Chapter 3
A Generalization of Suzuki's Lemma

Suppose that (X, d) is a metric space which contains a family $\mathfrak{L}$ of metric segments (isometric images of real line segments) such that distinct points $x, y \in X$ lie on exactly one member $S[x, y]$ of $\mathfrak{L}$. Let $\alpha \in [0, 1]$, we use the notation $\alpha x \oplus (1-\alpha)y$ to denote the point of the segment $S[x, y]$ with distance $\alpha d(x, y)$ from y, that is,

$$d(\alpha x \oplus (1-\alpha)y, y) = \alpha d(x, y). \tag{3.1.8}$$

We will say that $(X, d, \mathfrak{L})$ is of hyperbolic type if for each $p, x, y \in X$ and $\alpha \in [0, 1]$,

$$d(\alpha p \oplus (1-\alpha)x, \alpha p \oplus (1-\alpha)y) \leq (1-\alpha)d(x, y). \tag{3.1.9}$$

It is proved in [29] that (3.1.9) implies

$$d(p, \alpha x \oplus (1-\alpha)y) = \alpha d(p, x) + (1-\alpha)d(p, y). \tag{3.1.10}$$

It is well-known that Banach spaces are of hyperbolic type. Notice also that CAT(0) spaces and hyperconvex metric spaces are of hyperbolic type (see [28] and [22]).

In 1983, Goebel and Kirk [16] proved that if $\{z_n\}$ and $\{w_n\}$ are sequences in a metric space of hyperbolic type (X, d) and $\{\alpha_n\} \subset [0, 1]$ which satisfy for all $i, n \in \mathbb{N}$,

(i) $z_{n+1} = \alpha_n w_n \oplus (1-\alpha_n)z_n$,

(ii) $d(w_{n+1}, w_n) \leq d(z_{n+1}, z_n)$,

(iii) $d(w_{i+n}, x_i) \leq a < \infty$,

(iv) $\alpha_n \leq b < 1$,

(v) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

then $\lim_{n\to\infty} d(w_n, z_n) = 0$. It was proved by Suzuki [48] that one obtains the same conclusion if the conditions (i)-(v) are replaced by the conditions (S1)-(S4) as follows:

(S1) $z_{n+1} = \alpha_n w_n \oplus (1-\alpha_n) z_n$,

(S2) $\limsup_{n\to\infty} (d(w_{n+1}, w_n) - d(z_{n+1}, z_n)) \leq 0$,

(S3) $\{z_n\}$ and $\{w_n\}$ are bounded sequences,

(S4) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$.

Both Goebel-Kirk's and Suzuki's results have been used to prove weak and strong convergence theorems for approximating fixed points of various types of mappings. The purpose of this chepter is to generalize Suzuki's result by relaxing the condition (S1), namely, we can define z_{n+1} in terms of w_n and v_n such that $\lim_{n\to\infty} d(z_n, v_n) = 0$. Precisely, we are going to prove the following lemma.

Lemma 3.1.7. *Let $\{z_n\}$, $\{w_n\}$ and $\{v_n\}$ be bounded sequences in a metric space of hyperbolic type (X, d) and let $\{\alpha_n\}$ be a sequence in $[0,1]$ with satisfy for all $n \in \mathbb{N}$,*

(C1) $z_{n+1} = \alpha_n w_n \oplus (1-\alpha_n) v_n$,

(C2) $\lim_{n\to\infty} d(z_n, v_n) = 0$,

(C3) $\limsup_{n\to\infty} \big(d(w_{n+1}, w_n) - d(z_{n+1}, z_n)\big) \leq 0$,

(C4) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$.

Then $\lim_{n\to\infty} d(w_n, z_n) = 0$.

In the proof of lemma 3.1.7 we will need the following technical result.

Lemma 3.1.8. *Let $\{z_n\}$, $\{w_n\}$ and $\{v_n\}$ be sequences in a metric space of hyperbolic type (X, d) and let $\{\alpha_n\}$ be a sequence in $[0,1]$ with $\limsup_{n\to\infty} \alpha_n < 1$. Put*

$$r = \limsup_{n\to\infty} d(w_n, z_n) \ \ \text{or} \ \ r = \liminf_{n\to\infty} d(w_n, z_n).$$

Suppose that $r < \infty$, $z_{n+1} = \alpha_n w_n \oplus (1-\alpha_n) v_n$ for all $n \in \mathbb{N}$, $\lim_{n\to\infty} d(z_n, v_n) = 0$, and

$$\limsup_{n\to\infty} \big(d(w_{n+1}, w_n) - d(z_{n+1}, z_n)\big) \leq 0.$$

Then

$$\liminf_{n\to\infty} |d(w_{n+k}, z_n) - (1 + \alpha_n + \alpha_{n+1} + \ldots + \alpha_{n+k-1}) r| = 0$$

holds for all $k \in \mathbb{N}$.

Proof. (This proof is patterned after the proof of [48, Lemma 1.1]). For each $n \in \mathbb{N}$, let $u_n = \alpha_n w_n \oplus (1-\alpha_n) z_n$, then by (3.1.9) we have

$$d(u_n, z_{n+1}) \leq (1-\alpha_n) d(z_n, v_n) \leq d(z_n, v_n). \tag{3.1.11}$$

This implies

$$\begin{aligned} d(w_{n+1}, z_{n+1}) &\leq d(w_{n+1}, w_n) + d(w_n, u_n) + d(u_n, z_{n+1}) \\ &\leq d(w_{n+1}, w_n) + d(w_n, u_n) + d(z_n, v_n). \end{aligned}$$

Since $d(w_n, u_n) + d(u_n, z_n) = d(w_n, z_n)$, we have

$$\begin{aligned} d(w_{n+1}, z_{n+1}) - d(w_n, z_n) &\le d(w_{n+1}, w_n) + d(w_n, u_n) + d(z_n, v_n) - d(w_n, u_n) - d(u_n, z_n) \\ &= d(w_{n+1}, w_n) + d(z_n, v_n) - d(u_n, z_n). \end{aligned}$$

This fact and (3.1.11) yield,

$$\begin{aligned} d(w_{n+1}, z_{n+1}) - d(w_n, z_n) - d(z_n, v_n) &\le d(w_{n+1}, z_{n+1}) - d(w_n, z_n) - d(u_n, z_{n+1}) \\ &\le d(w_{n+1}, w_n) + d(z_n, v_n) - d(u_n, z_n) - d(u_n, z_{n+1}) \\ &\le d(w_{n+1}, w_n) + d(z_n, v_n) - d(z_{n+1}, z_n) \\ &= d(w_{n+1}, w_n) - d(z_{n+1}, z_n) + d(z_n, v_n). \end{aligned}$$

Since $\lim_n d(z_n, v_n) = 0$, we have

$$\limsup_{n\to\infty} \big(d(w_{n+1}, z_{n+1}) - d(w_n, z_n)\big) \le \limsup_{n\to\infty} \big(d(w_{n+1}, w_n) - d(z_{n+1}, z_n)\big).$$

By using this fact we have, for $j \in \mathbb{N}$,

$$\begin{aligned} \limsup_{n\to\infty} \big(d(w_{n+j}, z_{n+j}) - d(w_n, z_n)\big) &= \limsup_{n\to\infty} \sum_{i=0}^{j-1} \big(d(w_{n+i+1}, z_{n+i+1}) - d(w_{n+i}, z_{n+i})\big) \\ &\le \sum_{i=0}^{j-1} \limsup_{n\to\infty} \big(d(w_{n+i+1}, z_{n+i+1}) - d(w_{n+i}, z_{n+i})\big) \\ &\le \sum_{i=0}^{j-1} \limsup_{n\to\infty} \big(d(w_{n+i+1}, w_{n+i}) - d(z_{n+i+1}, z_{n+i})\big) \\ &\le 0. \end{aligned}$$

Put $a = (1 - \limsup_n \alpha_n)/2$. We note that $0 < a \le \frac{1}{2}$. Fix $k, l \in \mathbb{N}$ and $\varepsilon > 0$. Then there exists $m' \ge l$ such that $a \le 1 - \alpha_n$, $d(z_n, v_n) \le \frac{\varepsilon}{2}$, $d(w_{n+1}, w_n) - d(z_{n+1}, z_n) \le \frac{\varepsilon}{2}$, and $d(w_{n+j}, z_{n+j}) - d(w_n, z_n) \le \frac{\varepsilon}{4}$, for all $n \ge m'$ and $j = 1, 2, ..., k$. In the case of $r = \limsup_n d(w_n, z_n)$, we choose $m \ge m'$ satisfying

$$d(w_{m+k}, z_{m+k}) \ge r - \frac{\varepsilon}{4}$$

and $d(w_n, z_n) \le r + \frac{\varepsilon}{2}$ for all $n \ge m$. We note that

$$d(w_{m+j}, z_{m+j}) \ge d(w_{m+k}, z_{m+k}) - \frac{\varepsilon}{4} \ge r - \frac{\varepsilon}{2}$$

for $j = 0, 1, ..., k-1$. In the case of $r = \liminf_n d(w_n, z_n)$, we choose $m \ge m'$ satisfying

$$d(w_m, z_m) \le r + \frac{\varepsilon}{4}$$

and $d(w_n, z_n) \ge r - \frac{\varepsilon}{2}$ for all $n \ge m$. We note that

$$d(w_{m+j}, z_{m+j}) \le d(w_m, z_m) + \frac{\varepsilon}{4} \le r + \frac{\varepsilon}{2}$$

for $j = 1, 2, ..., k$. In both cases, such m satisfies that $m \geq l$, $a \leq 1 - \alpha_n \leq 1$, $d(z_n, v_n) \leq \frac{\varepsilon}{2}$, $d(w_{n+1}, w_n) - d(z_{n+1}, z_n) \leq \frac{\varepsilon}{2}$ for all $n \geq m$, and

$$r - \frac{\varepsilon}{2} \leq d(w_{m+j}, z_{m+j}) \leq r + \frac{\varepsilon}{2}$$

for $j = 0, 1, ..., k$. We next show that

$$d(w_{m+k}, z_{m+j}) \geq (1 + \alpha_{m+j} + \alpha_{m+j+1} + ... + \alpha_{m+k-1})r - \frac{(k-j)(2k+2)}{a^{k-j}}\varepsilon \qquad (3.1.12)$$

for $j = 0, 1, ..., k-1$. Since

$$\begin{aligned}
r - \frac{\varepsilon}{2} &\leq d(w_{m+k}, z_{m+k}) \\
&= d(w_{m+k}, \alpha_{m+k-1}w_{m+k-1} \oplus (1-\alpha_{m+k-1})v_{m+k-1}) \\
&\leq \alpha_{m+k-1}d(w_{m+k}, w_{m+k-1}) + (1-\alpha_{m+k-1})d(w_{m+k}, v_{m+k-1}) \\
&\leq \alpha_{m+k-1}d(z_{m+k}, z_{m+k-1}) + \frac{\varepsilon}{2} \\
&\quad + (1-\alpha_{m+k-1})d(w_{m+k}, z_{m+k-1}) + (1-\alpha_{m+k-1})d(z_{m+k-1}, v_{m+k-1}) \\
&\leq \alpha_{m+k-1}\big(d(z_{m+k}, u_{m+k-1}) + d(u_{m+k-1}, z_{m+k-1})\big) + \frac{\varepsilon}{2} \\
&\quad + (1-\alpha_{m+k-1})d(w_{m+k}, z_{m+k-1}) + (1-\alpha_{m+k-1})d(z_{m+k-1}, v_{m+k-1}) \\
&\leq \alpha_{m+k-1}(1-\alpha_{m+k-1})d(z_{m+k-1}, v_{m+k-1}) + \alpha^2_{m+k-1}d(w_{m+k-1}, z_{m+k-1}) + \frac{\varepsilon}{2} \\
&\quad + (1-\alpha_{m+k-1})d(w_{m+k}, z_{m+k-1}) + (1-\alpha_{m+k-1})d(z_{m+k-1}, v_{m+k-1}) \\
&\leq \alpha^2_{m+k-1}\left(r + \frac{\varepsilon}{2}\right) + \frac{\varepsilon}{2} + (1-\alpha_{m+k-1})d(w_{m+k}, z_{m+k-1}) \\
&\quad + (1-\alpha^2_{m+k-1})d(z_{m+k-1}, v_{m+k-1}) \\
&\leq \alpha^2_{m+k-1}r + \varepsilon + (1-\alpha_{m+k-1})d(w_{m+k}, z_{m+k-1}) + (1-\alpha^2_{m+k-1})\frac{\varepsilon}{2}
\end{aligned}$$

and $a \leq 1 - \alpha_{m+k-1}$, we have

$$\begin{aligned}
d(w_{m+k}, z_{m+k-1}) &\geq \frac{(1-\alpha^2_{m+k-1})r - \frac{3}{2}\varepsilon - (1-\alpha^2_{m+k-1})\frac{\varepsilon}{2}}{1-\alpha_{m+k-1}} \\
&\geq (1+\alpha_{m+k-1})r - \frac{2k+1}{a}\varepsilon - \frac{\varepsilon}{a} \\
&= (1+\alpha_{m+k-1})r - \frac{2k+2}{a}\varepsilon.
\end{aligned}$$

Hence (3.1.12) holds for $j = k-1$. We assume that (3.1.12) holds for some $j \in \{1, 2, ..., k-1\}$. Then since

$$
\begin{aligned}
&\left(1+\textstyle\sum_{i=j}^{k-1}\alpha_{m+i}\right)r-\tfrac{(k-j)(2k+2)}{a^{k-j}}\varepsilon\\
&\le d(w_{m+k},z_{m+j})\\
&= d(w_{m+k},\alpha_{m+j-1}w_{m+j-1}\oplus(1-\alpha_{m+j-1})v_{m+j-1})\\
&\le \alpha_{m+j-1}d(w_{m+k},w_{m+j-1})+(1-\alpha_{m+j-1})d(w_{m+k},v_{m+j-1})\\
&\le \alpha_{m+j-1}\sum_{i=j-1}^{k-1}d(w_{m+i+1},w_{m+i})+(1-\alpha_{m+j-1})d(w_{m+k},v_{m+j-1})\\
&\le \alpha_{m+j-1}\sum_{i=j-1}^{k-1}\big(d(z_{m+i+1},z_{m+i})+\frac{\varepsilon}{2}\big)+(1-\alpha_{m+j-1})d(w_{m+k},v_{m+j-1})\\
&\le \alpha_{m+j-1}\sum_{i=j-1}^{k-1}d(z_{m+i+1},z_{m+i})+\frac{k\varepsilon}{2}+(1-\alpha_{m+j-1})d(w_{m+k},v_{m+j-1})\\
&\le \alpha_{m+j-1}\sum_{i=j-1}^{k-1}(\alpha_{m+i}d(w_{m+i},z_{m+i})+(1-\alpha_{m+i})d(z_{m+i},v_{m+i}))+\frac{k\varepsilon}{2}\\
&\quad+(1-\alpha_{m+j-1})d(w_{m+k},v_{m+j-1})\\
&\le \alpha_{m+j-1}\sum_{i=j-1}^{k-1}\alpha_{m+i}d(w_{m+i},z_{m+i})+\alpha_{m+j-1}\sum_{i=j-1}^{k-1}(1-\alpha_{m+i})d(z_{m+i},v_{m+i})+\frac{k\varepsilon}{2}\\
&\quad+(1-\alpha_{m+j-1})d(w_{m+k},z_{m+j-1})+(1-\alpha_{m+j-1})d(z_{m+j-1},v_{m+j-1})\\
&\le \alpha_{m+j-1}\sum_{i=j-1}^{k-1}\alpha_{m+i}(r+\frac{\varepsilon}{2})+(k+1)\frac{\varepsilon}{2}+\frac{k\varepsilon}{2}+(1-\alpha_{m+j-1})d(w_{m+k},z_{m+j-1})\\
&\le \alpha_{m+j-1}\sum_{i=j-1}^{k-1}\alpha_{m+i}r+\frac{(3k+1)\varepsilon}{2}+(1-\alpha_{m+j-1})d(w_{m+k},z_{m+j-1}),
\end{aligned}
$$

we obtain

$$
\begin{aligned}
d(w_{m+k},z_{m+j-1}) &\ge \frac{1+\sum_{i=j}^{k-1}\alpha_{m+i}-\alpha_{m+j-1}\sum_{i=j-1}^{k-1}\alpha_{m+i}}{1-\alpha_{m+j-1}}r\\
&\quad-\frac{(k-j)(2k+2)/a^{k-j}+(3k+1)/2}{1-\alpha_{m+j-1}}\varepsilon\\
&\ge \left(1+\sum_{i=j-1}^{k-1}\alpha_{m+i}\right)r-\frac{(k-j+1)(2k+2)}{a^{k-j+1}}\varepsilon.
\end{aligned}
$$

Hence (3.1.12) holds for $j:=j-1$. Therefore (3.1.12) holds for all $j=0,1,...,k-1$. Specially, we have

$$
d(w_{m+k},z_m)\ge(1+\alpha_m+\alpha_{m+1}+...+\alpha_{m+k-1})r-\frac{k(2k+2)}{a^k}\varepsilon. \tag{3.1.13}
$$

On the other hand, we have

$$\begin{aligned}
d(w_{m+k}, z_m) &\leq d(w_{m+k}, z_{m+k}) + \sum_{i=0}^{k-1} d(z_{m+i+1}, z_{m+i}) \\
&\leq d(w_{m+k}, z_{m+k}) + \sum_{i=0}^{k-1} d(z_{m+i+1}, u_{m+i}) + \sum_{i=0}^{k-1} d(u_{m+i}, z_{m+i}) \\
&\leq d(w_{m+k}, z_{m+k}) + \sum_{i=0}^{k-1} d(v_{m+i}, z_{m+i}) + \sum_{i=0}^{k-1} \alpha_{m+i} d(w_{m+i}, z_{m+i}) \\
&\leq r + \frac{\varepsilon}{2} + \frac{k\varepsilon}{2} + \sum_{i=0}^{k-1} \alpha_{m+i}(r + \frac{\varepsilon}{2}) \\
&= \left(1 + \sum_{i=0}^{k-1} \alpha_{m+i}\right) r + \left(\frac{2k+1}{2}\right) \varepsilon.
\end{aligned}$$

This fact and (3.1.13) imply

$$|d(w_{m+k}, z_m) - (1 + \alpha_m + \alpha_{m+1} + ... + \alpha_{m+k-1})r| \leq \frac{k(2k+2)}{a^k}\varepsilon.$$

Since $l \in \mathbb{N}$ and $\varepsilon > 0$ are arbitrary, we obtain the desired result. □

Now, we are ready to prove Lemma 3.1.7

Proof of Lemma 3.1.7. We put $a = \liminf_{n\to\infty} \alpha_n > 0$, $\quad r = \limsup_{n\to\infty} d(w_n, z_n) < \infty$, let $p \in X$ and $M = 2\sup\{d(z_n, p) + d(w_n, p) : n \in \mathbb{N}\}$. We assume that $r > 0$ and fix $k \in \mathbb{N}$ with $(1 + ka)r > M$. By Lemma 3.1.8, we have

$$\liminf_{n\to\infty} |d(w_{n+k}, z_n) - (1 + \alpha_n + \alpha_{n+1} + ... + \alpha_{n+k-1})r| = 0. \tag{3.1.14}$$

Thus, there exists a subsequence $\{n_i\}$ of a sequence $\{n\}$ in $\mathbb{N}$ such that

$$\lim_{i\to\infty} (d(w_{n_i+k}, z_{n_i}) - (1 + \alpha_{n_i} + \alpha_{n_i+1} + ... + \alpha_{n_i+k-1})r) = 0, \tag{3.1.15}$$

the limit of $\{d(w_{n_i+k}, z_{n_i})\}$ exists, and the limits of $\{\alpha_{n_i+j}\}$ exist for all $j \in \{0, 1, ..., k-1\}$. Put $\beta_j = \lim_{i\to\infty} \alpha_{n_i+j}$ for $j \in \{0, 1, ..., k-1\}$. It is obvious that $\beta_j \geq a$ for all $j \in \{0, 1, ..., k-1\}$. We have

$$\begin{aligned}
M &< (1 + ka)r \\
&\leq (1 + \beta_0 + \beta_1 + ... + \beta_{k-1})r \\
&= \lim_{i\to\infty} (1 + \alpha_{n_i} + \alpha_{n_i+1} + ... + \alpha_{n_i+k-1})r \\
&= \lim_{i\to\infty} d(w_{n_i+k}, z_{n_i}) \\
&\leq \limsup_{n\to\infty} d(w_{n+k}, z_n) \\
&\leq \limsup_{n\to\infty} (d(w_{n+k}, p) + d(z_n, p)) \\
&\leq M.
\end{aligned}$$

This is a contradiction. Therefore $r = 0$. □