, (G1)-(G3) and (H) . Let A_j be a δ_j-inverse-strongly monotone mapping of C into H for each $j = 1, 2, 3$.Let$\{T_i : C \to C\}$ be a family of infinitely nonexpansive mappings. Satisfy AKTT condition. Assume that $\Omega := FP \cap \Gamma \cap VI$ where $\bigcap_{i=1}^{\infty} F(T_i) = FP$ and $VI := VI(C, A_1) \cap VI(C, A_2)$. Let $u \in C$, $x_1 \in C$ and $\{x_n\}$ be a sequence generated in*

$$\begin{cases} F(u_n, y) + G(u_n, y) + \langle A_3 x_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_{n,} u_n - x_n\rangle \geqslant 0, \ \ \forall y \in C, \\ z_n = P_C(u_n - \lambda_n A_2 u_n), \\ y_n = P_C(z_n - \eta_n A_1 z_n), \\ x_{n+1} = \beta_n x_n + (1 - \beta_n) T_n[\alpha_n u + (1 - \alpha_n) y_n], \forall n \geqslant 1, \end{cases} \tag{4.1.1}$$

where $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in [0,1] and* $\{r_n\}$,$\{\lambda_n\}$ *and* $\{\eta_n\}$ *are positive number sequences. Assume that the above control sequences satisfy the following restrictions*

(C1) $0 < c \leq \{\beta_n\} \leq d < 1$, $0 < a \leq \{r_n\} \leq b < 2\delta_3$, $0 < a' \leq \lambda_n \leq b' < 2\delta_2$, $0 < a'' \leq \eta_n \leq b'' < 2\delta_1$, *for all* $n \geq 1$.

(C2) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\Sigma_{n=1}^{\infty}\alpha_n = \infty$,

(C3) $\lim_{n\to\infty}(r_n - r_{n+1}) = \lim_{n\to\infty}(\lambda_n - \lambda_{n+1}) = \lim_{n\to\infty}(\eta_n - \eta_{n+1}) = 0$

Then $\{u_n\}$ *and* $\{x_n\}$ *converge strongly to* $z = P_\Omega u$.

Proof. we have $x^* = T_{r_n}(x^* - r_nA_3x^*)$. Since A_3 are $\delta_3 - inverse - strongly$ monotone, condition (C1) and nonexpansive, we have, for any $n \geq 1$,

$$\begin{aligned}
\|u_n - x^*\|^2 &= \|T_{r_n}(x_n - r_nA_3x_n) - T_{r_n}(x^* - r_nA_3x^*)\|^2 \\
&\leq \|(x_n - r_nAx_n) - (x^* - r_nA_3x^*)\|^2 \\
&\leq \|(x_n - x^*) - r_n(A_3x_n - A_3x^*)\|^2 \\
&= \|x_n - x^*\|^2 - 2r_n\langle x_n - x^*, A_3x_n - A_3x^*\rangle + \|A_3x_n - A_3x^*\|^2 \\
&\leq \|x_n - x^*\|^2 - 2r_n\delta_3\|A_3x_n - A_3x^*\|^2 + r_n^2\|A_3x_n - A_3x^*\|^2 \\
&= \|x_n - x^*\|^2 + r_n(r_n - 2\delta_3)\|A_3x_n - A_3x^*\|^2 \\
&\leq \|x_n - x^*\|^2. \qquad (4.1.2)
\end{aligned}$$

Note that the mappings $I - \eta_nA_1, I - \lambda_nA_2, I - r_nA_3$ are nonexpansive. to see this, for $\forall x, y \in C$, we have from the condition $(C1)$ that

$$\begin{aligned}
\|(I - \eta_nA_1)x - (I - \eta_nA_1)y\|^2 &= \|(x - y) - \eta_n(A_1x - A_1y)\|^2 \\
&= \|x - y\|^2 - 2\eta_n\langle x - y, A_1x - A_1y\rangle \\
&\quad +\eta_n^2\|A_1x - A_1y\|^2 \\
&\leq \|x - y\|^2 - 2\eta_n\delta_1\|A_1x - A_1y\|^2 + \eta_n^2\|A_1x - A_1y\|^2 \\
&= \|x - y\|^2 + \eta_n(\eta_n - 2\delta_1)\|A_1x - A_1y\|^2 \\
&\leq \|x - y\|^2, \qquad (4.1.3)
\end{aligned}$$

which implies the mapping $I-\eta_n A_1$ is nonexpansive. By using the same argument as aboue, we can show that $I-\lambda_n A_2$ and $I-r_n A_3$ are nonexpansiveness of P_C and $I-\lambda_n A_2$, we have

$$\begin{aligned} \|z_n - x^*\| &= \|P_C(u_n - \lambda_n A_2 u_n) - P_C(x^* - \lambda_n A_2 x^*)\| \\ &\leq \|(u_n - \lambda_n A_2 u_n) - (x^* - \lambda_n A_2 x^*)\| \\ &= \|(I-\lambda_n A_2)u_n - (I-\lambda_n A_2)x^*\| \\ &\leq \|u_n - x^*\| \qquad (4.1.4) \end{aligned}$$

It follows from (4.1.2) that

$$\|z_n - x^*\| \leq \|x_n - x^*\|$$

Since $x^* = P_C(I-\eta_n A_1)x^*$, by nonexpansiveness of P_C and $I-\eta_n A_1$, we have

$$\begin{aligned} \|y_n - x^*\| &= \|P_C(z_n - \eta_n A_1 z_n) - P_C(x^* - \eta_n A_1 x^*)\| \\ &\leq \|(z_n - \eta_n A_1 z_n) - (x^* - \eta_n A_1 x^*)\| \\ &= \|(I-\eta_n A_1)z_n - (I-\eta_n A_1)x^*\| \\ &\leq \|z_n - x^*\| \\ &\leq \|x_n - x^*\| \qquad (4.1.5) \end{aligned}$$

From (4.1.1) and (4.1.5) , we arrive at

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|\beta_n x_n + (1-\beta_n)T_n[\alpha_n u + (1-\alpha_n)y_n] - x^*\| \\ &= \beta_n\|x_n - x^*\| + (1-\beta_n)\|T_n[\alpha_n u + (1-\alpha_n)y_n] - T_n x^*\| \\ &\leq \beta_n\|x_n - x^*\| + (1-\beta_n)[\alpha_n\|u - x^*\| + (1-\alpha_n)\|y_n - x^*\| \\ &\leq \beta_n\|x_n - x^*\| + (1-\beta_n)[\alpha_n\|u - x^*\| + (1-\alpha_n)\|x_n - x^*\| \\ &= \beta_n\|x_n - x^*\| + (1-\beta_n)\alpha_n\|u - x^*\| + (1-\beta_n)(1-\alpha_n)\|x_n - x^*\| \\ &= [1-\alpha_n(1-\beta_n)]\|x_n - x^*\| + \alpha_n(1-\beta_n)\|u - x^*\|. \end{aligned}$$

By induction, we obtain that

$$\|x_n - x^*\| \leq max\{\|x_1 - x^*\|, \|u - x^*\|\},$$

this implies $\{x_n\}$ bounded. It follows that $\{y_n\}$, $\{z_n\}$ and $\{u_n\}$ are also bounded. Without loss of generality, we can assume that there exists a bounded set $B \subseteq C$ such that

$$u_n, x_n, y_n, z_n \in B, \forall n \geq 1. \tag{4.1.6}$$

In view of nonexpansivity of $I - \lambda_n A_2$ and $I - r_n A_3$, we see from (4.1.1) that

$$\begin{aligned}
\|z_{n+1} - z_n\| &= \|P_C(u_{n+1} - \lambda_{n+1} A_2 u_{n+1}) - P_C(u_n - \lambda_n A_2 u_n)\| \\
&\leq \|u_{n+1} - \lambda_{n+1} A_2 u_{n+1} - (u_n - \lambda_n A_2 u_n)\| \\
&= \|(I - \lambda_{n+1} A_2) u_{n+1} - (I - \lambda_n A_2) u_n\| + |\lambda_n - \lambda_{n+1}| \|A_2 u_n\| \\
&\leq \|u_{n+1} - u_n\| + |\lambda_n - \lambda_{n+1}| \|A_2 u_n\| \\
&= \|Tr_{n+1}(x_{n+1} - r_{n+1} A_3 x_{n+1}) - Tr_n(x_n - r_n A_3 x_n)\| \\
&\quad + |\lambda_n - \lambda_{n+1}| \|A_2 u_n\| \\
&= \|Tr_{n+1}(x_{n+1} - r_{n+1} A_3 x_{n+1}) - Tr_n(x_n - r_n A_3 x_n) \\
&\quad + Tr_{n+1}(x_n - r_n A_3 x_n) - Tr_{n+1}(x_n - r_n A_3 x_n)\| + |\lambda_n - \lambda_{n+1}| \|A_2 u_n\| \\
&= \|Tr_{n+1}(x_{n+1} - r_{n+1} A_3 x_{n+1}) - Tr_{n+1}(x_n - r_n A_3 x_n)\| \\
&\quad + \|Tr_{n+1}(x_n - r_n A_3 x_n) - Tr_n(x_n - r_n A_3 x_n)\| + |\lambda_n - \lambda_{n+1}| \|A_2 u_n\| \\
&\leq \|(x_{n+1} - r_{n+1} A_3 x_{n+1}) - (x_n - r_n A_3 x_n)\| + \|Tr_{n+1}(x_n - r_n A_3 x_n) \\
&\quad - Tr_n(x_n - r_n A_3 x_n)\| + |\lambda_n - \lambda_{n+1}| \|A_2 u_n\| \\
&= \|(x_{n+1} - r_{n+1} A_3 x_{n+1}) - (x_n - r_{n+1} A_3 x_n) + (x_n - r_{n+1} A_3 x_n) \\
&\quad - (x_n - r_n A_3 x_n)\| + \|Tr_{n+1}(x_n - r_n A_3 x_n) - Tr_n(x_n - r_n A_3 x_n)\| \\
&\quad + |\lambda_n - \lambda_{n+1}| \|A_2 u_n\| \\
&\leq \|(I - r_{n+1} A_3) x_{n+1} - (I - r_{n+1} A_3) x_n\| + |r_n - r_{n+1}| \|A_3 x_n\| \\
&\quad + \|Tr_{n+1}(x_n - r_n A_3 x_n) - Tr_n(x_n - r_n A_3 x_n)\| + |\lambda_n - \lambda_{n+1}| \|A_2 u_n\| \\
&\leq \|x_{n+1} - x_n\| + |r_n - r_{n+1}| \|A_3 x_n\| + \|Tr_{n+1}(x_n - r_n A_3 x_n) \\
&\quad - Tr_n(x_n - r_n A_3 x_n)\| + |\lambda_n - \lambda_{n+1}| \|A_2 u_n\|
\end{aligned} \tag{4.1.7}$$

On the other hand, we have

$$\begin{aligned}
\|y_{n+1} - y_n\| &= \|P_C(z_{n+1} - \eta_{n+1}A_1z_{n+1}) - P_C(z_n - \eta_nA_1z_n)\| \\
&\leq \|z_{n+1} - \eta_{n+1}A_1z_{n+1} - (z_n - \eta_nA_1z_n)\| \\
&= \|z_{n+1} - \eta_{n+1}A_1z_{n+1} - (z_n - \eta_{n+1}A_1z_n) + (z_n - \eta_{n+1}A_1z_n) \\
&\quad -(z_n - \eta_nA_1z_n)\| \\
&= \|(I - \eta_{n+1}A_1)z_{n+1} - (I - \eta_nA_1)z_n + (\eta_n - \eta_{n+1})A_1z_n\| \\
&= \|z_{n+1} - z_n\| + |\eta_n - \eta_{n+1}|\|A_1z_n\|. \qquad (4.1.8)
\end{aligned}$$

Substituting (4.1.7) into (4.1.8) we obtain that

$$\begin{aligned}
\|y_{n+1} - y_n\| &= \|z_{n+1} - z_n\| + |\eta_n - \eta_{n+1})|\|A_1z_n\| \\
&\leq \|x_{n+1} - x_n\| + |r_n - r_{n+1}|\|A_3x_n\| + \|Tr_{n+1}(x_n - r_nA_3x_n) \\
&\quad -Tr_n(x_n - r_nA_3x_n)\| + |\lambda_n - \lambda_n + 1|\|A_2u_n\| + |\eta_n - \eta_{n+1}|\|A_1z_n\| \\
&\leq \|x_{n+1} - x_n\| + M(|r_n - r_{n+1} + |\lambda_n - \lambda_n + 1| + |\eta_n - \eta_{n+1}|) \\
&\quad +\|Tr_{n+1}(x_n - r_nA_3x_n) - Tr_n(x_n - r_nA_3x_n)\|, \qquad (4.1.9)
\end{aligned}$$

where M is an appropriate constant such that

$$M = max\{sup_{n\geq 1}\{\|A_1z_n\|\}, sup_{n\geq 1}\{\|A_2u_n\|\}, sup_{n\geq 1}\{\|A_3x_n\|\}\}.$$

Putting $f_n = \alpha_nu + (1 - \alpha_n)y_n$, which implies that

$$\begin{aligned}
\|f_{n+1} - f_n\| &= \|\alpha_{n+1} - (1 - \alpha_{n+1})y_{n+1} - (\alpha_n - (1 - \alpha_n)y_n)\| \\
&\leq |\alpha_{n+1} - \alpha_n|\|u\| + |1 - \alpha_n|\|\gamma_{n+1} - \gamma_n\| \\
&\quad +|\alpha_n - \alpha_{n+1}|\|y_{n+1}\|. \qquad (4.1.10)
\end{aligned}$$

Substituting (4.1.9) into (4.1.10), we arrive at

$$\begin{aligned}
\|f_{n+1} - f_n\| &\leq |\alpha_{n+1} - \alpha_n|\|u\| + \|x_{n+1} - x_n\| + Q(|r_n - r_{n+1}| \\
&\quad +|\lambda_n - \lambda_{n+1}| + |\eta_n - \eta_{n+1}| + |\alpha_n - \alpha_{n+1}|) \\
&\quad +\|Tr_{n+1}(x_n - r_nA_3x_n) - Tr_n(x_n - r_nA_3x_n)\|, \qquad (4.1.11)
\end{aligned}$$

where Q is an appropriate constant such that

$$Q = \max\{M, sup_{n\geq 1}\{\|y_n\|\}\}$$

Let T be defined by $T_x = \lim_{n\to\infty} T_n x$. Notice that

$$\begin{aligned}\|T_{n+1}f_{n+1} - T_n f_n\| &= \|T_{n+1}f_{n+1} - Tf_{n+1} + Tf_{n+1} - Tf_n + Tf_n - T_n f_n\| \\ &\leq \|T_{n+1}f_{n+1} - Tf_{n+1}\| + \|Tf_{n+1} - Tf_n\| + \|Tf_n - T_n f_n\| \\ &\leq \sup_{x\in B}\{\|T_{n+1}x - Tx\|\} + \sup_{x\in B}\{\|Tx - T_n x\|\} \\ &\quad +\|f_{n+1} - f_n\|. \qquad (4.1.12)\end{aligned}$$

Let B be a bounded subset of C containing $\{f_n : n \in \mathbb{N}\}$ defined in (4.1.6). Combining (4.1.11) with (4.1.12), we arrive at

$$\begin{aligned}\|T_{n+1}f_{n+1} - T_n f_n\| &\leq \sup_{x\in B}\{\|T_{n+1}x - Tx\|\} + \sup_{x\in B}\{\|Tx - T_n x\|\} + \|f_{n+1} - f_n\| \\ &\leq \sup_{x\in B}\{\|T_{n+1}x - Tx\|\} + \sup_{x\in B}\{\|Tx - T_n x\|\} + \alpha_{n+1} - \alpha_n|\|u\| \\ &\quad +\|x_{n+1} - x_n\| + Q(|r_n - r_{n+1} + |\lambda_n - \lambda_{n+1}| \\ &\quad +|\eta_n - \eta_{n+1}| + |\alpha_n - \alpha_{n+1}|) \\ &\quad +\|Tr_{n+1}(x_n - r_n A_3 x_n) - Tr_n(x_n - r_n A_3 x_n)\|\end{aligned}$$

Notice that

$$\begin{aligned}\|T_{n+1}f_{n+1} - T_n f_n\| - \|x_{n+1} - x_n\| &\leq \sup_{x\in B}\{\|T_{n+1}x - Tx\|\} + \sup_{x\in B}\{\|Tx - T_n x\|\} \\ &\quad +|\alpha_{n+1} - \alpha_n|\|u\| + Q(|r_n - r_{n+1} + |\lambda_n - \lambda_{n+1}| \\ &\quad +|\eta_n - \eta_{n+1}| + |\alpha_n - \alpha_{n+1}|) + \|Tr_{n+1}v_n - Tr_n v_n\|,\end{aligned}$$

where $v_n = x_n - r_n A_3 x_n$. It follows from Lemma 2.5.1, Lemma 2.5.3 and the conditions $(C1) - (C3)$ that

$$\limsup_{n\to\infty}\{\|T_{n+1}f_{n+1} - T_n f_n\| - \|x_{n+1} - x_n\|\} \leq 0.$$

In view of lemma 2.5.2, we obtain that

$$\lim_{n\to\infty}\|T_n f_n - x_n\| = 0. \qquad (4.1.13)$$

Note that

$$\begin{aligned}\lim_{n\to\infty}\|x_{n+1} - x_n\| &= \lim_{n\to\infty}\|\beta_n x_n + (1-\beta_n)T_n f_n - x_n\| \\ &= \lim_{n\to\infty}(1-\beta_n)\|T_n f_n - x_n\| \\ &= 0. \qquad (4.1.14)\end{aligned}$$

For any $x^* \in \Omega$. we see from (4.1.1) that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\beta_n x_n + (1-\beta_n)T_n f_n - x^*\|^2 \\
&\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|f_n - x^*\|^2 \\
&= \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|(\alpha_n u + (1-\alpha_n)y_n) - x^*\|^2 \\
&= \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|(\alpha_n u + (1-\alpha_n)y_n) - x^* + \alpha_n x^* - \alpha_n x^*\|^2 \\
&\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)(\alpha_n\|u - x^*\|^2 \\
&\quad +(1-\alpha_n)\|y_n - x^*\|^2) \qquad (4.1.15)
\end{aligned}$$

Notice that

$$\begin{aligned}
\|y_n - x^*\|^2 &= \|P_C(z_n - \eta_n A_1 z_n) - P_C(x^* - \eta_n A_1 x^*)\|^2 \\
&\leq \|(z_n - \eta_n A_1 z_n) - (x^* - \eta_n A_1 x^*)\|^2 \\
&= \|(z_n - x^*) - \eta_n(A_1 z_n - A_1 x^*)\|^2 \\
&= \|z_n - x^*\|^2 - 2\eta_n\langle z_n - x^*, A_1 z_n - A_1 x^*\rangle + \eta_n^2\|A_1 z_n - A_1 x^*\|^2 \\
&\leq \|z_n - x^*\|^2 - 2\eta_n\delta_1\|A_1 z_n - A_1 x^*\|^2 + \eta_n^2\|A_1 z_n - A_1 x^*\|^2 \\
&= \|z_n - x^*\|^2 + \eta_n(\eta_n - 2\delta_1)\|A_1 z_n - A_1 x^*\|^2 \qquad (4.1.16)
\end{aligned}$$

Substituting (4.1.16) into (4.1.15),we arrive at

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)(\alpha_n\|u - x^*\|^2 + (1-\alpha_n)\|y_n - x^*\|^2) \\
&\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)(\alpha_n\|u - x^*\|^2 + (1-\alpha_n)(\|z_n - x^*\|^2 \\
&\quad +\eta_n(\eta_n - 2\delta_1)\|A_1 z_n - A_1 x^*\|^2) \\
&\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)(\alpha_n\|u - x^*\|^2 + (1-\alpha_n)(\|x_n - x^*\|^2 \\
&\quad +\eta_n(\eta_n - 2\delta_1)\|A_1 z_n - A_1 x^*\|^2) \\
&\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)(\alpha_n\|u - x^*\|^2 + \|x_n - x^*\|^2 \\
&\quad +(1-\alpha_n)\eta_n(\eta_n - 2\delta_1)\|A_1 z_n - A_1 x^*\|^2) \\
&= \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\alpha_n\|u - x^*\|^2 + (1-\beta_n)\|x_n - x^*\|^2 \\
&\quad +(1-\beta_n)(1-\alpha_n)\eta_n(\eta_n - 2\delta_1)\|A_1 z_n - A_1 x^*\|^2 \\
&\leq \|x_n - x^*\|^2 + \alpha_n\|u - x^*\|^2 + (1-\beta_n)(1-\alpha_n)\eta_n(\eta_n - 2\delta_1) \\
&\quad \|A_1 z_n - A_1 x^*\|^2. \qquad (4.1.17)
\end{aligned}$$

It follows from the condition (C1) that

$$
\begin{aligned}
(1-\alpha_n)(1-d)a''(2\delta_1-b'')\|A_1z_n-A_1x^*\|^2 &\le \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2 \\
&\quad +\alpha_n\|u-x^*\|^2 \\
&\le (\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_n-x_{n+1}\| \\
&\quad +\alpha_n\|u-x^*\|.
\end{aligned}
$$

It follows from (4.1.14) and the condition (C2) that

$$
\lim_{n\to\infty}\|A_1z_n-A_1x^*\| = 0 \qquad (4.1.18)
$$

Next, we show that

$$
\lim_{n\to\infty}\|A_2u_n-A_2x^*\| = 0 \qquad (4.1.19)
$$

In view of (4.1.15) , we arrive at

$$
\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le \beta_n\|x_n-x^*\|^2+(1-\beta_n)(\alpha_n\|u-x^*\|^2+(1-\alpha_n)\|y_n-x^*\|^2) \\
&\le \beta_n\|x_n-x^*\|^2+(1-\beta_n)(\alpha_n\|u-x^*\|^2 \\
&\quad +(1-\alpha_n)\|z_n-x^*\|^2) \qquad (4.1.20)
\end{aligned}
$$

Notice that

$$
\begin{aligned}
\|z_n-x^*\|^2 &= \|P_C(u_n-\lambda_nA_2u_n)-P_C(x^*-\lambda_nA_2x^*)\|^2 \\
&\le \|(u_n-\lambda_nA_2u_n)-(x^*-\lambda_nA_2x^*)\|^2 \\
&= \|u_n-x^*\|^2-2\lambda_n\langle u_n-x^*,A_2u_n-A_2x^*\rangle+\lambda_n^2\|A_2u_n-A_2x^*\|^2 \\
&\le \|u_n-x^*\|^2+\lambda_n(\lambda_n-2\delta_2)\|A_2u_n-A_2x^*\|^2 \qquad (4.1.21)
\end{aligned}
$$

Substituting (4.1.21) into (4.1.20), we arrive at

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)(\alpha_n\|u - x^*\|^2 + (1-\alpha_n)\|z_n - x^*\|^2) \\
&\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)(\alpha_n\|u - x^*\|^2 + (1-\alpha_n)\|u_n - x^*\|^2 \\
&\quad +\lambda_n(1-\alpha_n)(\lambda_n - 2\delta_2)\|A_2u_n - A_2x^*\|^2) \\
&\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)(\alpha_n\|u - x^*\|^2 + (1-\alpha_n)\|x_n - x^*\|^2 \\
&\quad +\lambda_n(1-\alpha_n)(\lambda_n - 2\delta_2)\|A_2u_n - A_2x^*\|^2) \\
&\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\alpha_n\|u - x^*\|^2 + (1-\beta_n)\|x_n - x^*\|^2 \\
&\quad +\lambda_n(1-\alpha_n)(1-\beta_n)(\lambda_n - 2\delta_2)\|A_2u_n - A_2x^*\|^2) \\
&\leq \|x_n - x^*\|^2 + \alpha_n\|u - x^*\|^2 + (1-\alpha_n)(1-\beta_n)\lambda_n(\lambda_n - 2\delta_2) \\
&\quad \|A_2u_n - A_2x^*\|^2
\end{aligned}$$

It follows from the condition (C1) that

$$\begin{aligned}
(1-\alpha_n)(1-d)a'(2\delta_2 - b')\|A_2u_n - A_2x^*\|^2 &\leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&\quad +\alpha_n\|u - x^*\|^2 \\
&\leq (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_n - x_{n+1}\| \\
&\quad +\alpha_n\|u - x^*\|^2
\end{aligned}$$

From(4.1.14) and the condition (C2), we obtain that (4.1.19) holds. it that

$$\lim_{n\to\infty}\|A_2u_n - A_2x^*\| = 0$$

In a similar way, we can prove that

$$\lim_{n\to\infty}\|A_3x_n - A_3x^*\| = 0 \tag{4.1.22}$$

On the other hand, we see from Lemma 2.4.7 that

$$\begin{aligned}
\|u_n - x^*\|^2 &= \|T_{r_n}(x_n - r_nA_3x_n) - T_{r_n}(x^* - r_nA_3x^*)\|^2 \\
&= \|T_{r_n}(I - r_nA_3)x_n - T_{r_n}(I - r_nA_3x)x^*\|^2 \\
&\leq \langle T_{r_n}(I - r_nA_3)x_n - T_{r_n}(I - r_nA_3x)x^*, (I - r_nA_3)x_n - (I - r_nA_3x)x^*\rangle \\
&= \langle T_{r_n}((I - r_nA_3)x_n - (I - r_nA_3x)x^*, u_n - x^*\rangle \\
&= \frac{1}{2}(\|(I - r_nA_3)x_n - (I - r_nA_3x)x^*\|^2 + \|u_n - x^*\|^2 - \|(I - r_nA_3)x_n \\
&\quad -(I - r_nA_3x)x^* - (u_n - x^*)\|^2) \\
&\leq \frac{1}{2}(\|x_n - x^*\|^2 + \|u_n - x^*\|^2 - \|x_n - r_nA_3x_n - x^* + r_nA_3x^* - u_n - x^*\|^2) \\
&= \frac{1}{2}(\|x_n - x^*\|^2 + \|u_n - x^*\|^2 - \|(x_n - u_n) - r_n(A_3x_n - A_3x^*)\|^2) \\
&= \frac{1}{2}(\|x_n - x^*\|^2 + \|u_n - x^*\|^2 - (\|x_n - u_n\|^2 + r_n^2\|A_3x_n - A_3x^*\|^2 \\
&\quad -2r_n\langle x_n - u_n, A_3x_n - A_3x^*\rangle)) \\
&= \frac{1}{2}(\|x_n - x^*\|^2 + \|u_n - x^*\|^2 - \|x_n - u_n\|^2 + 2r_n\langle x_n - u_n, A_3x_n - A_3x^*\rangle \\
&\quad -r_n\|A_3x_n - A_3x^*\|^2)
\end{aligned}$$

and hence

$$\begin{aligned}
\|u_n - x^*\|^2 &\leq \|x_n - x^*\|^2 - \|x_n - u_n\|^2 + 2r_n\langle x_n - u_n, A_3x_n - A_3x^*\rangle \\
&\quad -r_n\|A_3x_n - A_3x^*\|^2 \\
&\leq \|x_n - x^*\|^2 - \|x_n - u_n\|^2 \\
&\quad +2r_n\|x_n - u_n\|\|A_3x_n - A_3x^*\| \qquad (4.1.23)
\end{aligned}$$

From (4.1.20) , we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \beta_n\|x_n - x^*\|^2 + (1 - \beta_n)(\alpha_n\|u - x^*\|^2 \\
&\quad +(1 - \alpha_n)\|u_n - x^*\|^2) \qquad (4.1.24)
\end{aligned}$$

Substituting (4.1.23) into (4.1.24), we arrive at

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)(\alpha_n\|u - x^*\|^2 + (1-\alpha_n)(\|x_n - x^*\|^2 \\
&\quad -\|x_n - u_n\|^2 + 2r_n\|x_n - u_n\|\|A_3x_n - A_3x^*\|)) \\
&\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\alpha_n\|u - x^*\|^2 + (1-\beta_n)(1-\alpha_n)\|x_n - x^*\|^2 \\
&\quad -(1-\beta_n)(1-\alpha_n)\|x_n - u_n\|^2 + (1-\beta_n)(1-\alpha_n)2r_n\|x_n - u_n\| \\
&\quad \|A_3x_n - A_3x^*\| \\
&\leq \beta_n\|x_n - x^*\|^2 + \alpha_n\|u - x^*\|^2 - (1-\beta_n)(1-\alpha_n)\|x_n - u_n\|^2 \\
&\quad +(1-\beta_n)(1-\alpha_n)2r_n\|x_n - u_n\|\|A_3x_n - A_3x^*\|
\end{aligned}$$

and Hence

$$\begin{aligned}
(1-\beta_n)(1-\alpha_n)\|x_n - u_n\|^2 &\leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + alpha_n\|u - x^*\|^2 \\
&\quad +(1-\beta_n)(1-\alpha_n)2r_n\|x_n - u_n\|\|A_3x_n - A_3x^*\| \\
&\leq (\|x_n - x^*\| - \|x_{n+1} - x^*\|)(\|x_n - x^*\| + \|x_{n+1} - x^*\|) \\
&\quad +\alpha_n\|u - x^*\|^2 + (1-\beta_n)(1-\alpha_n)2r_n\|x_n - u_n\| \\
&\quad \|A_3x_n - A_3x^*\| \\
&\leq (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_n - x_{n+1}\| + \alpha_n\|u - x^*\|^2 \\
&\quad +(1-\beta_n)(1-\alpha_n)2r_n\|x_n - u_n\|\|A_3x_n - A_3x^*\|
\end{aligned}$$

In view of the conditions (C1) and (C2), we obtain from (4.1.14) and (4.1.22) that

$$\lim_{n\to\infty}\|x_n - u_n\| = 0 \tag{4.1.25}$$

On the other hand, we see from (4.1.1) that

$$\begin{aligned}
\|z_n - x^*\|^2 &= \|P_C(I-\lambda_n A_2)u_n - P_C(I-\lambda_n A_2)x^*\|^2 \\
&\leq \langle (I-\lambda_n A_2)u_n - (I-\lambda_n A_2)x^*, z_n - x^*\rangle \\
&= \frac{1}{2}(\|(I-\lambda_n A_2)u_n - (I-\lambda_n A_2)x^*\|^2 + \|z_n - x^*\|^2 - \|(I-\lambda_n A_2)u_n \\
&\quad -(I-\lambda_n A_2)x^* \|^2 - \|z_n - x^*\|^2) \\
&\leq \frac{1}{2}(\|u_n - x^*\|^2 + \|z_n - x^*\|^2 - \|(u_n - z_n) - \lambda_n(A_2u_n - A_2x^*)\|^2) \\
&\leq \frac{1}{2}(\|u_n - x^*\|^2 + \|z_n - x^*\|^2 - \|u_n - z_n\|^2 + 2\lambda_n\langle u_n - z_n, A_2u_n - A_2x^*\rangle \\
&\quad -\lambda_n^2\|A_2u_n - A_2x^*\|^2)
\end{aligned}$$

which implies that

$$\begin{aligned}
\|z_n - x^*\|^2 &\leq \|u_n - x^*\|^2 - \|u_n - z_n\| + 2\lambda_n\langle u_n - z_n, A_2u_n - A_2x^*\rangle \\
&\leq \|u_n - x^*\|^2 - \|u_n - z_n\| + 2\lambda_n\|u_n - z_n\|\|A_2u_n - A_2x^*\| \\
&\leq \|x_n - x^*\|^2 - \|u_n - z_n\| \\
&\quad +2\lambda_n\|u_n - z_n\|\|A_2u_n - A_2x^*\| \qquad (4.1.26)
\end{aligned}$$

Substituting (4.1.26) into (4.1.20), we arrive at

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)(\alpha_n\|u - x^*\|^2 + (1-\alpha_n)\|z_n - x^*\|^2) \\
&\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)(\alpha_n\|u - x^*\|^2 + (1-\alpha_n)(\|x_n - x^*\|^2 \\
&\quad -\|u_n - z_n\| + 2\lambda_n\|u_n - z_n\|\|A_2u_n - A_2x^*\|) \\
&= \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\alpha_n\|u - x^*\|^2 + (1-\beta_n)(1-\alpha_n)\|x_n - x^*\|^2 \\
&\quad -(1-\beta_n)(1-\alpha_n)\|u_n - z_n\| + (1-\beta_n)(1-\alpha_n)2\lambda_n\|u_n - z_n\| \\
&\quad \|A_2u_n - A_2x^*\|) \\
&\leq \|x_n - x^*\|^2 + \alpha_n\|u - x^*\|^2 - (1-\alpha_n)(1-\beta_n)\|u_n - z_n\|^2 \\
&\quad +2\lambda_n\|u_n - z_n\|\|A_2u_n - A_2x^*\|
\end{aligned}$$

from which it follows that

$$\begin{aligned}(1-\alpha_n)(1-\beta_n)\|u_n - z_n\|^2 &\leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n\|u - x^*\|^2 \\ &\quad +2\lambda_n\|u_n - z_n\|\|A_2u_n - A_2x^*\| \\ &\leq (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_n - x_{n+1}\| + \alpha_n\|u - x^*\|^2 \\ &\quad +2\lambda_n\|u_n - z_n\|\|A_2u_n - A_2x^*\|\end{aligned}$$

In view of the conditions (C1) and (C2), we obtain from (4.1.14) and (4.1.19) that

$$\lim_{n\to\infty}\|u_n - z_n\| = 0. \tag{4.1.27}$$

In a similar way, we can show that

$$\lim_{n\to\infty}\|y_n - z_n\| = 0. \tag{4.1.28}$$

In view of $f_n = \alpha_n u + (1-\alpha_n)y_n$. We get from the condition (C2) that

$$\lim_{n\to\infty}\|f_n - y_n\| = 0. \tag{4.1.29}$$

Notice that

$$\|x_{n+1} - x_n\| = (1-\beta_n)\|T_nf_n - x_n\|.$$

Which combines with (4.1.14) gives that

$$\lim_{n\to\infty}\|T_nf_n - x_n\| = 0. \tag{4.1.30}$$

On the other hand, we have

$$\begin{aligned}\|T_nf_n - f_n\| &= \|f_n - T_nf_n\| \\ &= \|f_n - y_n + y_n - z_n + z_n - u_n + u_n - x_n + x_n - T_nf_n\| \\ &\leq \|f_n - y_n\| + \|y_n - z_n\| + \|z_n - u_n\| + \|u_n - x_n\| + \|x_n - T_nf_n\|.\end{aligned}$$

From (4.1.25), (4.1.27), (4.1.28), (4.1.29) and (4.1.30) , we obtain that

$$\lim_{n\to\infty}\|T_nf_n - f_n\| = 0. \tag{4.1.31}$$

Next, we prove that

$$\limsup_{n\to\infty}\langle u - z, f_n - z\rangle \leq 0,$$

where $z = P_\Omega u$. To see this, we choose a subsequence $\{f_{n_i}\}$ of $\{f_n\}$ such that

$$\limsup_{n\to\infty}\langle u-z, f_n - z\rangle \;=\; \lim_{i\to\infty}\langle u-z, f_{n_i}-z\rangle. \tag{4.1.32}$$

Since $\{f_{n_i}\}$ is bounded, there exists a subsequence $\{f_{n_{i_j}}\}$ of $\{f_{n_i}\}$ which converges weakly to f. Without loss of generality, we may assume that $f_{n_i} \rightharpoonup f$. From (4.1.30), we also have $y_{n_i} \rightharpoonup f$. First, we prove $f \in VI(C, A_1)$. For the purpose, let T be the maximal monotone mapping defined by lemma 2.4.6

$$Tv = \begin{cases} Bv + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

For any given $(x, y) \in G(T)$, hence $y - A_1 x \in N_C x$. Since $y_n \in C$, we see from the definition of N_C that

$$\langle x - y_n, y - A_1 x\rangle \geq 0. \tag{4.1.33}$$

On the other hand, we have from $y_n = P_C(I - \eta_n A_1)z_n$ that

$$\langle x - y_n, y_n - (I-\eta_n A_1)z_n\rangle \geq 0$$

and hence

$$\langle x - y_n, \frac{y_n - z_n}{\eta_n} + A_1 z_n\rangle \geq 0.$$

From (4.1.33) and the δ_1-inverse monotonicity of A_1, we see that

$$\begin{aligned}
\langle x - y_{n_i}, y\rangle &\geq \langle x - y_{n_i}, A_1 x\rangle \\
&\geq \langle x - y_{n_i}, A_1 x\rangle - \langle x - y_{n_i}, A_1 z_{n_i}\rangle - \langle x - y_{n_i}, \frac{y_{n_i} - z_{n_i}}{\eta_{n_i}} + A_1 z_{n_i}\rangle \\
&= \langle x - y_{n_i}, A_1 x - A_1 y_{n_i}\rangle + \langle x - y_{n_i}, A_1 y_{n_i} - A_1 z_{n_i}\rangle - \langle x - y_{n_i}, \frac{y_{n_i} - z_{n_i}}{\eta_{n_i}}\rangle \\
&\geq \langle x - y_{n_i}, A_1 y_{n_i} - A_1 n_i\rangle - \langle x - y_{n_i}, \frac{y_{n_i} - z_{n_i}}{\eta_{n_i}}\rangle.
\end{aligned}$$

Since $y_{n_i} \rightharpoonup f$ and A_1 is Lipschitz continuous, we obtain that $\langle x - f, y\rangle \geq 0$. Notice that T is maximal monotone, hence $0 \in Tf$. This shows that $f \in VI(C, A_1)$. On the other hand, we have

$$\|z_n - f_n\| \;\leq\; \|z_n - y_n\| + \|y_n - f_n\|.$$

It follows from (4.1.29) and (4.1.29) that

$$\lim_{n\to\infty} \|z_n - f_n\| = 0.$$

Therefore, we also have $z_{n_i} \rightharpoonup f$. Similarly, we can prove $f \in VI(C, A_2)$. That is $f \in VI$. Next, we show that $f \in FP$. Suppose the contrary, $f \notin FP$,$Tf \neq f$. Notice that

$$\begin{aligned}\|f_n - Tf_n\| &\leq \|Tf_n - T_nf_n\| + \|T_nf_n - f_n\| \\ &\leq \sup_{x\in B}\{\|Tx - T_nx\|\} + \|T_nf_n - f_n\|.\end{aligned}$$

In view of Lemma 2.4.7, we obtain from (4.1.31) that $\lim_{n\to\infty} \|f_n - Tf_n\| = 0$. Since $f_{n_i} \rightharpoonup f$ and by the Opial condition, we see that

$$\begin{aligned}\liminf_{n\to\infty} \|f_{n_i} - f\| &< \liminf_{n\to\infty} \|f_{n_i} - Tf\| \\ &\leq \liminf_{n\to\infty}\{\|f_{n_i} - Tf_{n_i}\| + \|Tf_{n_i} - Tf\|\} \\ &\leq \liminf_{n\to\infty}\{\|f_{n_i} - Tf_{n_i}\| + \|Tf_{n_i} - f\|\} \\ &\leq \liminf_{n\to\infty} \|f_{n_i} - f\|,\end{aligned}$$

which derives a contradiction. Thus, we have $f \in FP$. Finally, we show that $f \in \Gamma$. Notice that

$$\|f_n - u_n\| \leq \|f_n - y_n\| + \|y_n - z_n\| + \|z_n - u_n\|.$$

From (4.1.27),(4.1.28),(4.1.29), we have

$$\lim_{n\to\infty} \|f_n - u_n\| = 0. \tag{4.1.34}$$

It follows that $u_n \rightharpoonup f$. Since $u_n = T_{r_n}(x_n - r_nA_3x_n)$, for any $y \in C$,we have

$$F(u_n, y) + G(u_n, y) + \langle A_3x_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0$$

From the monotonicity of F, we have

$$G(u_n, y) + \langle A_3x_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F(y, u_n), \quad \forall y \in C.$$

Hence,

$$G(u_{n_i}, y) + \langle A_3x_{n_i}, y - u_{n_i}\rangle + \langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle \geq F(y, u_n), \tag{4.1.35}$$

where $y \in C$. Put $\rho_t = ty + (1-t)f$ for all$t \in (0,1]$, and $y \in C$. Then, we have $\rho_t \in C$. So, from (4.1.35) we have

$$\begin{aligned}\langle \rho_t - u_{n_i}, A_3\rho_t\rangle &\geq \langle \rho_t - u_{n_i}, A_3\rho_t\rangle - \langle A_3x_{n_i}, \rho_t - u_{n_i}\rangle + F(\rho_t, u_{n_i}) - G(u_{n_i}, \rho_t)\\ &\quad -\langle \rho_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle\\ &= \langle \rho_t - u_{n_i}, A_3\rho_t - A_3u_{n_i}\rangle + \langle \rho_t - u_{n_i}, A_3u_{n_i} - A_3x_{n_i}\rangle + F(\rho_t, u_{n_i})\\ &\quad -G(u_{n_i}, \rho_t) - \langle \rho_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle \qquad (4.1.36)\end{aligned}$$

From (4.1.25), we have $\|A_3u_{n_i} - A_3x_{n_i}\| \to 0$ as $i \to \infty$. On the other hand, we get from the monotonicity of A_3 that

$$\langle \rho_t - u_{n_i}, A_3\rho_t - A_3u_{n_i}\rangle \geq 0$$

It follows from (4.1.36) that

$$\langle \rho_t - u_{n_i}, A_3\rho_t\rangle \geq F(\rho_t, u_{n_i}) - G(u_{n_i}, \rho_t), \quad as \;\; i \to \infty \qquad (4.1.37)$$

From (F1), (F3), (G1), (G3) and (4.1.37), we also have

$$\begin{aligned}0 &= F(\rho_t, \rho_t) + G(\rho_t, \rho_t)\\ &\leq tF(\rho_t, y) + (1-t)F(\rho_t, f) + tG(\rho_t, y) + (1-t)G(\rho_t, f)\\ &\leq tF(\rho_t, y) + tG(\rho_t, y) + (1-t)[F(\rho_t, f) - G(f, \rho_t)]\\ &\leq tF(\rho_t, y) + tG(\rho_t, y) + (1-t)\langle \rho_t - f, A_3\rho_t\rangle\\ &\leq t[F(\rho_t, y) + G(\rho_t, y) + (1-t)\langle y - f, A_3\rho_t\rangle]\end{aligned}$$

and hence

$$0 \leq F(\rho_t, y) + G(\rho_t, y) + (1-t)\langle A_3\rho_t, y - f\rangle \qquad (4.1.38)$$

Letting $t \to 0$ in (4.1.38) , we have, for each $y \in C$,

$$0 \leq F(f, y) + G(f, y) + \langle y - f, A_3f\rangle$$

This implies that $f \in \Gamma$.Next we prove

$$\limsup_{n\to\infty}\langle u - z, f_n - z\rangle \leq 0.$$

where $z = P_\Omega u$. It follows from $f_n \to f$ and (1.1.2) that

$$\limsup_{n\to\infty}\langle u - z, f_n - z\rangle = \langle u - z, f - z\rangle \leq 0.$$

Finally, we show that $x_n \to z$, as $n \to \infty$. Note that

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|\beta_n x_n + (1-\beta_n)T_n f_n - z\|^2 \\
&\leq \beta_n\|x_n - z\|^2 + (1-\beta_n)\|f_n - z\|^2 \\
&= \beta_n\|x_n - z\|^2 + (1-\beta_n)\|\alpha_n u + (1-\alpha_n)y_n - z\|^2 \\
&= \beta_n\|x_n - z\|^2 + (1-\beta_n)\|\alpha_n u - \alpha_n z + (1-\alpha_n)y_n \\
&\quad -(1-\alpha_n)z\|^2 \\
&= \beta_n\|x_n - z\|^2 + (1-\beta_n)\|\alpha_n(u-z) + (1-\alpha_n)(y_n - z)\|^2 \\
&\leq \beta_n\|x_n - z\|^2 + (1-\beta_n)[(1-\alpha_n)^2\|y_n - z)\|^2 \\
&\quad +2\langle\alpha_n(u-z), \alpha_n(u-z) + (1-\alpha_n)(y_n - z)\rangle] \\
&= \beta_n\|x_n - z\|^2 + (1-\beta_n)[(1-\alpha_n)^2\|y_n - z)\|^2 \\
&\quad +2\alpha_n\langle u-z, f_n - z\rangle] \\
&\leq \beta_n\|x_n - z\|^2 + (1-\beta_n)[(1-\alpha_n)\|x_n - z)\|^2 \\
&\quad +2\alpha_n\langle u-z, f_n - z\rangle] \\
&= \beta_n\|x_n - z\|^2 + (1-\beta_n)(1-\alpha_n)\|x_n - z)\|^2 \\
&\quad +(1-\beta_n)2\alpha_n\langle u-z, f_n - z\rangle \\
&= [1-(1-\beta_n)\alpha_n]\|x_n - z\|^2 + 2\alpha_n(1-\beta_n) \\
&\quad \langle u-z, f_n - z\rangle. \qquad (4.1.39)
\end{aligned}$$

since $\sum_{n=1}^{\infty}(1-\beta_n)\alpha_n = \infty$ and $limsup_{n\to\infty} 2\langle u-z, f_n-z\rangle \leq 0$,we see from Lemma 2.3.13 that $\lim_{n\to\infty}\|x_n - z\| = 0$. □

4.2 Strong Convergence Theorems for Nonspreading Mappings

Theorem 4.2.1. *Let C be a nonempty closed convex subset of of a real Hilbert space. Let F and G be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (F1)-(F2),(G1)-(G3) and (H).Let A, B be a δ_j-inverse-strongly monotone mapping of C into H for each $j = 1, 2,$. let $T : C \to C$ be a k-strictly pseudononspreading mapping with a nonempty fixed point set and $\Omega := F(T) \cap \Gamma \cap VI(C, B) \neq \emptyset$. Let $\beta \in [k, 1)$ and let $T_\beta := \beta I + (1-\beta)T$. Let $u \in C$ and $\{x_n\}_{n=1}^{\infty}$, $\{u_n\}_{n=1}^{\infty}$, $\{z_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ be sequence in C generated from an arbitrary $x_1 \in C$ by*

$$\begin{cases} F(u_n, y) + G(u_n, y) + \langle Ax_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_{n,} u_n - x_n \rangle \geqslant 0, \quad \forall y \in C, \\ y_n = P_C(u_n - \lambda_n B u_n), \\ z_n = \frac{1}{n} \sum_{m=0}^{n-1} T_\beta^m y_n, \; n \geq 1. \\ x_{n+1} = \alpha_n u + (1-\alpha_n) z_n, \; n \geq 1, \end{cases} \tag{4.2.1}$$

where $\{\alpha_n\}_{n=1}^{\infty} \subset [0, 1)$ and $\{r_n\}_{n=1}^{\infty}, \{\lambda_n\}_{n=1}^{\infty}$ are positive number sequences. Assume that the above control sequences satisfy the following restrictions

(C1) $0 < a \leq \{r_n\} \leq b < 2\delta_1$, $0 < a' \leq \lambda_n \leq b' < 2\delta_2$ *for all* $n \geq 1$.

(C2) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\Sigma_{n=1}^{\infty} \alpha_n = \infty$,

(C3) $\lim_{n\to\infty}(r_n - r_{n+1}) = \lim_{n\to\infty}(\lambda_n - \lambda_{n+1}) = 0$

Then $\{x_n\}_{n=1}^{\infty}$, $\{u_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ converge strongly to $P_\Omega u$, where $P_\Omega : H \to \Omega$ is the metric projection of H onto Ω.

Proof. Let $x^* \in \Omega$. Put $u_n = T_{r_n}(x_n - r_n A x_n)$ and $x^* = T_{r_n}(x^* - r_n A x^*)$. for all $n \geq 0$. Since A are $\delta_1 - inverse - strongly$ monotone, condition (C1) and nonexpansive, we have, for any $n \geq 1$,

$$\begin{aligned}
\|u_n - x^*\|^2 &= \|T_{r_n}(x_n - r_nAx_n) - T_{r_n}(x^* - r_nAx^*)\|^2 \\
&\leq \|(x_n - r_nAx_n) - (x^* - r_nAx^*)\|^2 \\
&\leq \|(x_n - x^*) - r_n(Ax_n - Ax^*)\|^2 \\
&= \|x_n - x^*\|^2 - 2r_n\langle x_n - x^*, Ax_n - Ax^*\rangle + \|Ax_n - Ax^*\|^2 \\
&\leq \|x_n - x^*\|^2 - 2r_n\delta_2\|Ax_n - Ax^*\|^2 + r_n^2\|Ax_n - Ax^*\|^2 \\
&= \|x_n - x^*\|^2 + r_n(r_n - 2\delta_2)\|Ax_n - Ax^*\|^2 \\
&\leq \|x_n - x^*\|^2. \qquad (4.2.2)
\end{aligned}$$

We show the mapping $I - \lambda_nB$ is nonexpansive. Indeed, since B is a δ_2-strongly monotone mapping, we have that for all $x, y \in C$,

$$\begin{aligned}
\|(I - \lambda_nB)x - (I - \lambda_nB)y\|^2 &= \|(x - y) - \lambda_n(Bx - By)\|^2 \\
&= \|x - y\|^2 - 2\lambda_n\langle x - y, Bx - By\rangle + \lambda_n^2\|Bx - By\|^2 \\
&\leq \|x - y\|^2 + \lambda_n(\lambda_n - 2\delta_2)\|Bx - By\|^2 \\
&\leq \|x - y\|^2, \qquad (4.2.3)
\end{aligned}$$

which implies that the mapping $I - \lambda_nB$ is nonexpansive. By using the sane argument as aboue, we can show that $I - \lambda_nA$ are nonexpansiveness.It follows that

$$\begin{aligned}
\|y_n - x^*\| &= \|P_C(u_n - \lambda_nBu_n) - P_C(x^* - \lambda_nBx^*)\| \\
&\leq \|(u_n - \lambda_nB)u_n - (I - \lambda_nB)x^*\| \\
&\leq \|(I - \lambda_nB)u_n - (I - \lambda_nB)x^*\| \\
&\leq \|u_n - x^*\|. \\
&\leq \|x_n - x^*\|. \qquad (4.2.4)
\end{aligned}$$

Let $T_\beta x := \beta x + (1 - \beta)Tx$. It is clear that $F(T_\beta) = F(T)$ and for all $x, y \in C$, we have

$$
\begin{aligned}
\|T_\beta x - T_\beta y\|^2 &= \|\beta(x-y) + (1-\beta)(Tx - Ty)\|^2 \\
&= \beta\|x-y\|^2 + (1-\beta)\|Tx-Ty\|^2 - \beta(1-\beta)\|x - Tx - (y - Ty)\|^2 \\
&\leq \beta\|x-y\|^2 - \beta(1-\beta)\|x - Tx - (y-Ty)\|^2 \\
&\quad +(1-\beta)\big[\|x-y\|^2 + k\|x - Tx - (y-Ty)\|^2 + 2\langle x - Tx, y - Ty\rangle\big] \\
&= \|x-y\|^2 - \beta(1-\beta)\|x - Tx - (y-Ty)\|^2 \\
&\quad +k(1-\beta)\|x - Tx - (y-Ty)\|^2 + 2(1-\beta)\langle x - Tx, y - Ty\rangle \\
&= \|x-y\|^2 - (1-\beta)(\beta - k)\|x - Tx - (y-Ty)\|^2 \\
&\quad +2(1-\beta)\langle x - Tx, y - Ty\rangle \\
&\leq \|x-y\|^2 + 2(1-\beta)\langle x - Tx, y - Ty\rangle \\
&= \|x-y\|^2 + \frac{2}{(1-\beta)}\langle x - T_\beta x, y - T_\beta y\rangle. \qquad (4.2.5)
\end{aligned}
$$

Using (4.2.1)and (4.2.4) we obtain

$$
\begin{aligned}
\|z_n - x^*\| &= \|\frac{1}{n}\sum_{m=0}^{n-1} T_\beta^m y_n - x^*\| \\
&\leq \frac{1}{n}\sum_{m=0}^{n-1}\|T_\beta^m y_n - x^*\| \\
&\leq \frac{1}{n}\sum_{m=0}^{n-1}\|y_n - x^*\|| \\
&= \|y_n - x^*\| \leq \|x_n - x^*\|. \qquad (4.2.6)
\end{aligned}
$$

Thus

$$
\begin{aligned}
\|x_{n+1} - x^*\| &= \|\alpha_n u + (1-\alpha_n)z_n - x^*\| \\
&\leq \alpha_n\|u - x^*\| + (1-\alpha_n)\|z_n - x^*\| \\
&\leq \alpha_n\|u - x^*\| + (1-\alpha_n)\|x_n - x^*\|. \qquad (4.2.7)
\end{aligned}
$$

By (4.2.7) and induction, we can conclude that for all $n \in \mathbb{N}$

$$\|x_n - x^*\| \leq \max\{\|u - x^*\|, \|x_1 - x^*\|\}.$$

This implies that $\{x_n\}$, $\{y_n\}$, $\{u_n\}$and $\{z_n\}$ are bounded.

Observe that since $\{z_n\}$ is bounded and $\lim_{n\to\infty}\alpha_n = 0$, we obtain

$$\begin{aligned}\|x_{n+1} - z_n\| &= \|\alpha_n u + (1-\alpha_n)z_n - z_n\| \\ &= \alpha_n\|u - z_n\| \to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{4.2.8}$$

Using (4.2.5) we obtain for all $m = 0, 1, 2, ..., n-1$ and for arbitrary $y \in C$

$$\begin{aligned}\|T_\beta^{m+1}y_n - T_\beta y\|^2 &= \|T_\beta(T_\beta^m y_n) - T_\beta y\|^2 \\ &\leq \|T_\beta^m y_n - y\|^2 + \frac{2}{1-\beta}\langle T_\beta^m y_n - T_\beta^{m+1}y_n, y - T_\beta y\rangle \\ &= \|T_\beta^m y_n - T_\beta y + T_\beta y - y\|^2 + \frac{2}{1-\beta}\langle T_\beta^m y_n - T_\beta^{m+1}y_n, y - T_\beta y\rangle \\ &= \|T_\beta^m y_n - T_\beta y\|^2 + \|T_\beta y - y\|^2 + 2\langle T_\beta^m y_n - T_\beta y, T_\beta y - y\rangle \\ &\quad + \frac{2}{1-\beta}\langle T_\beta^m y_n - T_\beta^{m+1}y_n, y - T_\beta y\rangle. \end{aligned} \tag{4.2.9}$$

Summing (4.2.9) from $m = 0$ to $n-1$ and dividing by n we obtain.

$$\begin{aligned}\frac{1}{n}\|T_\beta^n y_n - T_\beta y\|^2 &\leq \frac{1}{n}\|y_n - T_\beta y\|^2 + \|T_\beta y - y\|^2 + 2\langle z_n - T_\beta y, T_\beta y - y\rangle \\ &\quad + \frac{2}{n(1-\beta)}\langle y_n - T_\beta^n y_n, y - T_\beta y\rangle. \end{aligned} \tag{4.2.10}$$

Since z_n is bounded, there is a subsequence z_{n_j} of z_n and $w \in C$ such that $z_n \to w$. Replacing n by n_j in (4.2.10), we obtain

$$\begin{aligned}\frac{1}{n_j}\|T_\beta^{n_j} y_{n_j} - T_\beta y\|^2 &\leq \frac{1}{n_j}\|y_{n_j} - T_\beta y\|^2 + \|T_\beta y - y\|^2 + 2\langle z_{n_j} - T_\beta y, T_\beta y - y\rangle \\ &\quad \frac{2}{n_j(1-\beta)}\langle y_{n_j} - T_\beta^{n_j} y_{n_j}, y - T_\beta y\rangle. \end{aligned} \tag{4.2.11}$$

Since $\{y_n\}$ and $\{T_\beta^n y_n\}$ are bounded, letting $j \to \infty$ in (4.2.11) yields

$$0 \leq \|T_\beta y - y\|^2 + 2\langle w - T_\beta y, T_\beta y - y\rangle. \tag{4.2.12}$$

Since $y \in C$ was arbitrary, if we set $y = w$ in (4.2.12) we obtain

$$0 \leq \|T_\beta w - w\|^2 - 2\|T_\beta w - w\|^2 = -\|T_\beta w - w\|^2,$$

from which it follows that $w \in F(T_\beta) = F(T)$. We may assume without loss of generality that there exists a subsequence $\{x_{m_j+1}\}$ of $\{x_{m+1}\}$ such that

$$\limsup_{m\to\infty}\langle u - P_{F(T)}u, x_{m+1} - P_{F(T)}u\rangle = \lim_{j\to\infty}\langle u - P_{F(T)}u, x_{m_j+1} - P_{F(T)}u\rangle,$$

and $x_{m_j+1} \rightharpoonup v$. From (4.2.8) , we have $z_j \rightharpoonup v$. From the above argument, we have $v \in F(T)$. Since$P_{F(T)} : H \to F(T)$ is the metric projection, we have

$$\lim_{j\to\infty}\langle u - P_{F(T)}u, x_{m_j+1} - P_{F(T)}u\rangle = \langle u - P_{F(T)}u, v - P_{F(T)}u\rangle \le 0.$$

This implies

$$\limsup_{j\to\infty}\langle u - P_{F(T)}u, x_{n+1} - P_{F(T)}u\rangle \le 0. \tag{4.2.13}$$

Using Lemma 2.3.14 and (4.2.6) we have

$$\begin{aligned}
\|x_{n+1} - P_{F(T)}u\|^2 &= \|\alpha_n u + (1-\alpha_n)z_n - P_{F(T)}u\|^2 \\
&= \|\alpha_n u - \alpha_n P_{F(T)}u + (1-\alpha_n)z_n - (1-\alpha_n)P_{F(T)}u\|^2 \\
&= \|\alpha_n(u - P_{F(T)}u) + (1-\alpha_n)(z_n - P_{F(T)}u)\|^2 \\
&\le (1-\alpha_n)^2\|z_n - P_{F(T)}u\|^2 + 2\alpha_n\langle u - P_{F(T)}u, x_{n+1} - P_{F(T)}u\rangle \\
&\le (1-\alpha_n)^2\|x_n - P_{F(T)}u\|^2 + 2\alpha_n\langle u - P_{F(T)}u, x_{n+1} - P_{F(T)}u\rangle
\end{aligned} \tag{4.2.14}$$

Since $\alpha_n \to 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$ and (4.2.13), it follows from Lemma 2.3.14 that

$$\lim_{n\to\infty}\|x_n - P_{F(T)}u\| = 0. \tag{4.2.15}$$

From (4.2.15),that

$$\|x_{n+1} - x_n\| \to 0. \tag{4.2.16}$$

Since (4.2.15), we have

$$0 \le \|z_n - P_{F(T)}u\| \le \|z_n - x_{n+1}\| + \|x_{n+1} - P_{F(T)}u\| \to 0 \text{ as } n \to \infty.$$

Hence $\lim_{n\to\infty}\|z_n - P_{F(T)}u\| = 0$. Science (4.2.15) that

$$\|z_n - x_n\| \to 0. \tag{4.2.17}$$

Since $z_{n_j} \to w$, We obtain $w = P_{F(T)}u$. For any $x^* \in \Omega$. we see from (4.2.1) that

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|\alpha_n u_n + (1-\alpha_n) z_n - x^*\|^2 \\ &\leq \alpha_n \|u - x^*\|^2 + (1-\alpha_n)\|z_n - x^*\|^2 \end{aligned} \tag{4.2.18}$$

From (4.2.4) and (4.2.6),we arrive at

$$\|x_{n+1} - x^*\|^2 \leq \alpha_n \|u - x^*\|^2 + (1-\alpha_n)\|u_n - x^*\|^2 \tag{4.2.19}$$

Notice that

$$\begin{aligned} \|u_n - x^*\|^2 &= \|T_{r_n}(x_n - r_n A z_n) - T_{r_n}(x^* - r_n A x^*)\|^2 \\ &\leq \|(x_n - r_n A z_n) - (x^* - r_n A x^*)\|^2 \\ &= \|x_n - x^*\|^2 - 2r_n \langle x_n - x^*, Ax_n - Ax^* \rangle + r_n^2 \|Ax_n - Ax^*\|^2 \\ &\leq \|x_n - x^*\|^2 - 2r_n \delta_1 \|Ax_n - Ax^*\|^2 + r_n^2 \|Ax_n - Ax^*\|^2 \\ &= \|x_n - x^*\|^2 + r_n(r_n - 2\delta_1)\|Ax_n - Ax^*\|^2 \end{aligned} \tag{4.2.20}$$

Substituting (4.2.20) into (4.2.19),we arrive at

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq \alpha_n \|u - x^*\|^2 + (1-\alpha_n)(\|x_n - x^*\|^2 + r_n(r_n - 2\delta_1)\|Ax_n - Ax^*\|^2) \\ &\leq \alpha_n \|u - x^*\|^2 + (1-\alpha_n)\|x_n - x^*\|^2 + (1-\alpha_n) r_n (r_n - 2\delta_1) \\ &\quad \times \|Ax_n - Ax^*\|^2 \end{aligned}$$

It follows from the condition (C1) that

$$\begin{aligned} (1-\alpha_n) a (2\delta_1 - b)\|Ax_n - Ax^*\|^2 &\leq \alpha_n \|u - x^*\|^2 - \alpha_n \|x_n - x^*\|^2 + \|x_n - x^*\|^2 \\ &\quad + \|x_{n+1} - x^*\|^2 \\ &= \alpha_n \|u - x^*\|^2 - \alpha_n \|x_n - x^*\|^2 \\ &\quad + (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_n - x_{n+1}\| \end{aligned}$$

It follows from (4.2.16) and the condition (C2) that

$$\lim_{n\to\infty} \|Ax_n - Ax^*\|^2 = 0. \tag{4.2.21}$$

Next, we show that

$$\lim_{n\to\infty} \|Bu_n - Bx^*\| \;=\; 0 \tag{4.2.22}$$

In view of (4.2.6) and (4.2.18) , we arrive at

$$\|x_{n+1} - x^*\|^2 \;\leq\; \alpha_n\|u - x^*\|^2 + (1-\alpha_n)\|y_n - x^*\|^2 \tag{4.2.23}$$

Notice that

$$\begin{aligned} \|y_n - x^*\|^2 &= \|P_C(u_n - \lambda_n Bu_n) - P_C(x^* - \lambda_n Bx^*)\|^2 \\ &\leq \|(u_n - \lambda_n Bu_n) - (x^* - \lambda_n Bx^*)\|^2 \\ &= \|u_n - x^*\|^2 - 2\lambda_n\langle u_n - x^*, Bu_n - Bx^*\rangle + \lambda_n^2\|Bu_n - Bx^*\|^2 \\ &\leq \|u_n - x^*\|^2 + \lambda_n(\lambda_n - 2\delta_2)\|Bu_n - Bx^*\|^2 \\ &\leq \|x_n - x^*\|^2 + \lambda_n(\lambda_n - 2\delta_2)\|Bu_n - Bx^*\|^2 \qquad (4.2.24) \end{aligned}$$

Substituting (4.2.24) into (4.2.23), we arrive at

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq \alpha_n\|u - x^*\|^2 + (1-\alpha_n)(\|x_n - x^*\|^2 + \lambda_n(\lambda_n - 2\delta_2)\|Bu_n - Bx^*\|^2) \\ &= \alpha_n\|u - x^*\|^2 + (1-\alpha_n)\|x_n - x^*\|^2 + (1-\alpha_n)\lambda_n(\lambda_n - 2\delta_2) \\ &\quad \times\|Bu_n - Bx^*\|^2 \end{aligned}$$

It follows from the condition (C1) that

$$\begin{aligned} (1-\alpha_n)a'(2\delta_1 - b')\|Bu_n - Bx^*\|^2 &\leq \alpha_n\|u - x^*\|^2 - \alpha_n\|x_n - x^*\|^2 + \|x_n - x^*\|^2 \\ &\quad +\|x_{n+1} - x^*\|^2 \\ &= \alpha_n\|u - x^*\|^2 - \alpha_n\|x_n - x^*\|^2 \\ &\quad +(\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_n - x_{n+1}\| \end{aligned}$$

It follows from (4.2.18) and the condition (C2), we obtain that (4.2.21) holds.

We show that $\|x_n - u_n\| \to 0$, as $n \to \infty$. For $x^* \in \Omega$, we have from Lemma 2.4.7 that

$$\begin{aligned}
\|u_n - x^*\|^2 &= \|T_{r_n}(x_n - r_nAx_n) - T_{r_n}(x^* - r_nAx^*)\|^2 \\
&= \|T_{r_n}(I - r_nA)x_n - T_{r_n}(I - r_nAx)x^*\|^2 \\
&\leq \langle T_{r_n}(I - r_nA)x_n - T_{r_n}(I - r_nAx)x^*, (I - r_nA)x_n - (I - r_nAx)x^*\rangle \\
&= \langle (I - r_nA)x_n - (I - r_nAx)x^*, u_n - x^*\rangle \\
&= \frac{1}{2}(\|(I - r_nA)x_n - (I - r_nAx)x^*\|^2 + \|u_n - x^*\|^2 - \|(I - r_nA)x_n \\
&\quad -(I - r_nAx)x^* - (u_n - x^*)\|^2) \\
&= \frac{1}{2}(\|x_n - x^*\|^2 + \|u_n - x^*\|^2 - \|x_n - r_nAx_n - x^* + r_nAx^* - u_n - x^*\|^2) \\
&= \frac{1}{2}(\|x_n - x^*\|^2 + \|u_n - x^*\|^2 - \|(x_n - u_n) - r_n(Ax_n - Ax^*)\|^2) \\
&= \frac{1}{2}(\|x_n - x^*\|^2 + \|u_n - x^*\|^2 - (\|x_n - u_n\|^2 + r_n^2\|Ax_n - Ax^*\|^2 \\
&\quad -2r_n\langle x_n - u_n, Ax_n - Ax^*\rangle)) \\
&= \frac{1}{2}(\|x_n - x^*\|^2 + \|u_n - x^*\|^2 - \|x_n - u_n\|^2 + 2r_n\langle x_n - u_n, Ax_n - Ax^*\rangle \\
&\quad -r_n\|Ax_n - Ax^*\|^2)
\end{aligned}$$

and hence

$$\begin{aligned}
\|u_n - x^*\|^2 &\leq \|x_n - x^*\|^2 - \|x_n - u_n\|^2 + 2r_n\langle x_n - u_n, Ax_n - Ax^*\rangle \\
&\quad -r_n\|Ax_n - Ax^*\|^2 \\
&\leq \|x_n - x^*\|^2 - \|x_n - u_n\|^2 + 2r_n\|x_n - u_n\| \\
&\quad \times\|Ax_n - Ax^*\|. \qquad (4.2.25)
\end{aligned}$$

Substituting (4.2.25) into (4.2.19), we arrive at

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \alpha_n\|u - x^*\|^2 + (1 - \alpha_n)(\|x_n - x^*\|^2 - \|x_n - u_n\|^2 \\
&\quad +2r_n\|x_n - u_n\|\|Ax_n - Ax^*\|) \\
&= \alpha_n\|u - x^*\|^2 + (1 - \alpha_n)\|x_n - x^*\|^2 - (1 - \alpha_n)\|x_n - u_n\|^2 \\
&\quad +2r_n(1 - \alpha_n)\|x_n - u_n\|\|Ax_n - Ax^*\|.
\end{aligned}$$

It follow that

$$
\begin{aligned}
(1-\alpha_n)\|x_n-u_n\|^2 &\le \alpha_n\|u-x^*\|^2+(1-\alpha_n)\|x_n-x^*\|^2+2r_n(1-\alpha_n)\\
&\quad \times\|x_n-u_n\|\|Ax_n-Ax^*\|-\|x_{n+1}-x^*\|^2\\
&= \alpha_n(\|u-x^*\|^2-\|x_n-x^*\|^2)+(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\\
&\quad \times\|x_n-x_{n+1}\|+2r_n(1-\alpha_n)\|x_n-u_n\|\|Ax_n-Ax^*\|.
\end{aligned}
$$

In view of the conditions (C1) and (C2), we obtain from (4.2.18) and(4.2.20) That

$$\lim_{n\to\infty}\|x_n-u_n\| = 0. \tag{4.2.26}$$

So, we have $\|x_n-u_n\|\to 0$, and $\|u_n-P_{F(T)}u\|\le\|u_n-x_n\|+\|x_n-P_{F(T)}u\|\to 0$ *as* $n\to\infty$. Next, we show that

$$\lim_{n\to\infty}\|y_n-u_n\| = 0. \tag{4.2.27}$$

We have

$$
\begin{aligned}
\|y_n-x^*\|^2 &= \|P_C(I-\lambda_nB)u_n-P_C(I-\lambda_nAB)x^*\|^2\\
&\le \langle(I-\lambda_nB)u_n-(I-\lambda_nB)x^*,y_n-x^*\rangle\\
&= \frac{1}{2}(\|(I-\lambda_nB)u_n-(I-\lambda_nB)x^*\|^2+\|y_n-x^*\|^2-\|(I-\lambda_nB)u_n\\
&\quad -(I-\lambda_nB)x^*\|^2-\|y_n-x^*\|^2)\\
&\le \frac{1}{2}(\|u_n-x^*\|^2+\|y_n-x^*\|^2-\|(u_n-y_n)-\lambda_n(Bu_n-Bx^*)\|^2)\\
&\le \frac{1}{2}(\|u_n-x^*\|^2+\|y_n-x^*\|^2-\|u_n-y_n\|^2+2\lambda_n\langle u_n-y_n,Bu_n-Bx^*\rangle\\
&\quad -\lambda_n^2\|Bu_n-Bx^*\|^2)
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|y_n-x^*\|^2 &\le \|u_n-x^*\|^2-\|u_n-y_n\|+2\lambda_n\langle u_n-y_n,Bu_n-Bx^*\rangle\\
&\le \|u_n-x^*\|^2-\|u_n-y_n\|+2\lambda_n\|u_n-y_n\|\|Bu_n-Bx^*\|\\
&\le \|x_n-x^*\|^2-\|u_n-y_n\|+2\lambda_n\|u_n-y_n\|\\
&\quad \times\|Bu_n-Bx^*\|
\end{aligned} \tag{4.2.28}
$$

Substituting (4.2.28) into (4.2.23), we arrive at

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le \alpha_n\|u-x^*\|^2+(1-\alpha_n)\|y_n-x^*\|^2\\ &\le \alpha_n\|u-x^*\|^2+(1-\alpha_n)(\|x_n-x^*\|^2-\|u_n-y_n\|\\ &\quad +2\lambda_n\|u_n-y_n\|\|Bu_n-Bx^*\|)\\ &\le \alpha_n\|u-x^*\|^2+(1-\alpha_n)\|x_n-x^*\|^2-(1-\alpha_n)\|u_n-y_n\|\\ &\quad +2\lambda_n(1-\alpha_n)\|u_n-y_n\|\|Bu_n-Bx^*\|\end{aligned}$$

from which it follows that

$$\begin{aligned}(1-\alpha_n)\|u_n-y_n\| &\le \alpha_n\|u-x^*\|^2+(1-\alpha_n)\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\\ &\quad +2\lambda_n(1-\alpha_n)\|u_n-y_n\|\|Bu_n-Bx^*\|\\ &\le \alpha_n\|u-x^*\|^2+\alpha_n\|x_n-x^*\|^2+(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\\ &\quad \times\|x_n-x_{n+1}\|+2\lambda_n(1-\alpha_n)\|u_n-y_n\|\|Bu_n-Bx^*\|\end{aligned}$$

In view of the conditions (C1) and (C2), we obtain from (4.2.18) and (4.2.22) that

$$\lim_{n\to\infty}\|y_n-u_n\| = 0.$$

From (4.2.26) ,we obtain that

$$\lim_{n\to\infty}\|x_n-y_n\| = 0. \tag{4.2.29}$$

This implies $y_n \to w$ as $n\to\infty$.

We prove $w\in VI(C,B)$. For the purpose, let T be the maximal monotone mapping defined by Lemma 2.4.6

$$Tv=\begin{cases} Bx+N_Cx, & x\in C,\\ \emptyset, & x\notin C.\end{cases}$$

For any given $(x,y)\in G(T)$, hence $y-Bx\in N_Cx$. Since $y_n\in C$, we see from the definition of N_C that

$$\langle x-y_n, y-Bx\rangle \ge 0. \tag{4.2.30}$$

On the other hand, we have from $y_n = P_C(I - \lambda_n B)u_n$ that

$$\langle x - y_n, y_n - (I - \lambda_n B)u_n\rangle \geq 0$$

and hence

$$\langle x - y_n, \frac{y_n - u_n}{\lambda_n} + Bu_n\rangle \geq 0.$$

From (4.2.30) and the δ_2-inverse monotonicity of B, we see that

$$\begin{aligned}
\langle x - y_n, y\rangle &\geq \langle x - y_n, Bx\rangle \\
&\geq \langle x - y_n, Bx\rangle - \langle x - y_n, Bu_n\rangle - \langle x - y_n, \frac{y_n - u_n}{\lambda_n} + Bu_n\rangle \\
&= \langle x - y_n, Bx - By_n\rangle + \langle x - y_n, By_n - Bu_n\rangle - \langle x - y_n, \frac{y_n - u_n}{\lambda_n}\rangle \\
&\geq \langle x - y_n, By_n - Bu_n\rangle - \langle x - y_n, \frac{y_n - u_n}{\lambda_n}\rangle.
\end{aligned}$$

Since $y_n \to w$ and B is Lipschitz continuous, we obtain that $\langle x - w, y\rangle \geq 0$. Notice that T is maximal monotone, hence $0 \in Tw$. This shows that $w \in VI(C, B)$.

Finally, we show that $w \in \Gamma$. Since $u_n = T_{r_n}(x_n - r_n Ax_n)$, for any $y \in C$,we have

$$F(u_n, y) + G(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0$$

From the monotonicity of F, we have

$$G(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F(y, u_n), \quad \forall y \in C.$$

Hence,

$$G(u_n, y) + \langle Ax_n, y - u_n\rangle + \langle y - u_n, \frac{u_n - x_{n_i}}{r_n}\rangle \geq F(y, u_n), \quad \forall y \in C. \quad (4.2.31)$$

Put $\rho_t = ty + (1-t)w$ for all $t \in (0, 1]$, and $y \in C$. Then, we have $\rho_t \in C$. So, from (4.2.31) we have

$$\begin{aligned}
\langle \rho_t - u_n, A\rho_t\rangle &\geq \langle \rho_t - u_n, A\rho_t\rangle - \langle Ax_n, \rho_t - u_n\rangle + F(\rho_t, u_n) - G(u_n, \rho_t) \\
&\quad -\langle \rho_t - u_n, \frac{u_n - x_n}{r_n}\rangle \\
&= \langle \rho_t - u_n, A\rho_t - Au_n\rangle + \langle \rho_t - u_n, Au_n - Ax_n\rangle + F(\rho_t, u_n) \\
&\quad -G(u_n, \rho_t) - \langle \rho_t - u_n, \frac{u_n - x_n}{r_n}\rangle \qquad (4.2.32)
\end{aligned}$$

From (4.2.26), we have $\|Au_n - Ax_n\| \to 0$ as $n \to \infty$. On the other hand, we get from the monotonicity of A that

$$\langle \rho_t - u_n, A\rho_t - Au_n \rangle \;\geq\; 0$$

It follows from (4.2.32) that

$$\langle \rho_t - u_{n_i}, A\rho_t \rangle \;\geq\; F(\rho_t, u_{n_i}) - G(u_{n_i}, \rho_t), \quad as \;\; i \to \infty \tag{4.2.33}$$

From (F1), (F3), (G1), (G3) and (4.2.33), we also have

$$\begin{aligned} 0 &= F(\rho_t, \rho_t) + G(\rho_t, \rho_t) \\ &\leq tF(\rho_t, y) + (1-t)F(\rho_t, w) + tG(\rho_t, y) + (1-t)G(\rho_t, w) \\ &\leq tF(\rho_t, y) + tG(\rho_t, y) + (1-t)[F(\rho_t, w) - G(w, \rho_t)] \\ &\leq tF(\rho_t, y) + tG(\rho_t, y) + (1-t)\langle \rho_t - w, A\rho_t \rangle \\ &\leq t[F(\rho_t, y) + G(\rho_t, y) + (1-t)\langle y - w, A\rho_t \rangle] \end{aligned}$$

and hence

$$0 \;\leq\; F(\rho_t, y) + G(\rho_t, y) + (1-t)\langle A\rho_t, y - w \rangle \tag{4.2.34}$$

Letting $t \to 0$ in (4.2.34), we have, for each $y \in C$,

$$0 \;\leq\; F(w, y) + G(w, y) + \langle y - w, Aw \rangle$$

This implies that $w \in \Gamma$. Hence we have $w \in \Omega$

Since $w = P_{F(T)}u$, we have

$$\|w - u\| = \inf_{y \in F(T)} \|u - y\| \leq \inf_{z \in \Omega} \|u - z\| \leq \|u - w\| \tag{4.2.35}$$

Hence $\|w - u\| = \inf_{z \in \Omega} \|u - z\|$, that is $w = P_\Omega u$. This implies $x_n \to P_\Omega u$ $\square$