Chapter 2
Basic Concepts and Preliminaries

The purpose of this chapter is to collect notations, terminologies and elementary results used throughout the thesis.

2.1 Basic Definitions

2.1.1 Metric Spaces

In calculus we study functions defined on the real line $\mathbb{R}$. A little reflection shows that in limit processes and many other considerations we use the fact that on $\mathbb{R}$ we have available a distance function, call it d, which associates a *distance* $d(x, y) = |x - y|$ with every pair of points $x, y \in \mathbb{R}$.

Definition 2.1.1. (Metric space, metric). A metric space is a pair (X, d), where X is a set and d is a metric on X (or *distance function on* X), that is a function defined on $X \times X$ such that for all $x, y, z \in X$ we have:

(M1) d is real valued, finite and nonnegative.

(M2) $d(x, y) = 0$ if and only if $x = y$.

(M3) $d(x, y) = d(y, x)$ (Symmetry).

(M4) $d(x, y) \leq d(x, z) + d(z, y)$ (Triangle inequality).

For fixed $x, y \in X$, we call the nonnegative number $d(x, y)$ the *distance* from x to y. Properties (M1) to (M4) are the *axioms of a metric*. The name "triangle inequality" is motivated by elementary geometry as shown in Figure 2.1.

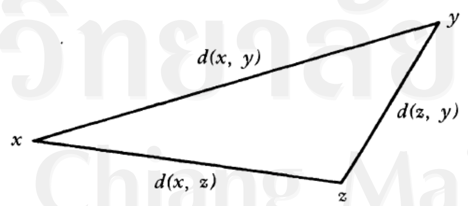


Figure 2.1: Triangle inequality in the plane

From (M4) we obtain by induction the *generalized triangle inequality*

$$d(x_1, x_n) \leq d(x_1, x_2) + d(x_2, x_3) + ... + d(x_{n-1}, x_n). \tag{2.1.1}$$

Instead of (X, d) we may simply write X if there is no danger of confusion.

Example 2.1.2. Real line $\mathbb{R}$. This is the set of all real numbers, taken with the usual metric defined by

$$d(x, y) = |x - y|. \tag{2.1.2}$$

Example 2.1.3. Euclidean plane $\mathbb{R}^2$. The metric space $\mathbb{R}^2$, called the *Euclidean plane*, is obtained if we take the set of ordered pairs of real numbers, written $x = (\xi_1, \xi_2)$, $y = (\eta_1, \eta_2)$, etc., and the *Euclidean metric* defined by

$$d(x, y) = \sqrt{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2}, \tag{2.1.3}$$

see Figure 2.2. Another metric space is obtained if we choose the same set as before but another metric d_1 defined by

$$d_1(x, y) = |\xi_1 - \eta_1| + |\xi_2 - \eta_2|. \tag{2.1.4}$$

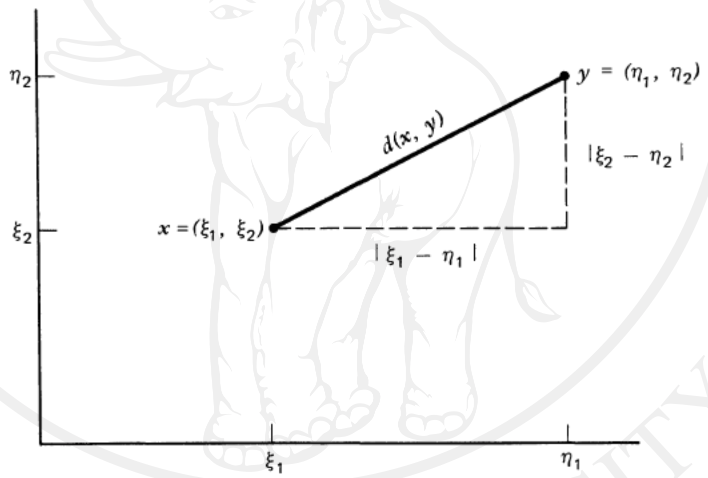


Figure 2.2: Euclidean metric and the metric d_1

Example 2.1.4. Euclidean space $\mathbb{R}^n$, unitary space $\mathbb{C}^n$, complex plane $\mathbb{C}$. The previous examples are special cases of $n-$dimensional Euclidean space $\mathbb{R}^n$. This space is obtained if we take the set of all ordered $n-$tuples of real numbers, written

$$x = (\xi_1, ..., \xi_n), \qquad y = (\eta_1, ..., \eta_n)$$

etc., and the *Euclidean metric* defined by

$$d(x, y) = \sqrt{(\xi_1 - \eta_1)^2 + ... + (\xi_n - \eta_n)^2}. \tag{2.1.5}$$

n-dimensional unitary space $\mathbb{C}^n$ is the space of all ordered n-tuples of *complex* numbers with metric defined by

$$d(x, y) = \sqrt{|\xi_1 - \eta_1|^2 + ... + |\xi_n - \eta_n|^2}. \tag{2.1.6}$$

When $n = 1$ this is the complex plane $\mathbb{C}$ with the usual metric defined by

$$d(x, y) = |x - y|. \tag{2.1.7}$$

Example 2.1.5. Sequence space ℓ^p. Let $1 \leq p < \infty$ be a fixed real number. By definition, each element in the space ℓ^p is a sequence $x = \{\xi_1, \xi_2, ...\}$ of numbers such that $|\xi_1|^p + |\xi_2|^p + ...$ converges; thus

$$\sum_{j=1}^{\infty} |\xi_j|^p < \infty \tag{2.1.8}$$

and the metric is defined by

$$d(x, y) = \left(\sum_{j=1}^{\infty} |\xi_j - \eta_j|^p \right)^{1/p} \tag{2.1.9}$$

where $y = \{\eta_j\}$ and $\sum |\eta_j|^p < \infty$.

Example 2.1.6. Sequence space ℓ^∞. This example and the next one give a first impression of how surprisingly general the concept of a metric space is. As a set X we take the set of all bounded sequences of complex numbers; that is, every element of X is a complex sequence

$$x = \{\xi_1, \xi_2, ...\}$$

such that for all $j = 1, 2, ...$ we have

$$|\xi_j| \leq c_x$$

where c_x is a real number which may depend on x, but does not depend on j. We choose the metric defined by

$$d(x, y) = \sup_{j \in \mathbb{N}} |\xi_j - \eta_j| \tag{2.1.10}$$

where $y = \{\eta_j\} \in X$ and $\mathbb{N} = \{1, 2, ...\}$, and *sup* denotes the *supremum*. The metric space thus obtained is generally denoted by ℓ^∞, it is a *sequence space* because each element of X is a sequence.

Example 2.1.7. Function space $C[a, b]$. As a set X we take the set of all real-valued functions $x, y, ...$ which are functions of an independent real variable t and are defined and continuous on a given closed interval $J = [a, b]$. Choosing the metric defined by

$$d(x, y) = \max_{t \in J} |x(t) - y(t)|, \tag{2.1.11}$$

where *max* denotes the *maximum*, we obtain a metric space which is denoted by $C[a, b]$. The letter C suggests "continuous". This is a function space because every point of $C[a, b]$ is a function.

Example 2.1.8. Space $B(A)$ of bounded functions. By definition, each element $x \in B(A)$ is a function defined and bounded on a given set A, and the metric is defined by

$$d(x, y) = \sup_{t \in A} |x(t) - y(t)|. \tag{2.1.12}$$

We write $B[a, b]$ for $B(A)$ in the case of an interval $A = [a, b] \subset \mathbb{R}$.

We shall now define the concept of completeness of a metric space, which will be basic in our further work. We shall see that completeness does not follow from (M1) to (M4), since there are incomplete (not complete) metric spaces. In other words, completeness is an additional property which a metric space may or may not have.

Definition 2.1.9. A sequence $\{x_n\}$ in a metric space (X, d) is said to be a *Cauchy sequence* if for each $\epsilon > 0$, there exists a positive integer N such that $d(x_n, x_m) < \epsilon$ for all $m, n \geq N$.

The space (X, d) is said to be *complete* if every Cauchy sequence in (X, d) converges (that is, has a limit which is an element of X).

We first consider important types of subsets of a given metric space $X = (X, d)$.

Definition 2.1.10. Given a point $x_0 \in X$ and a real number $r > 0$, we define three types of sets:

(i) $B(x_0, r) = \{x \in X : d(x, x_0) < r\}$ (Open ball)

(ii) $\overline{B}(x_0, r) = \{x \in X : d(x, x_0) \leq r\}$ (Closed ball)

(iii) $S(x_0, r) = \{x \in X : d(x, x_0) = r\}$ (Sphere)

In all three cases, x_0 is called the *center* and r the *radius.*

Definition 2.1.11. A subset G of a metric space X is said to be *open* if it contains a ball about each of its points. A subset C of X is said to be *closed* if its complement in X is open, that is, $X - C$ is open.

The following is an important characterization of closed sets in a metric space.

Theorem 2.1.12. *A subset C of a metric space X is closed if and only if*

$$\{x_n\} \subset C \quad \textit{and} \quad \lim_{n\to\infty} x_n = x \Rightarrow x \in C.$$

The following is a characterization of compactness.

Theorem 2.1.13. *A subset C of a metric space X is compact if and only if any sequence $\{x_n\}$ in C has a subsequence $\{x_{n_k}\}$ which converges to a point in C.*

2.1.2 Banach Spaces and Hilbert Spaces

In this section, we state fundamental theorems relating to Banach spaces and Hilbert spaces. We begin with the concept of vector spaces as follows:

Vector spaces play a role in many branches of mathematics and its applications. In fact, in various practical (and theoretical) problems we have a set X whose elements may be vectors in three-dimensional space, or sequences of numbers, or functions, and these elements can be added and multiplied by constants (numbers) in a natural way, the result being again an element of X. Such concrete situations suggest the concept of a vector space as defined below. The definition will involve a general field $\mathbb{K}$, but in functional analysis, $\mathbb{K}$ will be $\mathbb{R}$ or $\mathbb{C}$. The elements of $\mathbb{K}$ are called scalars; hence in our case they will be real or complex numbers.

Definition 2.1.14. A vector space (or linear space) over a field $\mathbb{K}$ is a nonempty set X of elements $x, y, ...$ (called *vectors*) together with two algebraic operations. These operations are called *vector addition* and *multiplication of vectors by scalars,* that is, by elements of $\mathbb{K}$.

Vector addition

(1) $x + y = y + x$;

(2) $x + (y + z) = (x + y) + z$;

(3) there exists in X a unique element, denoted by 0 and called the *zero element*, such that $x + 0 = x$ for every x;

(4) to each element x in X there corresponds a unique element in X, denoted by $-x$ and called the *negative* of x, such that $x + (-x) = 0$.

Multiplication by scalars

(5) $\alpha(x + y) = \alpha x + \alpha y$;

(6) $(\alpha + \beta)x = \alpha x + \beta x$;

(7) $(\alpha\beta)x = \alpha(\beta x)$;

(8) $1x = x$.

The algebraic system X defined by these operations and axioms is called a *linear space.* Depending on the numbers admitted as scalars (only the real numbers, or all the complex numbers), we distinguish when necessary between *real linear spaces* and *complex linear spaces.* A linear space is often called a *vector space*, and its elements are spoken of as *vectors.*

A vector space X may at the same time be a metric space because a metric d is defined on X. However, if there is no relation between the algebraic structure and the metric, we cannot expect a useful and applicable theory that combines algebraic and metric concepts. To guarantee such a relation between "algebraic" and "geometric" properties of X we define on X a metric d in a special way as follows. We first introduce an auxiliary concept, the norm (definition below), which uses the algebraic operations of vector space. Then we employ the norm to obtain a metric d that is of the desired kind. This idea leads to the concept of a normed space. It turns out that normed spaces are special enough to provide a basis for a rich and interesting theory, but general enough to include many concrete models of practical importance.

Definition 2.1.15. (Banach space). A *normed space* X is a vector space (or linear space) with a norm defined on it. A *Banach space* is a complete normed space (complete in the metric defined by the norm). Here a *norm* on a (real or complex) vector space X is a real-valued function on X whose value at an $x \in X$ is denoted by $\|x\|$ and which has the properties:

(N1) $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$;

(N2) $\|\alpha x\| = |\alpha|\|x\|$;

(N3) $\|x + y\| \leq \|x\| + \|y\|$;

here x and y are arbitrary vectors in X and α is any scalar.

A norm on X defines a metric d on X which is given by

$$d(x, y) = \|x - y\| \tag{2.1.13}$$

and is called the *metric induced by the norm.*

Example 2.1.16. Euclidean space $\mathbb{R}^n$ and unitary space $\mathbb{C}^n$. These spaces were defined in Example 2.1.4. They are Banach spaces with norm defined by

$$\|x\| = \left(\sum_{j=1}^{n} |\xi_j|^2\right)^{1/2} = \sqrt{|\xi_1|^2 + ... + |\xi_n|^2} \tag{2.1.14}$$

Example 2.1.17. Space ℓ^p. This space was defined in Example 2.1.5. It is a Banach space with norm given by

$$\|x\| = \left(\sum_{j=1}^{\infty} |\xi_j|^p\right)^{1/p}. \tag{2.1.15}$$

In fact, this norm induces the metric in Example 2.1.5:

$$d(x, y) = \|x - y\| = \left(\sum_{j=1}^{\infty} |\xi_j - \eta_j|^p\right)^{1/p}.$$

Example 2.1.18. Space ℓ^∞. This space was defined in Example 2.1.6 and is a Banach space since its metric is obtained from the norm defined by

$$\|x\| = \sup_j |\xi_j|$$

Example 2.1.19. Space C[a, b]. This space was defined in Example 2.1.7 and is a Banach space with norm given by

$$\|x\| = \max_{t \in J} |x(t)|,$$

where $J = [a, b]$.

Definition 2.1.20. A subset C of a real vector space X is called *convex* if, for any pair of points x, y in C, the closed segment with the extremities x, y is contained in C, that is, the set $\{\lambda x + (1 - \lambda y) : \lambda \in [0, 1]\}$ is contained in C. A subset C of a real normed space is called *bounded* if there exists $M > 0$ such that $\|x\| \leq M$, for all $x \in C$.

Definition 2.1.21. A Banach space $(X, \|.\|)$ is said to be *strictly convex* if

$$x, y \in S_X \quad \text{with} \quad x \neq y \Rightarrow \left\| \frac{x+y}{2} \right\| < 1.$$

We use S_X to denote the unit sphere $S_X = \{x \in X : \|x\| = 1\}$ on Banach space X.

Definition 2.1.22. A Banach space $(X, \|.\|)$ is called *uniformly convex* if given any $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in X$ with $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| \geq \epsilon$, we have

$$\frac{1}{2}\|x + y\| < 1 - \delta.$$

A function $f : X \to \mathbb{R}$ is said to be linear if $f(\alpha x + y) = \alpha f(x) + f(y)$ for all $x, y \in X$ and $\alpha \in \mathbb{R}$. In addition, if there is $M > 0$ such that $|f(x)| \leq M\|x\|$ for all $x \in X$, we say that f is a bounded linear functional. It is not difficult to see that the class of all bounded linear functionals of X, denoted by X^*, is a Banach space equipped with the norm defined by

$$\|f\| = \sup\{|f(x)| : x \in B_X\} = \sup\{|f(x)| : x \in S_X\},$$

where $B_X = \{x \in X : \|x\| \leq 1\}$ is the unit ball of X.

The most well-known theorem in Banach space theory is the Hahn-Banach theorem: For each $x \in X$ there exists $f \in X^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$.

Definition 2.1.23. A sequence $\{x_n\}$ in a normed space X is said to be *strongly convergent* (or *convergent in the norm*) if there exists a point $x \in X$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$. In this case, we write either $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.

Definition 2.1.24. (Inner product space, Hllbert space). An *inner product space* (or *pre-Hilbert* space) is a vector space X with an inner product defined on X. A *Hilbert space* is a complete inner product space (complete in the metric defined by the inner product; see (2.1.17), below). Here, an *inner product on* X is a mapping of $X \times X$ into the scalar field K of X; that is, with every pair of vectors x and y there is an associated scalar, which is written

$$\langle x, y \rangle$$

and is called the *inner product* of x and y, such that for all vectors x, y, z and scalars α we hvae

(IP1) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

(IP2) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$

(IP3) $\langle x, y \rangle = \overline{\langle y, x \rangle}$

(IP4) $\langle x, x \rangle \geq 0$ *and* $\langle x, x \rangle = 0 \Longleftrightarrow x = 0.$

An inner product on X defines a *norm* on X given by

$$\|x\| = \sqrt{\langle x, x\rangle} \tag{2.1.16}$$

and a *metric* on X given by

$$d(x,y) = \|x-y\| = \sqrt{\langle x-y, x-y\rangle}. \tag{2.1.17}$$

Hence *inner product spaces are normed spaces, and Hilbert spaces are Banach spaces.*

In **(IP3)**, the bar denotes complex conjugation. Consequently, if X is a *real* vector space, we simply have

$$\langle x, y\rangle = \langle y, x\rangle.$$

From **(IP1)** to **(IP3)** we obtain the formula

$$\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle \tag{2.1.18}$$

$$\langle x, \alpha y\rangle = \overline{\alpha}\langle x, y\rangle \tag{2.1.19}$$

$$\langle x, \alpha y + \beta z\rangle = \overline{\alpha}\langle x, y\rangle + \overline{\beta}\langle x, z\rangle \tag{2.1.20}$$

which we shall use quite often. (2.1.18) shows that the inner product is linear in the first factor. Since in (2.1.20) we have complex conjugates $\overline{\alpha}$ and $\overline{\beta}$ on the right, we say that the inner product is *conjugate linear* in the second factor.

Theorem 2.1.25. (Parallelogram Law) *For any inner product space X, the following holds:*

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad \textit{for all } x, y \in X.$$

This name is suggested by elementary geometry, as we see from Figure 2.3.

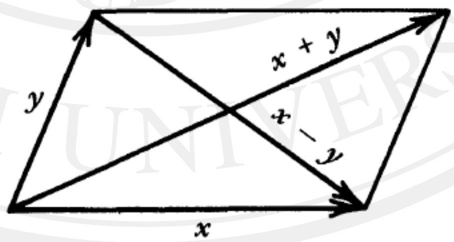


Figure 2.3: Parallelogram with sides x and y in the plane

Example 2.1.26. Euclidean space $\mathbb{R}^n$. The space $\mathbb{R}^n$ is a Hilbert space with inner product defined by

$$\langle x, y\rangle = \xi_1\eta_1 + \ldots + \xi_n\eta_n \tag{2.1.21}$$

where $x = (\xi_1, \ldots, \xi_n)$ and $y = (\eta_1, \ldots, \eta_n)$.

In fact, from (2.1.21) we obtain

$$\|x\| = \langle x, x\rangle^{1/2} = (\xi_1^2 + \ldots + \xi_n^2)^{1/2}$$

and from this the Euclidean metric defined by

$$d(x,y) = \|x-y\| = \langle x-y, x-y\rangle^{1/2} = [(\xi_1-\eta_1)^2 + \ldots + (\xi_n-\eta_n)^2]^{1/2}.$$

Example 2.1.27. Unitary space $\mathbb{C}^n$. The space $\mathbb{C}^n$ is a Hilbert space with inner product given by

$$\langle x, y\rangle = \xi_1\overline{\eta_1} + ... + \xi_n\overline{\eta_n}. \tag{2.1.22}$$

Example 2.1.28. Hilbert sequence space ℓ^2. The space ℓ^2 is a Hilbert space with inner product defined by

$$\langle x, y\rangle = \sum_{j=1}^{\infty} \xi_j\overline{\eta_j}. \tag{2.1.23}$$

2.1.3 CAT(0) Spaces

A metric space X is a CAT(0) space if it is geodesically connected, and if every geodesic triangle in X is at least as "thin" as its comparison triangle in the Euclidean plane. The precise definition is given below. It is well-known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a CAT(0) space. Other examples include Pre-Hilbert spaces, (see [5]), $\mathbb{R}-$trees (see [26]), Euclidean buildings (see [7]), the complex Hilbert ball with a hyperbolic metric (see [17]), and many others. It is necessary to state that the results in CAT(0) spaces can be applied to any CAT(k) space with $k \leq 0$ since any CAT(k) space is a CAT(k') space for every $k' \geq k$ (see [5], p. 165). For a thorough discussion of these spaces and of the fundamental role they play in geometry, we refer the reader to Bridson and Haefliger [5].

Fixed point theory in CAT(0) spaces was first studied by Kirk (see [27] and [28]) as follows:

Theorem 2.1.29. *Let C be a bounded closed convex subset of a complete CAT(0) space X. Suppose $T : C \to C$ is a nonexpansive mapping. Then the fixed point set of T is nonempty, closed, and convex.*

Since then the fixed point theory for single-valued and multivalued mappings in CAT(0) spaces has been rapidly developed and many papers have appeared (see e.g., [12, 10, 32, 13, 44, 14, 20, 31, 41, 23, 24, 1] and the references therein). It is worth mentioning that fixed point theorems in CAT(0) spaces (specially in $\mathbb{R}-$trees) can be applied to graph theory, biology and computer science (see e.g., [4, 43, 26, 15, 30]).

Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ is a mapping $c : [0, l] \to X$ such that

(1) $c(0) = x$,

(2) $c(l) = y$,

(3) $d(c(t_1), c(t_2)) = |t_1 - t_2|$ for any $t_1, t_2 \in [0, l]$.

In particular, c is an isometry. The image of c is called a *geodesic* (or metric) *segment* joining x and y (see Figure 2.4). X is called a (uniquely) *geodesic metric space* if any two points are connected by a (unique) geodesic.

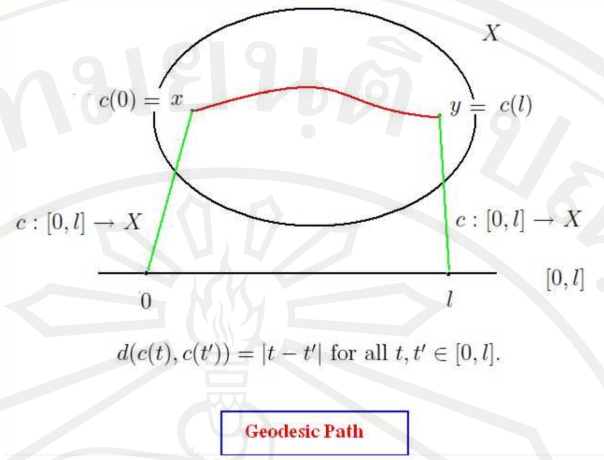


Figure 2.4: Geodesic path joining x and y

Proposition 2.1.30. *(see [28]) Every normed vector space X is a geodesic space. It is a uniquely geodesic if and only if the unit ball in X is strictly convex.*

A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points $x_1, x_2, x_3 \in X$ (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (see Figure 2.5). A *comparison triangle* for the geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\overline{x_i}, \overline{x_j}) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

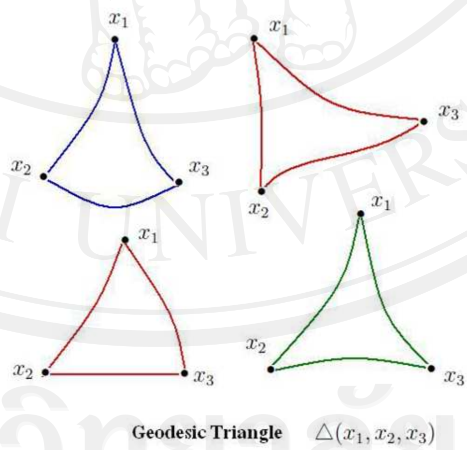


Figure 2.5: Geodesic triangle

A geodesic space is said to be a *CAT(0) space* if all geodesic triangles satisfy the following comparison axiom.

CAT(0): Let $\triangle$ be a geodesic triangle in X and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the *CAT(0) inequality* if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$,

$$d(x, y) \le d_{\mathbb{E}^2}(\overline{x}, \overline{y}),$$

see Figure 2.6.

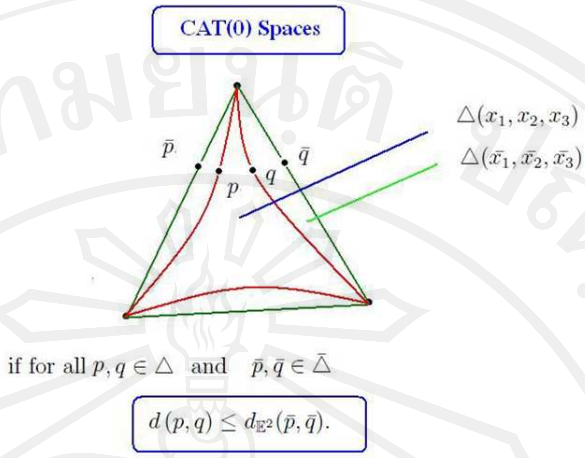


Figure 2.6: Figure of comparison axiom

Similarly, one defines CAT(1) and CAT(-1) spaces by comparing geodesic triangles in X with the comparison triangles in the standard 2-sphere S^2, the set of all 3-tuples (x_1, x_2, x_3) of real numbers such that the sum $x_1^2 + x_2^2 + x_3^2 = 1$ (S^1 is a circle; S^2 is the surface of an ordinary ball of radius one in 3 dimensions), and the hyperbolic plane H^2, (the maximally symmetric,simply connected, 2-dimensional Riemannian manifold with constant sectional curvature -1), respectively. In the case of CAT(1) we only consider geodesic triangles of total perimeter length less than 2π.

Let (X, d) be a CAT(0) space, if $x, y \in X$ and $t \in [0, 1]$ then we use the notation $(1-t)x \oplus ty$ for the point z in $[x, y]$ which satisfied

$$d(x, z) = td(x, y) \quad \text{and} \quad d(y, z) = (1-t)d(x, y) \quad \text{(see Figure 2.7).}$$

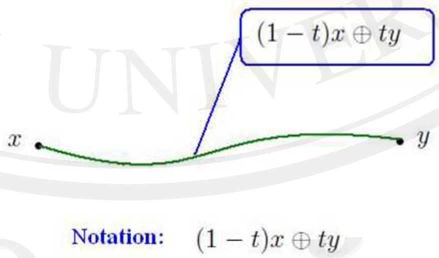


Figure 2.7: A point in line segment

In fact, a geodesic space is a CAT(0) space if and only if it satisfies the (CN) inequality (see [5], p. 163)

Lemma 2.1.31. *A geodesic metric space (X, d) is a CAT(0) space if and only if for $z, x, y \in X$ and if $m[x, y]$ is the midpoint of the segment $[x, y]$ then*

$$d(z, m[x, y])^2 \leq \frac{1}{2}d(z, x)^2 + \frac{1}{2}d(z, y)^2 - \frac{1}{4}d(x, y)^2. \tag{CN}$$

This is the (CN) inequality of Bruhat and Tits [8].

Example 2.1.32. Standard examples of CAT(0) spaces.

- Euclidean space, E^n.
- Hyperbolic space, H^n.
- Hadamard manifold, i.e., complete, simply connected Riemannian manifolds of non-positive sectional curvature.
- Trees.
- products of CAT(0) spaces.
- Gluing CAT(0) spaces in a certain way.
- Euclidean buildings.

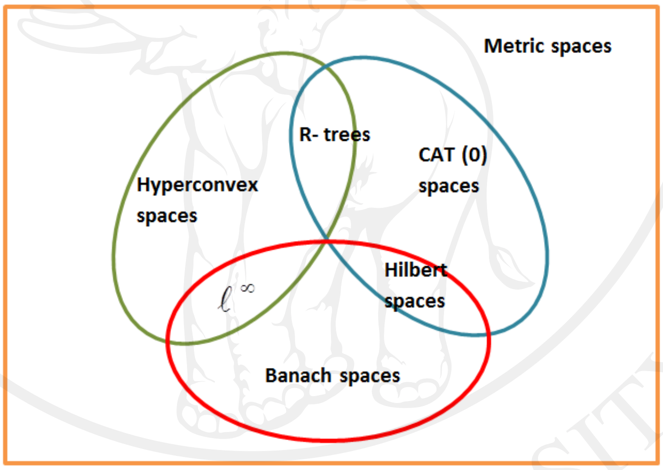


Figure 2.8: The relation of each space

2.1.4 Banach Limits

In this section, we study Banach limits (see also [51]), which is indispensable in the proofs of our main results. Let S be a nonempty set and let $B(S)$ be the Banach space of all bounded real valued functions on S with supremum norm. Let X be a subspace of $B(S)$ and let μ be an element of X^* (the dual space of X). Then, we denote by $\mu(f)$ the value of μ at $f \in X$. If $e(s) = 1$ for every $s \in S$, sometimes $\mu(e)$ will be denoted by $\mu(1)$. When X contains constants, a linear functional μ on X is called a *mean* on X if $\|\mu\| = \mu(1) = 1$.

Theorem 2.1.33. *Let X be a subspace of $B(S)$ containing constants and let μ be a linear functional on X. Then the following conditions are equivalent:*

(1) $\|\mu\| = \mu(1) = 1$, i.e., μ is a mean on X;

(2) the inequalities

$$\inf_{s\in S} f(s) \leq \mu(f) \leq \sup_{s\in S} f(s)$$

hold for each $f \in X$.

Let $S = \mathbb{N}$. Then $B(S) = \ell^\infty$. Let μ be a linear continuous functional on ℓ^∞ and let $x = (x_1, x_2, ...) \in \ell^\infty$. Then, sometimes, we denote by $\mu_n(x_n)$ the value $\mu(x)$.

Theorem 2.1.34. (the existence of Banach limits). *There exists a linear continuous functional μ on ℓ^∞ such that $\|\mu\| = \mu(1) = 1$ and $\mu_n(x_n) = \mu_n(x_{n+1})$ for every $x = (x_1, x_2, ...) \in \ell^\infty$.*

A linear continuous functional on ℓ^∞ such that $\|\mu\| = \mu(1) = 1$ and $\mu_n(x_n) = \mu_n(x_{n+1})$ for each $x = (x_1, x_2, ...) \in \ell^\infty$ is called a *Banach limit.*

Theorem 2.1.35. *Let μ be a Banach limit. Then,*

$$\liminf_{n\to\infty} x_n \leq \mu(x) \leq \limsup_{n\to\infty} x_n$$

for each $x = (x_1, x_2, ...) \in \ell^\infty$. Specially, if $x_n \to a$, then $\mu(x) = a$.

Remark 2.1.36. If μ is a linear functional on ℓ^∞ satisfying

$$\liminf_{n\to\infty} x_n \leq \mu(x) \leq \limsup_{n\to\infty} x_n$$

for each $x = (x_1, x_2, ...) \in \ell^\infty$, then μ is a mean of ℓ^∞.

2.2 Useful Lemmas

We now collect some elementary facts about CAT(0) spaces which will be used in our results.

Lemma 2.2.1. *(see [46], Proposition 2) Let $(a_1, a_2, \ldots) \in \ell^\infty$ be such that $\mu_n(a_n) \leq 0$ for all Banach limits μ and $\limsup_{n\to\infty}(a_{n+1} - a_n) \leq 0$. Then $\limsup_{n\to\infty} a_n \leq 0$.*

Lemma 2.2.2. *(see [41], Lemma 2.1) Let C be a closed convex subset of a complete CAT(0) space X and $T : C \to C$ be a nonexpansive mapping. Let $u \in C$ be fixed. For each $t \in (0,1)$, the mapping $S_t : C \to C$ defined by*

$$S_t x = tu \oplus (1-t)Tx \quad \text{for } x \in C \tag{2.2.1}$$

has a unique fixed point $z_t \in C$, that is,

$$z_t = S_t(z_t) = tu \oplus (1-t)T(z_t). \tag{2.2.2}$$

Lemma 2.2.3. *(see [41], Lemma 2.2) Let C, T be as the preceding lemma. Then $F(T) \neq \emptyset$ if and only if $\{z_t\}$ given by the formula (2.2.2) remains bounded as $t \to 0$. In this case, the following statements hold:*

(i) $\{z_t\}$ converges to the unique fixed point z of T which is nearest to u;

(ii) $d(u,z)^2 \leq \mu_n d(u,x_n)^2$ for all Banach limits μ and all bounded sequences $\{x_n\}$ with $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Lemma 2.2.4. *Let X be a CAT(0) space.*

(i) (see [5], Proposition 2.4) Let C be a closed convex subset of X. Then, for every $x \in X$ there exists a unique point $Px \in C$ such that $d(x,Px) = \inf\{d(x,y) : y \in C\}$. The mapping $P : X \to C$ is called the nearest point (or metric) projection from X onto C.

(ii) (see [13], Lemma 2.4) For each $x,y,z \in X$ and $t \in [0,1]$, one has

$$d\big((1-t)x \oplus ty, z\big) \leq (1-t)d(x,z) + td(y,z) \quad \text{(see Figure 2.9a)}. \tag{2.2.3}$$

(iii) (see [10]) For each $x,y \in X$ and $t,s \in [0,1]$, one has

$$d\big((1-t)x \oplus ty, (1-s)x \oplus sy\big) = |t-s|d(x,y) \quad \text{(see Figure 2.9b)}. \tag{2.2.4}$$

(iv) (see [28], Lemma 3)For each $x,y,z \in X$ and $t \in [0,1]$, one has

$$d\big((1-t)z \oplus tx, (1-t)z \oplus ty\big) \leq td(x,y) \quad \text{(see Figure 2.9c)}. \tag{2.2.5}$$

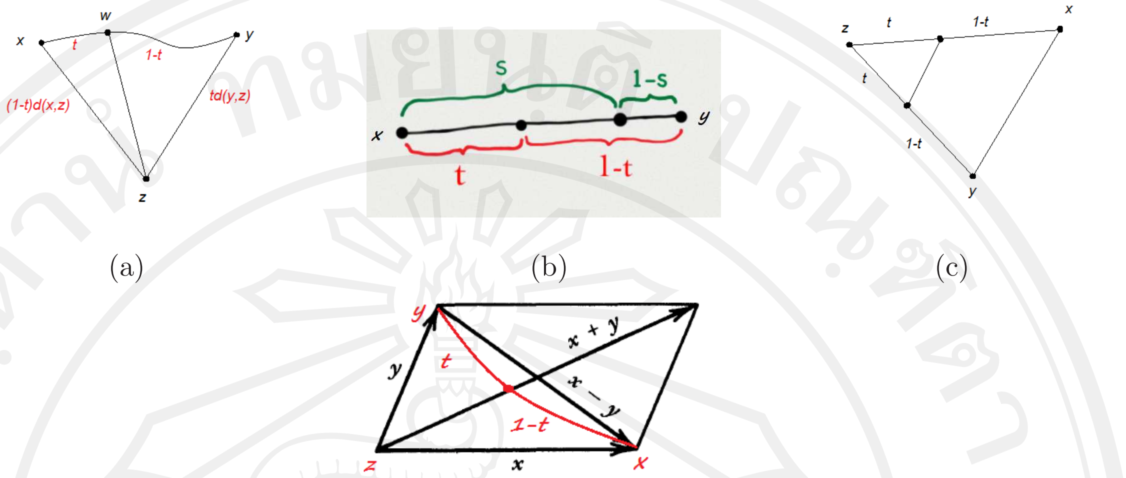
(v) (see [13], Lemma 2.5) For each $x,y,z \in X$ and $t \in [0,1]$, one has

$$d\big((1-t)x \oplus ty, z\big)^2 \leq (1-t)d(x,z)^2 + td(y,z)^2 - t(1-t)d(x,y)^2 \quad \text{(see Figure 2.9d)}. \tag{2.2.6}$$

Lemma 2.2.5. *(see [31, 48]) Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a $CAT(0)$ space X, and let $\{\alpha_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$. Suppose that $x_{n+1} = \alpha_n y_n \oplus (1-\alpha_n)x_n$ for all $n \in \mathbb{N}$ and*

$$\limsup_{n\to\infty} \left(d(y_{n+1}, y_n) - d(x_{n+1}, x_n)\right) \leq 0. \tag{2.2.7}$$

Then $\lim_{n\to\infty} d(x_n, y_n) = 0$.



(a) (b) (c)

(d)

Figure 2.9: Geometry in CAT(0) space

Lemma 2.2.6. *(see [55], Lemma 2.1) Let* $\{\alpha_n\}_{n=0}^{\infty}$ *be a sequence of nonnegative real numbers satisfying the property:* $\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \gamma_n\sigma_n, \quad n \geq 0$, *where* $\{\gamma_n\}_{n=0}^{\infty} \subset (0,1)$ *and* $\{\sigma_n\}_{n=0}^{\infty}$ *such that*

(i) $\lim_{n\to\infty} \gamma_n = 0$ *and* $\sum_{n=0}^{\infty} \gamma_n = \infty$,

(ii) either $\limsup_{n\to\infty} \sigma_n \leq 0$ *or* $\sum_{n=0}^{\infty} |\gamma_n\sigma_n| < \infty$.

Then $\{\alpha_n\}_{n=0}^{\infty}$ *converges to zero.*