Chapter 1
Introduction

This chapter is organized as follows: Section 1.1 presents elementary concepts, and a history in fixed point theory of nonexpansive mapping, variational inequality problems and mixed equilibrium problems. Section 1.2 also gives a history in fixed point theory of multivalued quasi-nonexpansive mapping.

1.1 Fixed Point Theory, Variational Inequality Problems and Mixed Equilibrium Problems

Fixed point theory plays an important role in nonlinear analysis. This is because many practical problems in applied science, economics, physics and engineering can be reformulated as a problem of finding fixed points of nonlinear mappings.

The study of fixed point theory is concerned with finding conditions on the structure that the set X must be endowed as well as on the properties of the operator $T : X \to X$, in order to obtain results on:

- the existence and the uniqueness of fixed points;
- the structure of fixed point sets;
- the approximation of fixed points.

Let X be a nonempty set and let $T : X \to X$ be a nonlinear mapping. We say that $x \in X$ is a *fixed point* of T if $Tx = x$ and denote by $F(T)$ the fixed points set of T. For any given $x \in X$, we define $T^n x$ inductively by $T^0 x = x$ and $T^{n+1}x = TT^n x$; we call $T^n x$ the *iterate of* x *under* T. The mapping $T^n (n \geq 1)$ is called the n^{th} *iterate* of T. For any $x_0 \in X$, the sequence $\{x_n\}$ given by

$$x_n = Tx_{n-1} = T^n x_0, \quad n = 1, 2, \cdots$$

is called *the sequence of successive approximations with the initial value* x_0. It is known as the **Picard's iteration** starting at x_0.

Iteration procedures are used in nearly every branch of applied mathematics, and convergence proofs and error estimates are very often obtained by an application of Banach fixed point theorem (or more difficult fixed point theorems).

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and let C be a nonempty closed convex subset of H. A self-mapping $f : H \to H$ is a *contraction* on H if there is a constant $\alpha \in [0, 1)$ such that $\|f(x) - f(y)\| \leq \alpha \|x - y\|$ for all $x, y \in H$. A mapping T of H into itself is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in H$.

In 1922, Banach proved the following famous theorem in fixed point theory for a contraction.

Theorem 1.1.1. (**The Banach contraction principle**) *Let (X, d) be a complete metric space and let $T : X \to X$ be a contraction, that is, there exists $\alpha \in [0, 1)$ such that, for all $x, y \in X$,*

$$d(Tx, Ty) \leq \alpha d(x, y).$$

Then T has a unique fixed point. Moreover, for each $x \in X$, the sequence $\{T^n x\}$ converges strongly to this fixed point.

Several authors have studied methods for the iterative approximation of fixed points of nonlinear mappings T. Three classical iteration processes are often used to approximate a fixed point of T. The first one was introduced by Mann [33], in 1953, the iteration process was known as **Mann's iteration** which is defined by: $x_0 \in C$,

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \forall n \geq 0,$$

where $\{\alpha_n\}$ is a real sequence in $[0, 1]$. He proved a weak convergence for a nonexpansive mapping under the control conditions $\{\alpha_n\}$ is chosen such that $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$. Due to the weak convergence of Mann iteration, in 1967, Halpern [18] introduced the modified Mann iteration so-called **Halpern's iteration** which is defined as follows: $x_0 \in C$,

$$x_{n+1} = \alpha_n x_0 + (1 - \alpha_n)Tx_n, \forall n \geq 0,$$

where $\{\alpha_n\}$ is a real sequence in $[0, 1]$. He proved the sequence $\{u_n\}$ converges strongly to a fixed point of T where $\alpha_n := n^{-a}$, $a \in (0, 1)$ in a real Hilbert space. In 1974, Ishikawa [21] introduced a generalization of Mann iteration so-called **Ishikawa's iteration** which is defined by: $x_0 \in C$,

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T(\beta_n x_n + (1 - \beta_n)Tx_n), \forall n \geq 0,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1]$. The Ishikawa's iteration was first used to establish the strong convergence to a fixed point for a Lipschitzian and pseudo-contractive mapping of a compact convex subset of a Hilbert space.

Fixed point iteration procedures are mainly designed for applying in solving concrete nonlinear operator equations, variational equations, variational inequalities etc. Variational inequality theory, a powerful computational algorithm, is one of them which has numerous applications in various disciplines of sciences such as mathematical programming, game theory, mechanics and geometry. For instance, an iteration method of one mapping can be applied for solving solution of equation. Iteration methods of finite family of mappings can be applied for solving solutions of system of equations, and iteration methods of infinite family of mappings can be applied for studying system of infinite equations in quantum mechanics.

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems. A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on H:

$$\min_{x \in F} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle, \tag{1.1.1}$$

where F is the fixed point set of a nonexpansive mapping T on H and b is a given point in H.

The variational inequality problem is to find $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0 \text{ for all } y \in C.$$

The set of solutions of the variational inequality is denoted by $VI(C, A)$ (see Stampacchia [53]).

Recall that a bounded linear operator A on H is *strongly positive*, that is, there is a constant $\overline{\gamma} > 0$ such that $\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2$ for all $x \in H$. A mapping B of C to H is called *inverse strongly monotone* if there exists a positive real number β such that $\langle x - y, Bx - By\rangle \geq \beta\|Bx - By\|^2$ for all $x, y \in C$. For every point $x \in H$, there exists a unique nearest point in C, denoted by P_Cx, such that $\|x - P_Cx\| \leq \|x - y\|$ for all $y \in C$ and P_Cx is called the *metric projection* of H onto C. We know that P_C is a nonexpansive mapping of H onto C. It is also known that P_C satisfies $\langle x - y, P_Cx - P_Cy\rangle \geq \|P_Cx - P_Cy\|^2$ for every $x, y \in H$. Moreover, P_Cx is characterized by the properties: $P_Cx \in C$ and $\langle x - P_Cx, P_Cx - y\rangle \geq 0$ for all $y \in C$. In the context of the variational inequality problem, this implies that

$$u \in VI(C, A) \Leftrightarrow u = P_C(u - \lambda Au), \quad \forall\lambda > 0.$$

Starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n b \qquad n \geq 0. \tag{1.1.2}$$

It is proved by Xu [60] that the sequence $\{x_n\}$ generated by (1.1.2) converges strongly to the unique solution of the minimization problem (1.1.1) provided the sequence $\{\alpha_n\}$ satisfies certain conditions.

On the other hand, Moudafi [37] introduced the viscosity approximation method for nonexpansive mappings. Let f be a contraction on H with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = (1 - \sigma_n)Tx_n + \sigma_n f(x_n) \qquad n \geq 0, \tag{1.1.3}$$

where $\{\sigma_n\}$ is a sequence in $(0, 1)$. It is proved by Moudafi [37] and Xu [63] that under certain appropriate conditions imposed on $\{\sigma_n\}$, the sequence $\{x_n\}$ generated by (1.1.3) strongly converges to the unique solution x^* in C of the variational inequality

$$\langle (I - f)x^*, x - x^*\rangle \geq 0 \qquad x \in C.$$

Recently, Marino and Xu [34] combined the iterative method (1.1.2) with the viscosity approximation method (1.1.3) and considered the following general iteration process:

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n \gamma f(x_n) \qquad n \geq 0 \tag{1.1.4}$$

and proved that if the sequence $\{\alpha_n\}$ satisfies appropriate conditions, the sequence $\{x_n\}$ generated by (1.1.4) converges strongly to the unique solution of the variational inequality

$$\langle (A - \gamma f)x^*, x - x^*\rangle \geq 0 \qquad x \in C$$

which is the optimality condition for the minimization problem

$$\min_{x\in C}\frac{1}{2}\langle Ax,x\rangle - h(x),$$

where h is a potential function for γf (i.e., $h'(x)=\gamma f(x)$ for $x\in H$).

Chen, Zhang and Fan [10] introduced the following iterative process: $x_0\in C$,

$$x_{n+1}=\alpha_n f(x_n)+(1-\alpha_n)TP_C(x_n-\lambda_n Bx_n),\qquad n\geq 0, \tag{1.1.5}$$

where $\{\alpha_n\}\subset(0,1)$ and $\{\lambda_n\}\subset[a,b]$ for some a,b with $0<a<b<2\beta$. They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$ and $\{\lambda_n\}$, the sequence $\{x_n\}$ generated by (1.1.5) converges strongly to a common element of the set of fixed points of nonexpansive mapping and the set of solutions of the variational inequality for an inverse strongly monotone mapping (say $\overline{x}\in C$), which solves the variational inequality

$$\langle (I-f)\overline{x}, x-\overline{x}\rangle\geq 0\qquad \forall x\in F(T)\cap VI(C,B).$$

Klin-eam and Suantai [29] modified the iterative methods (1.1.4) and (1.1.5) by proposing the following general iterative method: $x_0\in C$,

$$x_{n+1}=P_C(\alpha_n\gamma f(x_n)+(I-\alpha_n A)TP_C(x_n-\lambda_n Bx_n)),\qquad n\geq 0, \tag{1.1.6}$$

where P_C is the projection of H onto C, f is a contraction, A is a strongly positive linear bounded operator, B is a β-inverse strongly monotone mapping, $\{\alpha_n\}\subset(0,1)$, and $\{\lambda_n\}\subset[a,b]$ for some a,b with $0<a<b<2\beta$. They noted that when $A=I$ and $\gamma=1$, the iterative scheme (1.1.6) reduced to the iterative scheme (1.1.5).

Let H be a real Hilbert space and D be a nonempty closed convex subset of H. Let $F: D\times D\to\mathbb{R}$ be a bifunction and $\varphi: D\to\mathbb{R}\cup\{+\infty\}$ be a function such that $D\cap\operatorname{dom}\varphi\neq\emptyset$, where $\mathbb{R}$ is the set of real numbers and $\operatorname{dom}\varphi=\{x\in H:\varphi(x)<+\infty\}$.

Flores-Bazán [17] introduced the following mixed equilibrium problem:

$$\text{Find } x\in D \text{ such that } F(x,y)+\varphi(y)\geq\varphi(x),\quad \forall y\in D. \tag{1.1.7}$$

The set of solutions of (1.1.7) is denoted by $MEP(F,\varphi)$.

If $\varphi\equiv 0$, then the mixed equilibrium problem (1.1.7) reduces to the following equilibrium problem:

$$\text{Find } x\in D \text{ such that } F(x,y)\geq 0,\quad \forall y\in D. \tag{1.1.8}$$

The set of solutions of (1.1.8) is denoted by $EP(F)$ (see Combettes and Hirstoaga [15]).

If $F\equiv 0$, then the mixed equilibrium problem (1.1.7) reduces to the following convex minimization problem:

$$\text{Find } x\in D \text{ such that } \varphi(y)\geq\varphi(x),\quad \forall y\in D. \tag{1.1.9}$$

The set of solutions of (1.1.9) is denoted by $CMP(\varphi)$.

The mixed equilibrium problem (MEP) includes several important problems arising in optimization, economics, physics, engineering, transportation, network, Nash equilibrium problems in noncooperative games, and others. Variational inequalities and mathematical programming problems are also viewed as the abstract equilibrium problems (EP) (e.g., [7, 41]). Many authors have proposed several methods to solve the EP and MEP, see, for instance, [7, 41, 38, 45, 22, 32, 44, 31, 8] and the references therein.

Fixed point problems for multivalued mappings are more difficult than those of single valued mappings and play very important role in applied science and economics. Recently, there are many authors have proposed their fixed point methods for finding a fixed point of both multivalued mapping and a family of multivalued mappings. All of those methods have only weak convergence.

In an infinite dimensional Hilbert space, the Mann iteration algorithm have only weak convergence. In 2003, Nakajo and Takahashi [40] introduced the method, called CQ method, to modify Mann's iteration to obtain the strong convergence theorem for nonexpansive mapping in a Hilbert space. The CQ method has been studied extensively by many authors, for instance, [35, 65, 64].

In 2008, Takahashi et al. [58] introduced the following iteration scheme which is usually called the shrinking projection method. Let $\{\alpha_n\}$ be a sequence in $(0,1)$ and $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, define a sequence $\{x_n\}$ of D as follows:

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n T_n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \geq 1 \end{cases}$$

where P_{C_n} is the metric projection of H onto C_n and $\{T_n\}$ is a family of nonexpansive mappings. They proved that the sequence $\{x_n\}$ converges strongly to $z = P_{F(T)}x_0$, where $F(T) = \cap_{n=1}^{\infty} F(T_n)$. The shrinking projection method is now become a hot topics in nonlinear analysis because one can get strong convergence of the generated sequence from its construction.

In 2009, Wangkeeree and Wangkeeree [59] proved a strong convergence theorem of an iterative algorithm based on extragradient method for finding a common element of the set of solutions of a mixed equilibrium problem, the set of common fixed points of a family of infinitely nonexpansive mappings and the set of the variational inequality for a monotone Lipschitz continuous mapping in a Hilbert space.

In 2011, Rodjanadid [46] introduced another iterative method modified from an iterative scheme of Klin-eam and Suantai [29] for finding a common element of the set of solutions of mixed equilibrium problems and the set of common fixed points of countable family of nonexpansive mappings in real Hilbert spaces.

Motivated by above research works, we are interested to construct new hybrid method using fixed point method and shrinking projection method for finding a common element of the set of solutions of mixed equilibrium problem and the set of common fixed points of a countable family of multivalued nonexpansive mappings in Hilbert spaces.

1.2 Fixed Points of Multivalued Mappings

The approximating fixed point sequence has a fundamental role in the study of fixed point theory of nonexpansive mappings. Let X be a Banach space and $T : X \to 2^X$ be a multivalued mapping. An element $p \in X$ is called a *fixed point* of T if $p \in Tp$. The set of fixed points of T is denoted by $F(T)$. Let $CB(X)$ be a family of nonempty closed bounded subsets of X. The *Hausdorff metric* on $CB(X)$ is defined by

$$H(A, B) = \max\left\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\right\}, \quad \forall A, B \in CB(X),$$

where $d(x, B) = \inf\{\|x - y\| : y \in B\}$ is the distance from the point x to the set B.

A multivalued mapping $T : K \to CB(K)$ is said to be:

(1) *L-Lipschitzian* if there exists a constant $L > 0$ such that $H(Tx, Ty) \le L\|x - y\|$ for all $x, y \in K$;

(2) *contraction* if there exists a constant $0 < k < 1$ such that $H(Tx, Ty) \le k\|x - y\|$ for all $x, y \in X$;

(3) *nonexpansive* if $H(Tx, Ty) \le \|x - y\|$ for all $x, y \in K$;

(4) *quasi-nonexpansive* if $F(T) \ne \emptyset$ and $H(Tx, Tp) \le \|x - p\|$ for all $x \in K$ and $p \in F(T)$.

In 1969, Nadler [39] combined the ideas of multivalued mapping and Lipschitz mapping and proved some fixed point theorems for multivalued contraction mappings of the nonempty closed and bounded subsets of a complete metric space.

Markin [36] started to study fixed points of multivalued contractions and nonexpansive mapping by using the Hausdorff metric (see also Assad and Kirk [5]).

Later in 1997, Hu et al. [19] obtained common fixed point of two nonexpansive multivalued mappings satisfying certain contractive condition.

In 2005, Sastry and Babu [47] extended the convergence results from single valued mappings to multivalued mappings by defining Ishikawa and Mann iterates for multivalued mappings with a fixed point. They also gave example which shows that the limit of the sequence of Ishikawa iterates depends on the choice of the fixed point p and the initial choice of x_0 (fixed point q may be different from p).

In 2007, there is paper which generalized results of Sastry and Babu [47] to uniformly convex Banach spaces by Panyanak [42] and proved a strong convergence theorem of Mann iterates for a mapping defined on a noncompact domain and satisfies some conditions. He also obtained strong convergence result of Ishikawa iterates for a mapping defined on a compact domain. Furthermore, he gave an open question which was answered by Song and Wang [52]. Song and Wang [52] shown that strong convergence for Mann and Ishikawa iterates of multivalued nonexpansive mapping T under some appropriate conditions.

In 2009, Shahzad and Zegeye [49] proved strong convergence theorems of quasi-nonexpansive multivalued mapping for the Ishikawa iteration. They also relaxed compactness of the domain of T and constructed an iteration scheme which removes the restriction of T, namely, $Tp = \{p\}$ for any $p \in F(T)$. The results provided an affirmative answer to Panyanak's [42] question in a more general setting.

Then, Abkar and Eslamian [2] generalized and modified the iteration of Shahzad and Zegeye [49] from two step of quasi-nonexpansive multivalued maps to multi-step of finite family of multivalued maps and removed the restriction $Tp = \{p\}$ by used nonexpansiveness of P_{T_i}. They also proved strong convergence theorem of a common fixed point of $\{T_i\}_{i=1}^m$ in a complete CAT(0) space.

On the other hand, Song and Cho [51] modified and improved the proofs of the main results given by Shahzad and Zegeye [49]. They also proved strong convergence theorems of Ishikawa iterative scheme for a multivalued mapping with P_T is quasi-nonexpansive.

In 2010, Khan et al. [26] proved weak and strong convergence theorems of a one-step iterative scheme for two multivalued nonexpansive mappings, say S and T. Although this scheme is simpler, yet it needs the so called Condition (C) : $d(x, y) \leq d(z, y)$ for $y \in Sx$ and $z \in Tx$ in the course of proof of the results.

Very recently, Khan and Yildirim [27] have proved some results for multivalued mappings using an iteration scheme faster than Ishikawa and without the condition $Tp = \{p\}$.

Hussain and Khan [20], in 2003, introduced the best approximation operator P_T where $P_T(x) = \{y \in Tx : \|x - y\| = d(x, Tx)\}$ to find fixed points of *-nonexpansive multivalued mapping and proved strong convergence of its iterates on a closed convex unbounded subset of a Hilbert space.

Recently, Cholamjiak and Suantai [12] introduced two new iterative procedures with errors for two quasi-nonexpansive multivalued maps and proved strong convergence theorems of the proposed iterations in uniformly convex Banach spaces.

They [14] also introduced another new two-step iterative scheme with errors for finding a common fixed point of two quasi-nonexpansive multivalued maps in Banach spaces. The results obtained in [14] are extensions of those of Shahzad and Zegeye [49].

Later, Cholamjiak et al. [11] introduced a modified Mann iteration for a countable family of multivalued mappings by using the best approximation operator to obtain weak and strong convergence theorems in a Banach space and applied the main results to the problem of finding a common fixed point of a countable family of nonexpansive multivalued mappings. In this work, they also gave some examples of multi-valued mappings T such that P_T are nonexpansive.

In 2011, Abbas et al. [1] introduced a new one-step iterative process to approximate common fixed points of two multivalued nonexpansive mappings in a real uniformly convex Banach space and established weak and strong convergence theorems for the proposed process under some basic boundary conditions. Let $S, T : K \to CB(K)$ be two multivalued nonexpansive mappings. They introduced the following iterative scheme:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = a_n x_n + b_n y_n + c_n z_n, \quad n \in \mathbb{N} \end{cases}$$

where $y_n \in Tx_n$ and $z_n \in Sx_n$ such that $\|y_n - p\| \leq d(p, Sx_n)$ and $\|z_n - p\| \leq d(p, Tx_n)$ whenever p is a fixed point of any one of the mappings S and T, and $\{a_n\}, \{b_n\}, \{c_n\}$

are sequences of numbers in $(0,1)$ satisfying $a_n + b_n + c_n = 1$.

Later, Eslamian and Abkar [16] generalized and modified the iteration of Abbas et al. [1] from two mapping to the infinite family mappings $\{T_i : i \in \mathbb{N}\}$ of multivalued mapping as follow: $x_0 \in E$,

$$x_{n+1} = a_{n,0}x_n + a_{n,1}z_{n,1} + a_{n,2}z_{n,2} + \ldots + a_{n,m}z_{n,m}, \quad n \geq 0,$$

where $z_{n,i} \in P_{T_i}(x_n)$ and $\{a_{n,k}\}$ are sequence of numbers in $[0,1]$ such that for every natural number n, $\sum_{k=0}^{m} a_{n,k} = 1$. They proved strong convergence theorem of this iterative scheme to a common fixed point of $\{T_i\}$ such that each P_{T_i} satisfies the condition (C).

The purpose of this thesis are four folds. Firstly, we construct and study iterative methods for finding solutions of variational inequality and common fixed points of a countable family of nonexpansive mappings in a Hilbert space. Secondly, we construct and study iterative methods for approximating a common fixed point of a family of quasi-nonexpansive multivalued mappings in a uniformly convex Banach space. Thirdly, we construct and study iterative methods for approximating a common fixed point of a family of multivalued mappings by using the best approximation operator in a uniformly convex Banach space. Finally, we construct and study hybrid methods for finding solutions of MEP and common fixed points of a countable family of nonexpansive multivalued mappings in a Hilbert space. We also find sufficient conditions for weak and strong convergence theorems of the iterative methods defined in the first, the second and the third purposes.

This thesis is divided into 4 chapters. Chapter 1 is an introduction of this thesis. Chapter 2 is devoted to basic definition, some lemmas and propositions which will be used in this thesis. Chapter 3 is the main results of this thesis and the conclusion is in Chapter 4.

To be more precise, Chapter 3 is organized as follows: in Section 3.1, we prove strong convergence theorems for variational inequality and fixed point problems and apply the main result for finding a common fixed point of countable family of nonexpansive mappings and strictly pseudocontractive mapping and inverse strongly monotone mapping. In Section 3.2, we generalize and modify the iteration of Abbas et al. [1] for construction of several iterations and prove weak and strong convergence theorems for multivalued mappings in a uniformly convex Banach space. We also give some examples of control sequences satisfies control condition in the main result. Finally, in Section 3.3, we prove strong convergence theorems for hybrid method of countable family of multivalued nonexpansive mappings by using shrinking projection method and fixed point method and then apply the main result to equilibrium problem and convex minimization problem. We also give some examples of control sequences satisfies control condition in the main result.