Chapter 1
Introduction

Let C be a nonempty subset of a metric space (X, d). A mapping $T : C \to C$ is said to be *nonexpansive* if

$$d(Tx, Ty) \le d(x, y), \text{ for all } x, y \in C.$$

We say that $x \in X$ is a fixed point of T if

$$Tx = x.$$

We denote the set of all fixed points of T by $F(T) = \{x \in X : x = Tx\}$. The problem of finding a point $x \in X$ such that $Tx = x$ is called a *fixed point problem* and the solution x is called *fixed point (or invariant point)* of the operator T.

Let X be any set and let $T : X \to X$ a self-map. For any given $x \in X$, we define $T^n x$ inductively by $T^0 x = x$ and $T^{n+1} x = T(T^n x)$; we call $T^n x$ the n^{th} *iterate of x under T*.

Let (X, d) be a metric space, $C \subset X$ be a closed subset of X and $T : C \to C$ be a self-map possessing at least one fixed point $p \in F(T)$. For a given $x_0 \in X$ we consider the sequence of iterates $\{x_n\}_{n=0}^{\infty}$ determined by the successive iteration method

$$x_n = Tx_{n-1} = T^n x_0, \qquad n = 1, 2, ... \tag{1.1.1}$$

The sequence $\{x_n\}$ defined by (1.1.1) is known as the *Picard iteration.* The fixed point theory is concerned with finding conditions on the structure that the set X must be endowed as well as on the properties of the operator $T : X \to X$, in order to obtain results on:

1) the existence (and uniqueness) of fixed points;

2) the structure of the fixed point sets;

3) the approximation of fixed points.

The ambient spaces X involved in fixed point theorems cover a variety of spaces: lattice, metric space, normed linear space, generalized metric space, uniform space, linear topological space etc., while the conditions imposed on the operator T are generally metrical or compactness type conditions.

The following theorem is of fundamental importance in the metrical fixed point theory. In 1922, S. Banach proved the famous theorem in fixed point theory for a contraction as follows:

Theorem 1.1.1 (**Contraction mapping principle**). *Let (X,d) be a complete metric space and $T : X \to X$ be a given contraction ($d(Tx,Ty) \le kd(x,y)$ for some $k \in [0,1)$). Then T has a unique fixed point p, and*

$$T^n x \to p \quad \text{as} \quad n \to \infty, \quad \text{for each} \quad x \in X.$$

There are various generalizations of the contraction mapping principle, roughly obtained in two ways:

1) by weakening the contractive properties of the map and, possibly, by simultaneously giving to the space a sufficiently rich structure, in order to compensate the relaxation of the contractiveness assumptions;

2) by extending the structure of the ambient space.

Fixed point theory is the most important tool to solve a problem in many branches of science and a new technology. When the problem in science has transformed into a mathematical model such that equality, inequality, equality system and inequality system.

Many important nonlinear problems of applied mathematics can be described in a unitary manner by the following scheme. For given object f, find another object x satisfying two conditions:

(i) The object x belong to a given class X of objects.

(ii) The object x is in a certain relation R to the object f.

An object x satisfying these conditions will be called the *solution* of the given problem. This problem can be described by

$$\{x \in X : xRf\}. \tag{1.1.2}$$

Example 1.1.2. Find a real solution of the equation $x^5 - x - 1 = 0$. Here $f \equiv f(x) = x^5 - x - 1$, $X = \mathbb{R}$ and the relation R expresses the fact that x and f are related by the given equation.

Example 1.1.3. The initial value problem for a first order ordinary differential equation

$$\begin{cases} y' &= \varphi(t,y), \\ y(t_0) &= y_0 \end{cases}$$

fit the scheme (1.1.2).

Indeed, here we have $f = (\varphi, t_0, y_0)$, $X = C(I)$, where $t_0 \in I \subset \mathbb{R}$, x is the function $y : I \to \mathbb{R}$ and R is given by the previous system of conditions.

In turn, any problem of the form (1.1.2) can be written equivalently as a fixed point problem

$$x = Tx \tag{1.1.3}$$

where $T : C \to C$ is a corresponding operator, that allows us to use constructive fixed point tools in obtaining the desired solution.

Consequently, the main aim of the present section is to illustrate, on some important typical functional equations from applied mathematics, how we can convert them into equivalent fixed point problems. This will, in part, motivate our interest in the study of fixed point iteration procedures.

Efficiently finding roots of nonlinear equations is of major importance and has significant applications in numerical mathematics. In contrast to the case of linear systems of equations, direct methods for solving nonlinear equations are usually available only for a few special cases. Consequently, we need to resort to iterative methods. According to the mathematical importance of this problem, there exists a vast and dense literature related to iterative methods. Basically, for the equation

$$F(x) = 0 \tag{1.1.4}$$

where $F : C \subset \mathbb{R}^n \to \mathbb{R}^n$ is a given operator, we can consider several iterative methods for computing approximate solutions of it.

One of the most used method is to write (1.1.4) equivalently in the form (1.1.3), where T is a certain operator associated to F, in such a way that, by considering a certain fixed point iteration scheme (usually the Picard iteration), we obtain a sequence that converges to a solution of (1.1.4).

The operator T is usually called *iteration function.* There are several methods for constructing iteration functions. If we restrict to real functions of a real single variable, then one of the most used algorithms for obtaining T is the well-known Newton's method, which is based on the iteration function

$$T_x = x - \frac{F(x)}{F'(x)}.$$

Example 1.1.4. Consider the polynomial equation

$$x^5 - x - 1 = 0 \tag{1.1.5}$$

that can be written in the form (1.1.3) in many different ways. Here there are three of them:

(i) $x = x^5 - 1$;

(ii) $x = \sqrt[5]{x+1}$;

(iii) $x = \frac{4x^5+1}{5x^4-1}$.

It is easy to see that (1.1.5) has a unique solution in the interval $[1, \infty)$.
Denote:

$$T_1(x) = x^5 - 1;$$
$$T_2(x) = \sqrt[5]{x+1};$$
$$T_3(x) = \frac{4x^5+1}{5x^4-1}, \quad x \in [1, \infty).$$

Then the Picard iteration associated to T_1 does not converge, whatever the initial approximation $x_0 \in [1, \infty)$, while in the case of T_2 or T_3, it does.

When the contractive conditions are slightly weaker, then the Picard iterations need not converge to a fixed point of the operator T, and some other iteration procedures must be considered.

The following classical iteration methods are often used to approximate a fixed point of a mapping T.

Let $(X, \|.\|)$ be a real normed space and $T : X \to X$ a self-map, $x_0 \in X$ and $\lambda \in [0, 1]$. The sequence $\{x_n\}_{n=0}^{\infty}$ given by

$$x_{n+1} = (1 - \lambda)x_n + \lambda T x_n, \quad n = 0, 1, 2, ... \tag{1.1.6}$$

will be called the *Krasnoselskij iteration procedure* or, simply, *Krasnoselskij iteration.*

In 1953, W. Robert Mann [34] has defined an iteration as follows: Let C be a compact convex subset of a Banach space X and $T : C \to C$ a continuous mapping,

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T x_n, \quad n \geq 1. \tag{1.1.7}$$

where $x_1 \in C$ and $\{\alpha_n\}$ is a sequence in [0,1]. He proved a weak convergence theorem for a nonexpansive mapping under the control condition $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$. For $\alpha_n = \lambda$ (constant), the iteration (1.1.7) reduces to Krasnoselskij iteration.

Due to the weak convergence of Mann iteration, in 1967, Halpern [18] introduced the modified Mann iteration as follows: A sequence $\{x_n\}$ defined by $x_1 \in C$ and

$$x_{n+1} = (1 - \alpha_n)u + \alpha_n T x_n, \quad n \geq 1, \tag{1.1.8}$$

where $u \in C$ is arbitrarily chosen and $\{\alpha_n\}$ is a sequence in [0,1] is called the *Halpern Iteration.* He proved, in a real Hilbert space, that the sequence $\{x_n\}$ converges strongly to a fixed point of T where $\alpha_n := n^{-a}, a \in (0, 1)$.

In 1977, Lions [33] obtained a strong convergence provide that the sequence $\{\alpha_n\}$ satisfies the control conditions $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \frac{\alpha_n - \alpha_{n-1}}{\alpha_n^2} = 0$. However, both Halpern's and Lion's conditions imposed on the real sequence $\{\alpha_n\}$ exclude the result of Halpern from Hilbert spaces to uniformly smooth Banach spaces. The concept of Halpern iteration has been widely used to approximate the fixed points of nonexpansive mappings (see e.g., [2, 11, 25, 56, 57]).

In 1974, Shiro Ishikawa [21] has defined a new iteration which is a generalization of Mann iteration by starting at $x_1 \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T[\beta_n x_n + (1 - \beta_n)T x_n], \quad n \geq 1. \tag{1.1.9}$$

He proved that if X is a Hilbert space and T is a lipschitzian pseudo-contractive mapping then the sequence $\{x_n\}$ converges strongly to a fixed point of T under some suitable conditions.

In the last four decades Mann, Halpern, and Ishikawa schemes have been successfully used by various authors to approximate fixed points of various classes of operators in Banach spaces (see e.g., [35, 25, 19, 47]).

In 2000, Noor [35] introduced a three-step iteration method, which is called the *Noor iteration,* is defined by $x_1 \in C$ and

$$\begin{cases} z_n &= (1-\gamma_n)x_n + \gamma_n T x_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T z_n, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \quad \text{for all} \quad n \in \mathbb{N}, \end{cases} \tag{1.1.10}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in [0,1]. Clearly, Mann and Ishikawa iterations are special cases of Noor iteration. In [35], the author first used Noor iteration to approximate solutions of variational inclusion in a Hilbert space.

In 2005, Kim and Xu [25] generalized Wittmann's result by introducing a modified Halpern iteration in a Banach space as follows. Let C be a closed convex subset of a uniformly smooth Banach space X and $T : C \to C$ be a nonexpansive mapping. For any points $u, x_1 \in C$, the sequence $\{x_n\}$ is defined by

$$x_{n+1} = \beta_n u + (1-\beta_n)T(\alpha_n x_n + (1-\alpha_n)T x_n), \text{ for } n \geq 1, \tag{1.1.11}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$. They proved, under the following control conditions:

$$(D1) \lim_{n\to\infty} \alpha_n = 0, \lim_{n\to\infty} \beta_n = 0,$$

$$(D2) \sum_{n=1}^{\infty} \alpha_n = \infty, \sum_{n=1}^{\infty} \beta_n = \infty \text{ and}$$

$$(D3) \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty,$$

that the sequence $\{x_n\}$ converges strongly to a fixed point of T.

In 2008, L-G. Hu [19] introduced a modified Halpern's iteration as follows: For any $u, x_0 \in C$, the sequence $\{x_n\}$ is defined by

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n T x_n, \quad n \geq 0, \tag{1.1.12}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are three real sequences in (0,1), satisfying $\alpha_n+\beta_n+\gamma_n = 1$. Clearly, the iterative sequence (1.1.12) is a natural generalization of the well-known iterations (1.1.6),(1.1.7),(1.1.8) (see also [50] and [53] for subsequence comments).

In 2009, Y. Song and H. Li [47] introduced new algorithms for finding fixed points of a nonexpansive mapping T generated as follows:

$$x_{n+1} = \lambda_n[\alpha_n u + (1-\alpha_n)x_n] + (1-\lambda_n)T x_n, \tag{1.1.13}$$

and

$$x_{n+1} = \lambda_n x_n + (1-\lambda_n)T[\alpha_n u + (1-\alpha_n)x_n]. \tag{1.1.14}$$

In 2010, Saejung extended the results of Halpern to a CAT(0) space as follows: Suppose $u, x_1 \in C$ are arbitrarily chosen and $\{x_n\}$ is the iteratively generated by

$$x_{n+1} = \alpha_n u \oplus (1-\alpha_n)T x_n, \quad \forall n \geq 1, \tag{1.1.15}$$

where $\{\alpha_n\}$ is a sequence in (0,1) satisfying $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$, and $\sum_{n=1}^{\infty}|\alpha_n - \alpha_{n+1}| < \infty$ or $\lim_{n\to\infty}\frac{\alpha_n}{\alpha_{n+1}} = 1$. Then $\{x_n\}$ converges to $z \in F(T)$ which is the nearest point of $F(T)$ to u.

Base on the result in the literature, we found that there is no any result in a CAT(0) space concerning to the convergence of a modified Halpern iteration for nonexpansive mappings.

The purpose of this thesis is to extend the results of Kim and Xu, Hu, Song and Li to CAT(0) spaces. Precisely, we will prove the following results. Let C be a nonempty closed convex subset of a complete CAT(0) space and $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$.

1) If $\{x_n\}$ is defined by

$$x_{n+1} = \beta_n u \oplus (1-\beta_n)(\alpha_n x_n \oplus (1-\alpha_n)Tx_n)$$

where $\{\alpha_n\}$, $\{\beta_n\}$ satisfy the following conditions:

(A1) $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$,

(A2) $\lim_{n\to\infty}\beta_n = 0$, $\sum_{n=1}^{\infty}\beta_n = \infty$, and $\sum_{n=1}^{\infty}|\beta_{n+1} - \beta_n| < \infty$,

then $\{x_n\}$ converges to a point $z \in F(T)$ which is nearest to u.

2) If $\{x_n\}$ is defined by

$$x_{n+1} = \beta_n x_n \oplus (1-\beta_n)(\alpha_n u \oplus (1-\alpha_n)Tx_n)$$

where $\{\alpha_n\}$, $\{\beta_n\}$ satisfy the following conditions:

(B1) $\lim_{n\to\infty}\alpha_n = 0$;

(B2) $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(B3) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$.

then $\{x_n\}$ converges to a point $z \in F(T)$ which is nearest to u.

3) If $\{x_n\}$ is defined by

$$x_{n+1} = \lambda_n x_n \oplus (1-\lambda_n)T(\alpha_n u \oplus (1-\alpha_n)x_n)$$

where $\{\alpha_n\}$, $\{\lambda_n\}$ satisfy the following conditions:

(C1) $\lim_{n\to\infty}\alpha_n = 0$,

(C2) $\sum_{n=1}^{\infty}\alpha_n = \infty$, and

(C3) $0 < \liminf_{n\to\infty}\lambda_n \leq \limsup_{n\to\infty}\lambda_n < 1$.

then $\{x_n\}$ converges to a point $z \in F(T)$ which is nearest to u.

4) If $\{x_n\}$ is defined by

$$x_{n+1} = \lambda_n(\alpha_n u \oplus (1-\alpha_n)x_n) \oplus (1-\lambda_n)Tx_n$$

where $\{\alpha_n\}$, $\{\lambda_n\}$ satisfy the following conditions:

(D1) $\lim_{n\to\infty} \alpha_n = 0$,

(D2) $\sum_{n=1}^{\infty} \alpha_n = \infty$, and

(D3) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 1$.

then $\{x_n\}$ converges to a point $z \in F(T)$ which is nearest to u.

This thesis is divided into 6 chapters. Chapter 1 is an introduction to the research problems. In Chapter 2, we collect some basic concepts and results which are needed in later chapters. In Chapter 3, we generalize a lemma of Suzuki and apply it to prove the strong convergence of the modified Halpern iterations in Chapter 4. Finally, in Chapter 5 we prove the strong convergence of the modified Noor iterations in CAT(0) spaces and the conclusion is in Chapter 6.