Chapter 5
Strong Convergence of Modified Noor Iterations in CAT(0) Spaces

In 2006, Su and Qin [49] introduced the composite iteration scheme:

$$\begin{cases} w_n &= \delta_n x_n + (1-\delta_n)Tx_n, \\ z_n &= \gamma_n x_n + (1-\gamma_n)Tw_n, \\ y_n &= \beta_n x_n + (1-\beta_n)Tz_n, \\ x_{n+1} &= \alpha_n u + (1-\alpha_n)y_n, \quad \forall n \geq 0 \end{cases} \tag{5.1.26}$$

where $x_0, u \in C$ are an arbitrarily chosen and $\{\alpha_n\}, \{\beta_n\}$ are two sequences in (0,1) and $\{\gamma_n\}, \{\delta_n\}$ are in [0,1]. They proved under certain appropriate assumptions on the sequences $\{\alpha_n\}, \{\beta_n\}$ $\{\gamma_n\}$ and $\{\delta_n\}$ that $\{x_n\}$ converges to a fixed point of T in the framework of a uniformly smooth Banach space.

In this section, we extend Su and Qin's result to a complete CAT(0) space. The following lemma is useful for our results.

From now on, we let C be a nonempty closed and convex subset of a complete CAT(0) space X and let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$.

Lemma 5.1.14. *Given a point $u \in C$, the initial guess $x_0 \in C$ is chosen arbitrarily and given sequences $\{\alpha_n\}$, $\{\beta_n\}$ in $(0,1)$ and $\{\gamma_n\}$, $\{\delta_n\}$ in $[0,1]$. The sequence $\{x_n\}$ is defined iteratively by*

$$\begin{cases} w_n &= \delta_n x_n \oplus (1-\delta_n)Tx_n, \\ z_n &= \gamma_n x_n \oplus (1-\gamma_n)Tw_n, \\ y_n &= \beta_n x_n \oplus (1-\beta_n)Tz_n, \\ x_{n+1} &= \alpha_n u \oplus (1-\alpha_n)y_n, \quad \forall n \geq 0. \end{cases} \tag{5.1.27}$$

Then $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{w_n\}$ are bounded sequences.

Proof. If we take a fixed point p of T, then from (5.1.27) we estimate as follows:

$$\begin{aligned} d(w_n,p) &= d(\delta_n x_n \oplus (1-\delta_n)Tx_n, p) \\ &\leq \delta_n d(x_n,p) + (1-\delta_n)d(Tx_n,p) \\ &\leq \delta_n d(x_n,p) + (1-\delta_n)d(x_n,p) \\ d(w_n,p) &\leq d(x_n,p). \end{aligned} \tag{5.1.28}$$

It follows from (5.1.27) and (5.1.28) that

$$\begin{aligned} d(z_n,p) &= d(\gamma_n x_n \oplus (1-\gamma_n)Tw_n, p) \\ &\leq \gamma_n d(x_n,p) + (1-\gamma_n) d(Tw_n,p) \\ &\leq \gamma_n d(x_n,p) + (1-\gamma_n) d(x_n,p) \\ &\leq d(x_n,p) \end{aligned} \tag{5.1.29}$$

and

$$\begin{aligned} d(y_n,p) &= d(\beta_n x_n \oplus (1-\beta_n)Tz_n, p) \\ &\leq \beta_n d(x_n,p) + (1-\beta_n) d(Tz_n,p) \\ &\leq \beta_n d(x_n,p) + (1-\beta_n) d(z_n,p) \\ &\leq \beta_n d(x_n,p) + (1-\beta_n) d(x_n,p) \\ &\leq d(x_n,p). \end{aligned} \tag{5.1.30}$$

Thus

$$\begin{aligned} d(x_{n+1},p) &= d(\alpha_n u \oplus (1-\alpha_n)y_n, p) \\ &\leq \alpha_n d(u,p) + (1-\alpha_n) d(y_n,p) \\ &\leq \alpha_n d(u,p) + (1-\alpha_n) d(x_n,p). \end{aligned}$$

This implies that

$$d(x_{n+1},p) \leq \max\{d(u,p), d(x_n,p)\}. \tag{5.1.31}$$

By induction, we have

$$d(x_n,p) \leq \max\{d(u,p), d(x_0,p)\}, \quad n \geq 0.$$

This proves the boundedness of the sequence $\{x_n\}$, which implies that the sequences $\{y_n\}, \{z_n\}$ and $\{w_n\}$ are also bounded. □

Lemma 5.1.15. *If C, T, $\{x_n\}$, $\{y_n\}, \{z_n\}$,$\{w_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ satisfy the following conditions:*

(C1) $\sum_{n=0}^{\infty} \alpha_n = \infty, \quad \lim_{n\to\infty} \alpha_n = 0,$

(C2) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \ \sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty,$

$\sum_{n=0}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty, \quad$ *and* $\quad \sum_{n=0}^{\infty} |\delta_{n+1} - \delta_n| < \infty,$

then $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0.$

Proof. By (5.1.27), we have

$$\begin{aligned} x_{n+1} &= \alpha_n u \oplus (1-\alpha_n) y_n, & x_n &= \alpha_{n-1} u \oplus (1-\alpha_{n-1}) y_{n-1} \\ y_n &= \beta_n x_n \oplus (1-\beta_n) T z_n, & y_{n-1} &= \beta_{n-1} x_{n-1} \oplus (1-\beta_{n-1}) T z_{n-1}. \end{aligned}$$

It follows from Lemma 2.2.4 that

$$
\begin{aligned}
d(x_{n+1},x_n) &= d\big(\alpha_n u\oplus(1-\alpha_n)y_n, \alpha_{n-1}u\oplus(1-\alpha_{n-1})y_{n-1}\big)\\
&\le d\big(\alpha_n u\oplus(1-\alpha_n)y_n, \alpha_n u\oplus(1-\alpha_n)y_{n-1}\big)\\
&\quad + d\big(\alpha_n u\oplus(1-\alpha_n)y_{n-1}, \alpha_{n-1}u\oplus(1-\alpha_{n-1})y_{n-1}\big)\\
&\le (1-\alpha_n)d(y_n,y_{n-1}) + |\alpha_n-\alpha_{n-1}|d(u,y_{n-1})\\
&= (1-\alpha_n)d\big(\beta_n x_n\oplus(1-\beta_n)Tz_n, \beta_{n-1}x_{n-1}\oplus(1-\beta_{n-1})Tz_{n-1}\big)\\
&\quad + |\alpha_n-\alpha_{n-1}|d\big(u, \beta_{n-1}x_{n-1}\oplus(1-\beta_{n-1})Tz_{n-1}\big)\\
&\le (1-\alpha_n)\Big[d\big(\beta_n x_n\oplus(1-\beta_n)Tz_n, \beta_n x_{n-1}\oplus(1-\beta_n)Tz_n\big)\\
&\quad + d\big(\beta_n x_{n-1}\oplus(1-\beta_n)Tz_n, \beta_n x_{n-1}\oplus(1-\beta_n)Tz_{n-1}\\
&\quad + d\big(\beta_n x_{n-1}\oplus(1-\beta_n)Tz_{n-1}, \beta_{n-1}x_{n-1}\oplus(1-\beta_{n-1})Tz_{n-1}\big)\Big]\\
&\quad + |\alpha_n-\alpha_{n-1}|\Big[\beta_{n-1}d(u,x_{n-1}) + (1-\beta_{n-1})d(u,Tz_{n-1})\Big]\\
&\le (1-\alpha_n)\Big[\beta_n d(x_n,x_{n-1}) + (1-\beta_n)d(Tz_n,Tz_{n-1})\\
&\quad + |\beta_n-\beta_{n-1}|d(x_{n-1},Tz_{n-1})\Big]\\
&\quad + |\alpha_n-\alpha_{n-1}|\Big[\beta_{n-1}d(u,x_{n-1}) + (1-\beta_{n-1})d(u,Tz_{n-1})\Big]\\
&= (1-\alpha_n)\beta_n d(x_n,x_{n-1}) + (1-\alpha_n)(1-\beta_n)d(Tz_n,Tz_{n-1})\\
&\quad + (1-\alpha_n)|\beta_n-\beta_{n-1}|d(x_{n-1},Tz_{n-1})\\
&\quad + |\alpha_n-\alpha_{n-1}|\beta_{n-1}d(u,x_{n-1}) + |\alpha_n-\alpha_{n-1}|(1-\beta_{n-1})d(u,Tz_{n-1})\\
&\le (1-\alpha_n)\beta_n d(x_n,x_{n-1}) + (1-\alpha_n)(1-\beta_n)d(Tz_n,Tz_{n-1})\\
&\quad + (1-\alpha_n)|\beta_n-\beta_{n-1}|d(x_{n-1},Tz_{n-1})\\
&\quad + |\alpha_n-\alpha_{n-1}|\beta_{n-1}d(u,x_{n-1}) - |\alpha_n-\alpha_{n-1}|\beta_{n-1}d(u,Tz_{n-1})\\
&\quad + |\alpha_n-\alpha_{n-1}|d(u,Tz_{n-1})
\end{aligned}
$$

$$
\begin{aligned}
&\leq (1-\alpha_n)\beta_n d(x_n, x_{n-1}) + (1-\alpha_n)(1-\beta_n) d(Tz_n, Tz_{n-1}) \\
&\quad + (1-\alpha_n)|\beta_n - \beta_{n-1}| d(x_{n-1}, Tz_{n-1}) + |\alpha_n - \alpha_{n-1}| d(u, Tz_{n-1}) \\
&\quad + |\alpha_n - \alpha_{n-1}|\beta_{n-1}\Big[d(u, Tz_{n-1}) + d(Tz_{n-1}, x_{n-1})\Big] \\
&\quad - |\alpha_n - \alpha_{n-1}|\beta_{n-1} d(u, Tz_{n-1}) \\
&\leq (1-\alpha_n)\beta_n d(x_n, x_{n-1}) + (1-\alpha_n)(1-\beta_n) d(Tz_n, Tz_{n-1}) \\
&\quad + (1-\alpha_n)|\beta_n - \beta_{n-1}| d(x_{n-1}, Tz_{n-1}) + |\alpha_n - \alpha_{n-1}| d(u, Tz_{n-1}) \\
&\quad + |\alpha_n - \alpha_{n-1}|\beta_{n-1} d(x_{n-1}, Tz_{n-1}) \\
&\leq (1-\alpha_n)(1-\beta_n) d(Tz_n, Tz_{n-1}) + (1-\alpha_n)\beta_n d(x_n, x_{n-1}) \\
&\quad + |(\beta_n - \beta_{n-1})(1-\alpha_n) + (\alpha_n - \alpha_{n-1})\beta_{n-1}| d(x_{n-1}, Tz_{n-1}) \\
&\quad + |\alpha_n - \alpha_{n-1}| d(u, Tz_{n-1}).
\end{aligned}
$$

This implies that

$$
\begin{aligned}
d(x_{n+1}, x_n) \leq\ & (1-\alpha_n)(1-\beta_n) d(z_n, z_{n-1}) + (1-\alpha_n)\beta_n d(x_n, x_{n-1}) \\
& + |(\beta_n - \beta_{n-1})(1-\alpha_n) + (\alpha_n - \alpha_{n-1})\beta_{n-1}| d(x_{n-1}, Tz_{n-1}) \\
& + |\alpha_n - \alpha_{n-1}| d(u, Tz_{n-1}). \qquad (5.1.32)
\end{aligned}
$$

Again, from (5.1.27), we have

$$
\begin{aligned}
w_n &= \delta_n x_n \oplus (1-\delta_n) T x_n, \\
w_{n-1} &= \delta_{n-1} x_{n-1} \oplus (1-\delta_{n-1}) T x_{n-1}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
d(w_n, w_{n-1}) &= d\Big(\delta_n x_n \oplus (1-\delta_n) T x_n, \delta_{n-1} x_{n-1} \oplus (1-\delta_{n-1}) T x_{n-1}\Big) \\
&\leq d\Big(\delta_n x_n \oplus (1-\delta_n) T x_n, \delta_n x_n \oplus (1-\delta_n) T x_{n-1}\Big) \\
&\quad + d\Big(\delta_n x_n \oplus (1-\delta_n) T x_{n-1}, \delta_n x_{n-1} \oplus (1-\delta_n) T x_{n-1}\Big) \\
&\quad + d\Big(\delta_n x_{n-1} \oplus (1-\delta_n) T x_{n-1}, \delta_{n-1} x_{n-1} \oplus (1-\delta_{n-1}) T x_{n-1}\Big) \\
&\leq (1-\delta_n) d(Tx_n, Tx_{n-1}) + |\delta_n - \delta_{n-1}| d(x_{n-1}, Tx_{n-1}) + \delta_n d(x_n, x_{n-1}) \\
&\leq (1-\delta_n) d(x_n, x_{n-1}) + |\delta_n - \delta_{n-1}| d(x_{n-1}, Tx_{n-1}) + \delta_n d(x_n, x_{n-1}).
\end{aligned}
$$

That is,

$$
d(w_n, w_{n-1}) \leq d(x_n, x_{n-1}) + |\delta_n - \delta_{n-1}| d(x_{n-1}, Tx_{n-1}). \qquad (5.1.33)
$$

Similarly, we have

$$
\begin{aligned}
z_n &= \gamma_n x_n \oplus (1-\gamma_n) T w_n, \\
z_{n-1} &= \gamma_{n-1} x_{n-1} \oplus (1-\gamma_{n-1}) T w_{n-1}.
\end{aligned}
$$

Thus,

$$\begin{aligned} d(z_n, z_{n-1}) &\leq (1-\gamma_n)d(Tw_n, Tw_{n-1}) + \gamma_n d(x_n, x_{n-1}) \\ &\quad + |\gamma_n - \gamma_{n-1}| d(x_{n-1}, Tw_{n-1}) \\ &\leq (1-\gamma_n)d(w_n, w_{n-1}) + \gamma_n d(x_n, x_{n-1}) \\ &\quad + |\gamma_n - \gamma_{n-1}| d(x_{n-1}, Tw_{n-1}). \end{aligned} \tag{5.1.34}$$

By substituting (5.1.33) into (5.1.34) we have

$$\begin{aligned} d(z_n, z_{n-1}) &\leq (1-\gamma_n)\Big[d(x_n, x_{n-1}) + |\delta_n - \delta_{n-1}| d(x_{n-1}, Tx_{n-1})\Big] \\ &\quad + \gamma_n d(x_n, x_{n-1}) + |\gamma_n - \gamma_{n-1}| d(x_{n-1}, Tw_{n-1}) \\ &\leq d(x_n, x_{n-1}) + |\delta_n - \delta_{n-1}| d(x_{n-1}, Tx_{n-1}) \\ &\quad + |\gamma_n - \gamma_{n-1}| d(x_{n-1}, Tw_{n-1}). \end{aligned}$$

It follows that

$$d(z_n, z_{n-1}) \leq d(x_n, x_{n-1}) + \big(|\delta_n - \delta_{n-1}| + |\gamma_n - \gamma_{n-1}|\big) M_1 \tag{5.1.35}$$

where M_1 is an appropriate constant such that

$$M_1 \geq \max\big\{d(x_{n-1}, Tx_{n-1}), d(x_{n-1}, Tw_{n-1})\big\}.$$

By substituting (5.1.35) into (5.1.32), we get that

$$\begin{aligned} d(x_{n+1}, x_n) &\leq (1-\alpha_n)d(x_n, x_{n-1}) + (1-\alpha_n)(1-\beta_n)\Big(|\delta_{n-1} - \delta_n| + |\gamma_{n-1} - \gamma_n|\Big) M_1 \\ &\quad + |(\beta_n - \beta_{n-1})(1-\alpha_n) + (\alpha_n - \alpha_{n-1})\beta_{n-1}| d(x_{n-1}, Tz_{n-1}) \\ &\quad + |\alpha_n - \alpha_{n-1}| d(u, Tz_{n-1}). \end{aligned} \tag{5.1.36}$$

Therefore

$$\begin{aligned} d(x_{n+1}, x_n) &\leq (1-\alpha_n)d(x_n, x_{n-1}) \\ &\quad + M\Big(|\gamma_n - \gamma_{n-1}| + |\beta_n - \beta_{n-1}| + 2|\alpha_n - \alpha_{n-1}| + |\delta_n - \delta_{n-1}|\Big) \end{aligned} \tag{5.1.37}$$

where M is an appropriate constat such that

$$M \geq \max\Big\{d(u, Tz_{n-1}), d(x_{n-1}, Tz_{n-1}), M_1\Big\}$$

for all n. By assumptions (C1) - (C2), we have

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \text{and}$$

$$\sum_{n=1}^{\infty} \Big(|\gamma_n - \gamma_{n-1}| + |\beta_n - \beta_{n-1}| + 2|\alpha_n - \alpha_{n-1}| + |\delta_n - \delta_{n-1}|\Big) < \infty.$$

Now, applying Lemma 2.2.6 to (5.1.37), we obtain

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0.$$

□

Lemma 5.1.16. *If C, T, $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, $\{w_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ satisfy conditions (C1), (C2) and*

C(3): $\beta_n + (1+\beta_n)(1-\gamma_n)(2-\delta_n) \in [0,a)$ *for some* $a \in (0,1)$,

then $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Proof. First, by using condition (C1), we get that

$$\begin{aligned} d(x_{n+1}, y_n) &= d(\alpha_n u \oplus (1-\alpha_n)y_n, y_n) \\ d(x_{n+1}, y_n) &\leq \alpha_n d(u, y_n) \to 0, \quad \text{as} \quad n \to \infty. \end{aligned} \tag{5.1.38}$$

We estimate as follows:

$$\begin{aligned} d(x_n, Tx_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, y_n) + d(y_n, Tz_n) + d(Tz_n, Tx_n) \\ &\leq d(x_n, x_{n+1}) + d(x_{n+1}, y_n) + d(y_n, Tz_n) + d(z_n, x_n) \\ &\leq d(x_n, x_{n+1}) + d(x_{n+1}, y_n) + \beta_n d(x_n, Tz_n) + d(z_n, x_n) \\ &\leq d(x_n, x_{n+1}) + d(x_{n+1}, y_n) + \beta_n d(x_n, Tx_n) \\ &\quad + \beta_n d(Tx_n, Tz_n) + d(z_n, x_n) \\ &\leq d(x_n, x_{n+1}) + d(x_{n+1}, y_n) + \beta_n d(x_n, Tx_n) \\ &\quad + \beta_n d(x_n, z_n) + d(z_n, x_n) \\ \\ &= d(x_n, x_{n+1}) + d(x_{n+1}, y_n) + \beta_n d(x_n, Tx_n) + (1+\beta_n) d(x_n, z_n) \\ &\leq d(x_n, x_{n+1}) + d(x_{n+1}, y_n) + \beta_n d(x_n, Tx_n) \\ &\quad + (1+\beta_n)(1-\gamma_n) d(x_n, Tw_n) \\ \\ &\leq d(x_n, x_{n+1}) + d(x_{n+1}, y_n) + \beta_n d(x_n, Tx_n) \\ &\quad + (1+\beta_n)(1-\gamma_n)\big[d(x_n, Tx_n) + d(Tx_n, Tw_n)\big] \\ &\leq d(x_n, x_{n+1}) + d(x_{n+1}, y_n) + \beta_n d(x_n, Tx_n) \\ &\quad + (1+\beta_n)(1-\gamma_n) d(x_n, Tx_n) + (1+\beta_n)(1-\gamma_n) d(x_n, w_n) \\ \\ &\leq d(x_n, x_{n+1}) + d(x_{n+1}, y_n) + \beta_n d(x_n, Tx_n) \\ &\quad + (1+\beta_n)(1-\gamma_n) d(x_n, Tx_n) \\ &\quad + (1+\beta_n)(1-\gamma_n)(1-\delta_n) d(x_n, Tx_n). \end{aligned}$$

It follows that

$$\begin{aligned} &\Big\{1 - \big[\beta_n + (1+\beta_n)(1-\gamma_n) + (1+\beta_n)(1-\gamma_n)(1-\delta_n)\big]\Big\} d(x_n, Tx_n) \\ &\qquad \leq d(x_n, x_{n+1}) + d(x_{n+1}, y_n). \end{aligned}$$

That is

$$\Big\{1 - \big[\beta_n + (1+\beta_n)(1-\gamma_n)(2-\delta_n)\big]\Big\} d(x_n, Tx_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, y_n).$$

Hence, from condition (C3), (5.1.38) and Lemma 5.1.15 we obtain

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

□

By combining the ideas and the techniques in [41] and [49], we can obtain the strong convergence theorem of the modified Noor iterative scheme (5.1.27) in the CAT(0) space.

Theorem 5.1.17. *Let C be a nonempty closed and convex subset of a complete CAT(0) space X and let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Given a point $u, x_0 \in C$ are arbitrarily chosen and given sequences $\{\alpha_n\}$, $\{\beta_n\}$ in $(0,1)$ and $\{\gamma_n\}$, $\{\delta_n\}$ in $[0,1]$, the following conditions are satisfied:*

(C1) $\sum_{n=0}^{\infty} \alpha_n = \infty, \quad \lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \ \sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty,$

$\sum_{n=0}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty,$ *and* $\sum_{n=0}^{\infty} |\delta_{n+1} - \delta_n| < \infty$;

(C3) $\beta_n + (1+\beta_n)(1-\gamma_n)(2-\delta_n) \in [0, a)$ *for some* $a \in (0,1)$.

The sequence $\{x_n\}$ is defined iteratively by (5.1.27). *Then $\{x_n\}$ converges to a fixed point $z \in F(T)$ which is nearest to u.*

Proof. From Lemma 5.1.14, we have $\{x_n\}, \{y_n\}, \{z_n\}$ and $\{w_n\}$ are bounded sequences, and it follows from Lemma 5.1.16 that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Next, we prove that $\{x_n\}$ converges to a fixed point $z \in F(T)$ which is nearest to u. From Lemma 2.2.2, let $z = \lim_{t\to 0} z_t$ where z_t is given by (2.2.2). Then, z is the point of $F(T)$ which is nearest to u. We observe that

$$\begin{aligned} d^2(x_{n+1}, z) &= d^2(\alpha_n u \oplus (1-\alpha_n) y_n, z) \\ &\leq \alpha_n d^2(u,z) + (1-\alpha_n) d^2(y_n, z) - \alpha_n(1-\alpha_n) d^2(u, y_n) \\ &\leq \alpha_n d^2(u,z) + (1-\alpha_n) d^2(x_n, z) - \alpha_n(1-\alpha_n) d^2(u, y_n) \\ &= (1-\alpha_n) d^2(x_n, z) + \alpha_n\Big(d^2(u,z) - (1-\alpha_n) d^2(u, y_n)\Big). \end{aligned}$$

That is

$$d^2(x_{n+1}, z) \leq (1-\alpha_n) d^2(x_n, z) + \alpha_n\Big(d^2(u,z) - (1-\alpha_n) d^2(u, y_n)\Big). \tag{5.1.39}$$

By Lemma 2.2.3, we have $\mu_n\big(d^2(u,z) - d^2(u,x_n)\big) \leq 0$ for all Banach limit μ. Moreover, since $d(x_{n+1}, x_n) \to 0$, we have

$$\limsup_{n\to\infty} \Big(d^2(u,z) + d^2(u, x_{n+1}) - d^2(u,z) - d^2(u, x_n)\Big) = 0.$$

It follows from $d(x_n, y_n) \to 0$ and Lemma 2.2.1 that

$$\limsup_{n\to\infty} \Big(d^2(u,z) - (1-\alpha_n) d^2(u, y_n)\Big) = \limsup_{n\to\infty} \Big(d^2(u,z) - d^2(u, x_n)\Big) \leq 0.$$

Applying Lemma 2.2.6 to (5.1.39), we get that $\lim_{n\to\infty} d(x_n, z) = 0$. □

The following results are immediately consequence of Theorem 5.1.17, by taking $\delta_n = 1$ in Theorem 5.1.17.

Corollary 5.1.18. *Let C be a nonempty closed and convex subset of a complete CAT(0) space X and let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Given a point u, $x_0 \in C$ are arbitrarily chosen and given sequences $\{\alpha_n\}$, $\{\beta_n\}$ in $(0,1)$ and $\{\gamma_n\}$ in $[0,1]$, the following conditions are satisfied:*

(C1) $\sum_{n=0}^{\infty} \alpha_n = \infty, \quad \lim_{n\to\infty} \alpha_n = 0;$

(C2) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty,$

$\sum_{n=0}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty;$

(C3) $\beta_n + (1+\beta_n)(1-\gamma_n) \in [0, a)$ *for some* $a \in (0,1)$.

The sequence $\{x_n\}$ is defined iteratively by

$$\begin{cases} z_n &= \gamma_n x_n \oplus (1-\gamma_n)Tx_n, \\ y_n &= \beta_n x_n \oplus (1-\beta_n)Tz_n, \\ x_{n+1} &= \alpha_n u \oplus (1-\alpha_n)y_n, \quad \forall n \geq 0. \end{cases} \tag{5.1.40}$$

Then $\{x_n\}$ converges to a fixed point $z \in F(T)$ which is nearest to u.

As a consequence of Corollary 5.1.18, we obtain the following result. It can be viewed as a generalization of Theorem 4.1.9.

Corollary 5.1.19. *Let C be a nonempty closed and convex subset of a complete CAT(0) space X and let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Given a point u, $x_0 \in C$ are arbitrarily chosen and given sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $(0,1)$, the following conditions are satisfied:*

(C1) $\sum_{n=0}^{\infty} \alpha_n = \infty, \quad \lim_{n\to\infty} \alpha_n = 0;$

(C2) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty;$

(C3) $\beta_n \in [0, a)$ *for some* $a \in (0,1)$.

The sequence $\{x_n\}$ is defined iteratively by

$$\begin{cases} y_n &= \beta_n x_n \oplus (1-\beta_n)Tx_n, \\ x_{n+1} &= \alpha_n u \oplus (1-\alpha_n)y_n, \quad \forall n \geq 0. \end{cases} \tag{5.1.41}$$

Then $\{x_n\}$ converges to a fixed point $z \in F(T)$ which is nearest to u.

Proof. By taking $\gamma_n = 1$, in Corollary 5.1.18, then $\{x_n\}$ converges strongly to a fixed point $z \in F(T)$ which is nearest to u. □