Chapter 4
Strong Convergence of Modified Halpern Iterations in CAT(0) Spaces

In this chepter, we prove four kinds of strong convergence theorems for the modified Halpern iterations of nonexpansive mappings in CAT(0) spaces. The results we obtain are analogs of the Banach space results of Kim-Xu [25], Hu [19] and Song and Li [47].

The following result is an analog of Theorem 1 of Kim and Xu [25]. They prove the theorem by using the concept of duality mapping while we use the concept of Banach limit. We also observe that the condition $\sum_{n=1}^{\infty}\alpha_n = \infty$ in [25, Theorem 1] is superfluous.

Theorem 4.1.9. *Let C be a nonempty closed convex subset of a complete CAT(0) space X and let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Given a point $u \in C$ and sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $[0,1]$ the following conditions are satisfied:*

(A1) $\lim_n \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$*;*

(A2) $\lim_n \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$ *and* $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Define a sequence $\{x_n\}$ in C by $x_1 = x \in C$ arbitrarily, and

$$x_{n+1} = \beta_n u \oplus (1-\beta_n)\left(\alpha_n x_n \oplus (1-\alpha_n)Tx_n\right), \quad \textit{for all } n \geq 1. \tag{4.1.16}$$

Then $\{x_n\}$ converges to a point $z \in F(T)$ which is nearest u.

Proof. For each $n \geq 1$, we let $y_n := \alpha_n x_n \oplus (1-\alpha_n)Tx_n$. We divide the proof into 3 steps. (i) We will show that $\{x_n\}$, $\{y_n\}$ and $\{Tx_n\}$ are bounded sequences. (ii) We show that $\lim_n d(x_n, Tx_n) = 0$. Finally, we show that (iii) $\{x_n\}$ converges to a fixed point $z \in F(T)$ which is nearest u.

(i): As in the first part of the proof of [25, Theorem 1], we can show that $\{x_n\}$ is bounded and so is $\{y_n\}$ and $\{Tx_n\}$. Notice also that

$$d(y_n, p) \leq d(x_n, p) \text{ for all } p \in F(T).$$

(ii): It suffices to show that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{4.1.17}$$

Indeed, if (4.1.17) holds, we obtain

$$\begin{aligned} d(x_n, Tx_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, y_n) + d(y_n, Tx_n) \\ &= d(x_n, x_{n+1}) + d(\beta_n u \oplus (1-\beta_n)y_n, y_n) + d(\alpha_n x_n \oplus (1-\alpha_n)Tx_n, Tx_n) \\ &\leq d(x_n, x_{n+1}) + \beta_n d(u, y_n) + \alpha_n d(x_n, Tx_n) \to 0 \text{ as } n \to \infty. \end{aligned}$$

By using Lemma 2.2.4, we get that

$$\begin{aligned}
d(x_{n+1},x_n) &= d(\beta_n u\oplus(1-\beta_n)y_n, \beta_{n-1}u\oplus(1-\beta_{n-1})y_{n-1})\\
&\le d(\beta_n u\oplus(1-\beta_n)y_n, \beta_n u\oplus(1-\beta_n)y_{n-1})\\
&\quad + d(\beta_n u\oplus(1-\beta_n)y_{n-1}, \beta_{n-1}u\oplus(1-\beta_{n-1})y_{n-1})\\
&\le (1-\beta_n)d(y_n,y_{n-1}) + |\beta_n-\beta_{n-1}|d(u,y_{n-1})\\
&= (1-\beta_n)d(\alpha_n x_n\oplus(1-\alpha_n)Tx_n, \alpha_{n-1}x_{n-1}\oplus(1-\alpha_{n-1})Tx_{n-1})\\
&\quad + |\beta_n-\beta_{n-1}|d(u,\alpha_{n-1}x_{n-1}\oplus(1-\alpha_{n-1})Tx_{n-1})\\
&\le (1-\beta_n)\Big[d(\alpha_n x_n\oplus(1-\alpha_n)Tx_n, \alpha_n x_{n-1}\oplus(1-\alpha_n)Tx_n)\\
&\quad + d(\alpha_n x_{n-1}\oplus(1-\alpha_n)Tx_n, \alpha_n x_{n-1}\oplus(1-\alpha_n)Tx_{n-1})\\
&\quad + d(\alpha_n x_{n-1}\oplus(1-\alpha_n)Tx_{n-1}, \alpha_{n-1}x_{n-1}\oplus(1-\alpha_{n-1})Tx_{n-1})\Big]\\
&\quad + |\beta_n-\beta_{n-1}|\Big[\alpha_{n-1}d(u,x_{n-1}) + (1-\alpha_{n-1})d(u,Tx_{n-1})\Big]\\
&\le (1-\beta_n)\Big[\alpha_n d(x_n,x_{n-1}) + (1-\alpha_n)d(Tx_n,Tx_{n-1})\\
&\quad + |\alpha_n-\alpha_{n-1}|d(x_{n-1},Tx_{n-1})\Big]\\
&\quad + |\beta_n-\beta_{n-1}|\Big[\alpha_{n-1}d(u,x_{n-1}) + (1-\alpha_{n-1})d(u,Tx_{n-1})\Big]\\
&= (1-\beta_n)d(x_n,x_{n-1}) + (1-\beta_n)|\alpha_n-\alpha_{n-1}|d(x_{n-1},Tx_{n-1})\\
&\quad + |\beta_n-\beta_{n-1}|\alpha_{n-1}d(u,x_{n-1}) + |\beta_n-\beta_{n-1}|(1-\alpha_{n-1})d(u,Tx_{n-1})\\
&\le (1-\beta_n)d(x_n,x_{n-1}) + (1-\beta_n)|\alpha_n-\alpha_{n-1}|d(x_{n-1},Tx_{n-1})\\
&\quad + |\beta_n-\beta_{n-1}|\alpha_{n-1}\Big[d(u,Tx_{n-1}) + d(Tx_{n-1},x_{n-1})\Big]\\
&\quad + |\beta_n-\beta_{n-1}|d(u,Tx_{n-1}) - |\beta_n-\beta_{n-1}|\alpha_{n-1}d(u,Tx_{n-1})\\
&= (1-\beta_n)d(x_n,x_{n-1}) + (1-\beta_n)|\alpha_n-\alpha_{n-1}|d(x_{n-1},Tx_{n-1})\\
&\quad + |\beta_n-\beta_{n-1}|\alpha_{n-1}d(x_{n-1},Tx_{n-1}) + |\beta_n-\beta_{n-1}|d(u,Tx_{n-1}).
\end{aligned}$$

Hence,

$$d(x_{n+1},x_n) \le (1-\beta_n)d(x_n,x_{n-1}) + \gamma\Big(|\alpha_n-\alpha_{n-1}| + 2|\beta_n-\beta_{n-1}|\Big), \tag{4.1.18}$$

where $\gamma > 0$ is a constant such that $\gamma \ge \max\{d(u,Tx_{n-1}), d(x_{n-1},Tx_{n-1})\}$ for all $n\in\mathbb{N}$.

By assumptions, we have

$$\lim_{n\to\infty} \beta_n = 0, \quad \sum_{n=1}^{\infty} \beta_n = \infty, \quad \textit{and} \quad \sum_{n=1}^{\infty} \Big(|\alpha_n - \alpha_{n-1}| + 2|\beta_n - \beta_{n-1}| \Big) < \infty.$$

Hence, Lemma 2.2.6 is applicable to (4.1.18) and we obtain $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$.
(iii): From Lemma 2.2.2, let $z = \lim_{t\to 0} z_t$ where z_t is given by the formula (2.2.2). Then z is the point of $F(T)$ which is nearest u. We observe that

$$\begin{aligned} d^2(x_{n+1}, z) &= d^2(\beta_n u \oplus (1-\beta_n) y_n, z) \\ &\leq \beta_n d^2(u,z) + (1-\beta_n) d^2(y_n, z) - \beta_n(1-\beta_n) d^2(u, y_n) \\ &\leq \beta_n d^2(u,z) + (1-\beta_n) d^2(x_n, z) - \beta_n(1-\beta_n) d^2(u, y_n) \\ &= (1-\beta_n) d^2(x_n, z) + \beta_n \Big[d^2(u,z) - (1-\beta_n) d^2(u, y_n) \Big]. \end{aligned}$$

By Lemma 2.2.3, we have $\mu_n\big(d^2(u,z) - d^2(u,x_n)\big) \leq 0$ for all Banach limit μ. Moreover since $\lim_n d(x_{n+1}, x_n) = 0$,

$$\limsup_{n\to\infty} \big[\big(d^2(u,z) - d^2(u, x_{n+1})\big) - \big(d^2(u,z) - d^2(u,x_n)\big) \big] = 0.$$

It follows from $\lim_n d(y_n, x_n) = 0$ and Lemma 2.2.1 that

$$\limsup_{n\to\infty} \big(d^2(u,z) - (1-\beta_n) d^2(u, y_n) \big) = \limsup_{n\to\infty} \big(d^2(u,z) - d^2(u, x_n) \big) \leq 0.$$

Hence the conclusion follows from Lemma 2.2.6. $\square$

By using the similar technique as in the proof of Theorem 4.1.9, we can obtain a strong convergence theorem which is an analog of [19, Theorem 3.1] (see also [50] and [53] for subsequence comments).

Theorem 4.1.10. *Let C be a nonempty closed and convex subset of a complete CAT(0) space X and let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Given a point $u \in C$ and an initial value $x_1 \in C$. The sequence $\{x_n\}$ is defined iteratively by*

$$x_{n+1} = \beta_n x_n \oplus (1-\beta_n)\big(\alpha_n u \oplus (1-\alpha_n) T x_n\big), \ \ n \geq 1. \tag{4.1.19}$$

Suppose that both $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ satisfying

(B1) $\lim_{n\to\infty} \alpha_n = 0$;

(B2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(B3) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then $\{x_n\}$ converges to a point $z \in F(T)$ which is nearest u.

Proof. Let $y_n := \alpha_n u \oplus (1-\alpha_n)Tx_n$. We divide the proof into 3 steps.

Step 1. We show that $\{x_n\}$, $\{y_n\}$ and $\{Tx_n\}$ are bounded sequences. Let $p \in F(T)$, then we have

$$\begin{aligned}
d(x_{n+1}, p) &= d(\beta_n x_n \oplus (1-\beta_n)(\alpha_n u \oplus (1-\alpha_n)Tx_n), p) \\
&\le \beta_n d(x_n, p) + (1-\beta_n) d(\alpha_n u \oplus (1-\alpha_n)Tx_n, p) \\
&\le \beta_n d(x_n, p) + (1-\beta_n)\alpha_n d(u, p) + (1-\beta_n)(1-\alpha_n) d(Tx_n, p) \\
&\le (\beta_n + (1-\beta_n)(1-\alpha_n)) d(x_n, p) + (1-\beta_n)\alpha_n d(u, p) \\
&= \big[1 - (1-\beta_n)\alpha_n\big] d(x_n, p) + (1-\beta_n)\alpha_n d(u, p) \\
&\le \max\big\{d(x_n, p), d(u, p)\big\}.
\end{aligned}$$

Now, an induction yields

$$d(x_{n+1}, p) \le \max\big\{d(x_1, p), d(u, p)\big\}, n \ge 1.$$

Hence, $\{x_n\}$ is bounded and so are $\{y_n\}$ and $\{Tx_n\}$.

Step 2. We show that $\lim_n d(x_n, Tx_n) = 0$. Since

$$\begin{aligned}
d(y_{n+1}, y_n) &= d(\alpha_{n+1} u \oplus (1-\alpha_{n+1})Tx_{n+1}, \alpha_n u \oplus (1-\alpha_n)Tx_n) \\
&\le \alpha_n d(\alpha_{n+1} u \oplus (1-\alpha_{n+1})Tx_{n+1}, u) \\
&\quad + (1-\alpha_n) d(\alpha_{n+1} u \oplus (1-\alpha_{n+1})Tx_{n+1}, Tx_n) \\
&\le \alpha_n(1-\alpha_{n+1}) d(Tx_{n+1}, u) + (1-\alpha_n)\alpha_{n+1} d(u, Tx_n) \\
&\quad + (1-\alpha_n)(1-\alpha_{n+1}) d(Tx_{n+1}, Tx_n) \\
&\le \alpha_n(1-\alpha_{n+1}) d(Tx_{n+1}, u) + (1-\alpha_n)\alpha_{n+1} d(u, Tx_n) \\
&\quad + (1-\alpha_n)(1-\alpha_{n+1}) d(x_{n+1}, x_n),
\end{aligned}$$

we have

$$\begin{aligned}
d(y_{n+1}, y_n) - d(x_{n+1}, x_n) &\le \alpha_n(1-\alpha_{n+1}) d(Tx_{n+1}, u) + (1-\alpha_n)\alpha_{n+1} d(u, Tx_n) \\
&\quad + \big[\alpha_n\alpha_{n+1} - \alpha_n - \alpha_{n+1}\big] d(x_{n+1}, x_n).
\end{aligned}$$

Since $\{x_n\}$ and $\{Tx_n\}$ are bounded and $\lim_{n\to\infty} \alpha_n = 0$, it follows that

$$\limsup_{n\to\infty} \Big(d(y_{n+1}, y_n) - d(x_{n+1}, x_n)\Big) \le 0.$$

Hence, by Lemma 2.2.5, we get

$$\lim_{n\to\infty} d(x_n, y_n) = 0. \tag{4.1.20}$$

On the other hand,

$$d(y_n, Tx_n) = d(\alpha_n u \oplus (1-\alpha_n)Tx_n, Tx_n) \le \alpha_n d(u, Tx_n) \to 0 \text{ as } n \to \infty. \tag{4.1.21}$$

Using (4.1.20) and (4.1.21), we get

$$d(x_n, Tx_n) \le d(x_n, y_n) + d(y_n, Tx_n) \to 0 \text{ as } n \to \infty.$$

Step 3. We show that $\{x_n\}$ converges to a fixed point of T. Let $z = \lim_{t\to 0} z_t$ where z_t is given by the formula (2.2.2) then $z \in F(T)$. Finally, we show that $\lim_{n\to\infty} x_n = z$.

$$\begin{aligned}
d^2(x_{n+1}, z) &= d^2(\beta_n x_n \oplus (1-\beta_n)y_n, z)\\
&\le \beta_n d^2(x_n, z) + (1-\beta_n)d^2(y_n, z) - \beta_n(1-\beta_n)d^2(x_n, y_n)\\
&\le \beta_n d^2(x_n, z) + (1-\beta_n)d^2(\alpha_n u \oplus (1-\alpha_n)Tx_n, z) - \beta_n(1-\beta_n)d^2(x_n, y_n)\\
&\le (1-\beta_n)\big[\alpha_n d^2(u, z) + (1-\alpha_n)d^2(Tx_n, z) - \alpha_n(1-\alpha_n)d^2(u, Tx_n)\big]\\
&\quad - \beta_n(1-\beta_n)d^2(x_n, y_n) + \beta_n d^2(x_n, z)\\
&\le \Big[\beta_n + (1-\beta_n)(1-\alpha_n)\Big]d^2(x_n, z)\\
&\quad + (1-\beta_n)\alpha_n\Big[d^2(u, z) - (1-\alpha_n)d^2(u, Tx_n)\Big]\\
&= \Big[1 - (1-\beta_n)\alpha_n\Big]d^2(x_n, z) + (1-\beta_n)\alpha_n\big[d^2(u, z) - (1-\alpha_n)d^2(u, Tx_n)\big].
\end{aligned}$$

By Lemma 2.2.3 we have $\mu_n\big(d^2(u, z) - d^2(u, x_n)\big) \le 0$ for all Banach limit μ. Moreover since

$$\begin{aligned}
d(x_{n+1}, x_n) &= d(\beta_n x_n \oplus (1-\beta_n)y_n, x_n)\\
&\le (1-\beta_n)d(y_n, x_n) \to 0 \text{ as } n \to \infty,
\end{aligned}$$

we have

$$\limsup_{n\to\infty} \big(d^2(u, z) + d^2(u, x_{n+1}) - d^2(u, z) - d^2(u, x_n)\big) = 0.$$

It follows from condition (B1), $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and Lemma 2.2.1 that

$$\limsup_{n\to\infty} \big(d^2(u, z) - (1-\alpha_n)d^2(u, Tx_n)\big) = \limsup_{n\to\infty} \big(d^2(u, z) - d^2(u, x_n)\big) \le 0.$$

Hence the conclusion follows by Lemma 2.2.6.

□

The following result is an analog of [47, Theorem 3.1].

Theorem 4.1.11. *Let C be a nonempty closed convex subset of a complete CAT(0) space X and let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Given a point $u \in C$ and sequences $\{\alpha_n\}$ and $\{\lambda_n\}$ in $[0, 1]$, the following conditions are satisfied:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$,

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$, *and*

(C3) $0 < \liminf_{n\to\infty} \lambda_n \le \limsup_{n\to\infty} \lambda_n < 1$.

Define a sequence $\{x_n\}$ in C by $x_1 = x \in C$ arbitrarily, and

$$x_{n+1} = \lambda_n x_n \oplus (1-\lambda_n)T(\alpha_n u \oplus (1-\alpha_n)x_n), \quad \textit{for all } n \ge 1. \tag{4.1.22}$$

Then $\{x_n\}$ converges to a fixed point Pu of T, where P is the nearest point projection from C onto $F(T)$.

Proof. For each $n \geq 1$, we let $y_n = T\left(\alpha_n u \oplus (1-\alpha_n)x_n\right)$. We divide the proof into 3 steps. (i) We show that $\{x_n\}$ and $\{y_n\}$ are bounded sequences. (ii) We show that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. (iii) We show that $\{x_n\}$ converges to a point $z \in F(T)$ which is nearest to u.

(i): Let $p \in F(T)$, then we have

$$\begin{aligned} d(x_{n+1}, p) &= d\left(\lambda_n x_n \oplus (1-\lambda_n)y_n), p\right) \\ &\leq \lambda_n d(x_n, p) + (1-\lambda_n) d\left(T(\alpha_n u \oplus (1-\alpha_n)x_n), p\right) \\ &\leq \lambda_n d(x_n, p) + (1-\lambda_n)\alpha_n d(u, p) + (1-\lambda_n)(1-\alpha_n) d(x_n, p) \\ &\leq (\lambda_n + (1-\lambda_n)(1-\alpha_n)) d(x_n, p) + (1-\lambda_n)\alpha_n d(u, p) \\ &= \left[1 - (1-\lambda_n)\alpha_n\right] d(x_n, p) + (1-\lambda_n)\alpha_n d(u, p) \\ &\leq \max\left\{d(x_n, p), d(u, p)\right\}. \end{aligned}$$

Now, an induction yields

$$d(x_n, p) \leq \max\left\{d(x_1, p), d(u, p)\right\}, n \geq 1.$$

Hence, $\{x_n\}$ is bounded and so is $\{y_n\}$.

(ii): First, we show that $\lim_{n\to\infty} d(x_n, y_n) = 0$. Consider

$$\begin{aligned} d(y_{n+1}, y_n) &= d\big(T\left(\alpha_{n+1} u \oplus (1-\alpha_{n+1})x_{n+1}\right), T\left(\alpha_n u \oplus (1-\alpha_n)x_n\right)\big) \\ &\leq d\big(\alpha_{n+1} u \oplus (1-\alpha_{n+1})x_{n+1}, \alpha_n u \oplus (1-\alpha_n)x_n\big) \\ &\leq \alpha_{n+1} d\big(u, \alpha_n u \oplus (1-\alpha_n)x_n\big) + (1-\alpha_{n+1}) d\big(x_{n+1}, \alpha_n u \oplus (1-\alpha_n)x_n\big) \\ &\leq \alpha_{n+1}(1-\alpha_n) d(u, x_n) + (1-\alpha_{n+1})\alpha_n d(u, x_{n+1}) \\ &\quad + (1-\alpha_{n+1})(1-\alpha_n) d(x_{n+1}, x_n). \end{aligned}$$

This implies

$$d(y_{n+1}, y_n) - d(x_{n+1}, x_n) \leq \alpha_{n+1}(1-\alpha_n) d(u, x_n) + (1-\alpha_{n+1})\alpha_n d(u, x_{n+1}).$$

By the condition (C1) we have

$$\limsup_{n\to\infty} (d(y_{n+1}, y_n) - d(x_{n+1}, x_n)) \leq 0.$$

It follows from Lemma 3.1.7 that $\lim_{n\to\infty} d(x_n, y_n) = 0$. Now,

$$\begin{aligned} d(x_n, Tx_n) &\leq d(x_n, y_n) + d(y_n, Tx_n) \\ &\leq d(x_n, y_n) + d\big(T\big(\alpha_n u \oplus (1-\alpha_n)x_n\big), Tx_n\big) \\ &\leq d(x_n, y_n) + d\big(\alpha_n u \oplus (1-\alpha_n)x_n, x_n\big) \\ &\leq d(x_n, y_n) + \alpha_n d(u, x_n) \to 0 \text{ as } n \to \infty. \end{aligned}$$

(iii): From Lemma 2.2.2, let $z = \lim_{t\to 0} z_t$ where z_t is given by the formula (2.2.2).

Then z is the point of $F(T)$ which is nearest u. By applying Lemma 3.1.7 we have

$$\begin{aligned} d^2(x_{n+1}, z) &= d^2(\lambda_n x_n \oplus (1-\lambda_n)y_n, z) \\ &\le \lambda_n d^2(x_n, z) + (1-\lambda_n)d^2(y_n, z) - \lambda_n(1-\lambda_n)d^2(x_n, y_n) \\ &= \lambda_n d^2(x_n, z) + (1-\lambda_n)d^2(\alpha_n u \oplus (1-\alpha_n)x_n, z) - \lambda_n(1-\lambda_n)d^2(x_n, y_n) \\ &\le \lambda_n d^2(x_n, z) + (1-\lambda_n)\Big[\alpha_n d^2(u, z) + (1-\alpha_n)d^2(x_n, z) - \alpha_n(1-\alpha_n)d^2(u, x_n)\Big] \\ &\le \Big[\lambda_n + (1-\lambda_n)(1-\alpha_n)\Big]d^2(x_n, z) + \alpha_n(1-\lambda_n)\Big[d^2(u, z) - (1-\alpha_n)d^2(u, x_n)\Big] \\ &= \big(1 - (1-\lambda_n)\alpha_n\big)d^2(x_n, z) + (1-\lambda_n)\alpha_n\Big[d^2(u, z) - (1-\alpha_n)d^2(u, x_n)\Big]. \end{aligned}$$

By Lemma 2.2.3, we have $\mu_n\big(d^2(u, z) - d^2(u, x_n)\big) \le 0$ for all Banach limit μ. Moreover since

$$\begin{aligned} d(x_{n+1}, x_n) &= d(\lambda_n x_n \oplus (1-\lambda_n)y_n, x_n) \\ &\le (1-\lambda_n)d(y_n, x_n) \to 0 \text{ as } n \to \infty, \end{aligned}$$

we have

$$\limsup_{n\to\infty} \big[\big(d^2(u, z) - d^2(u, x_{n+1})\big) - \big(d^2(u, z) - d^2(u, x_n)\big)\big] = 0.$$

It follows from condition (C1) and Lemma 2.2.1 that

$$\limsup_{n\to\infty} \big(d^2(u, z) - (1-\alpha_n)d^2(u, x_n)\big) = \limsup_{n\to\infty} \big(d^2(u, z) - d^2(u, x_n)\big) \le 0.$$

Hence the conclusion follows from Lemma 2.2.6. □

Remark 4.1.12. In the proof of Theorem 4.1.11, one may observe that it is not necessary to use Lemma 3.1.7 because Suzuki's original lemma is sufficient. However, in [47], there is a strong convergence theorem for another type of modified Halpern iteration (see [47, Theorem 3.2]). We show that the proof is quite easy when we use Lemma 3.1.7.

Theorem 4.1.13. *Let C be a nonempty closed convex subset of a complete CAT(0) space X and let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Given a point $u \in C$ and sequences $\{\alpha_n\}$ and $\{\lambda_n\}$ in $[0, 1]$, the following conditions are satisfied:*

(D1) $\lim_{n\to\infty} \alpha_n = 0$,

(D2) $\sum_{n=1}^{\infty} \alpha_n = \infty$, *and*

(D3) $0 < \liminf_{n\to\infty} \lambda_n \le \limsup_{n\to\infty} \lambda_n < 1$.

Define a sequence $\{x_n\}$ in C by $x_1 = x \in C$ arbitrarily, and

$$x_{n+1} = \lambda_n(\alpha_n u \oplus (1-\alpha_n)x_n) \oplus (1-\lambda_n)Tx_n, \quad \textit{for all } n \ge 1. \tag{4.1.23}$$

Then $\{x_n\}$ converges to a fixed point Pu of T, where P is the nearest point projection from C onto $F(T)$.

Proof. Using the same technique as in the proof of Theorem 4.1.11, we easily obtain that both $\{x_n\}$ and $\{Tx_n\}$ are bounded. Let $y_n = \alpha_n u \oplus (1-\alpha_n)x_n$, then $x_{n+1} = \lambda_n y_n \oplus (1-\lambda_n)Tx_n$. By the condition (D1) we have

$$d(x_n, y_n) = d(x_n, \alpha_n u \oplus (1-\alpha_n)x_n) \leq \alpha_n d(x_n, u) \to 0 \quad \text{as } n \to \infty. \tag{4.1.24}$$

It follows from the nonexpansiveness of T that

$$\limsup_{n\to\infty} \Big(d\left(Tx_{n+1}, Tx_n\right) - d(x_{n+1}, x_n)\Big) \leq 0.$$

By Lemma 3.1.7 we have

$$\lim_{n\to\infty} d(Tx_n, x_n) = 0. \tag{4.1.25}$$

From (4.1.24) and (4.1.25), we get that

$$\begin{aligned} d(x_{n+1}, x_n) &= d(\lambda_n y_n \oplus (1-\lambda_n)Tx_n, x_n) \\ &\leq \lambda_n d(y_n, x_n) + (1-\lambda_n) d(Tx_n, x_n) \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Let $z = \lim_{t\to 0} z_t$ where z_t is given by (2.2.2). Then z is the point of $\in F(T)$ which is nearest u. Consider

$$\begin{aligned} d^2(x_{n+1}, z) &= d^2(\lambda_n y_n \oplus (1-\lambda_n)Tx_n, z) \\ &\leq \lambda_n d^2(y_n, z) + (1-\lambda_n) d^2(Tx_n, z) - \lambda_n(1-\lambda_n) d^2(y_n, Tx_n) \\ &\leq \lambda_n d^2(\alpha_n u \oplus (1-\alpha_n)x_n, z) + (1-\lambda_n) d^2(Tx_n, z) \\ &\leq \lambda_n \Big(\alpha_n d^2(u, z) + (1-\alpha_n) d^2(x_n, z) - \alpha_n(1-\alpha_n) d^2(u, x_n)\Big) \\ &\quad + (1-\lambda_n) d^2(x_n, z) \\ &\leq \big(\lambda_n(1-\alpha_n) + (1-\lambda_n)\big) d^2(x_n, z) + \lambda_n\alpha_n d^2(u, z) - \lambda_n\alpha_n(1-\alpha_n) d^2(u, x_n) \\ &= \big(1-\lambda_n\alpha_n\big) d^2(x_n, z) + \lambda_n\alpha_n\big(d^2(u, z) - (1-\alpha_n) d^2(u, x_n)\big) \end{aligned}$$

By Lemma 2.2.3, we have $\mu_n\big(d^2(u, z) - d^2(u, x_n)\big) \leq 0$ for all Banach limit μ. Moreover since $d(x_{n+1}, x_n) \to 0$, then

$$\limsup_{n\to\infty} \left[\big(d^2(u, z) - d^2(u, x_{n+1})\big) - \big(d^2(u, z) - d^2(u, x_n)\big)\right] = 0.$$

It follows from condition (D1) and Lemma 2.2.1 that

$$\limsup_{n\to\infty} \big(d^2(u, z) - (1-\alpha_n) d^2(u, x_n)\big) = \limsup_{n\to\infty} \big(d^2(u, z) - d^2(u, x_n)\big) \leq 0.$$

Hence the conclusion follows from Lemma 2.2.6. $\square$