Chapter 6
Conclusion

In this chapter, we conclude all main results obtained in the thesis. It is organized by dividing into 3 sections and each section gives the main result obtained in this study.

6.1 A Generalization of Suzuki's Lemma

In 1983, Goebel and Kirk [16] proved that if $\{z_n\}$ and $\{w_n\}$ are sequences in a metric space of hyperbolic type (X, d) and $\{\alpha_n\} \subset [0, 1]$ which satisfy for all $i, n \in \mathbb{N}$, (i) $z_{n+1} = \alpha_n w_n \oplus (1 - \alpha_n)z_n$, (ii) $d(w_{n+1}, w_n) \leq d(z_{n+1}, z_n)$, (iii) $d(w_{i+n}, x_i) \leq a < \infty$, (iv) $\alpha_n \leq b < 1$, (v) $\sum_{n=1}^{\infty} \alpha_n = \infty$, then $\lim_{n\to\infty} d(w_n, z_n) = 0$. It was proved by Suzuki [48] that one obtains the same conclusion if the conditions (i)-(v) are replaced by the conditions (S1)-(S4) as follows:

(S1) $z_{n+1} = \alpha_n w_n \oplus (1 - \alpha_n)z_n$,

(S2) $\limsup_{n\to\infty} (d(w_{n+1}, w_n) - d(z_{n+1}, z_n)) \leq 0$,

(S3) $\{z_n\}$ and $\{w_n\}$ are bounded sequences,

(S4) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$.

The result we obtain is a generalization of Suzuki's Lemma by relaxing the condition (S1), namely, we can define z_{n+1} in terms of w_n and v_n such that $\lim_{n\to\infty} d(z_n, v_n) = 0$. Precisely, the lemma is as follow:

Lemma 6.1.1. *Let $\{z_n\}$, $\{w_n\}$ and $\{v_n\}$ be bounded sequences in a metric space of hyperbolic type (X, d) and let $\{\alpha_n\}$ be a sequence in $[0, 1]$ with satisfy for all $n \in \mathbb{N}$,*

(C1) $z_{n+1} = \alpha_n w_n \oplus (1 - \alpha_n)v_n$,

(C2) $\lim_{n\to\infty} d(z_n, v_n) = 0$,

(C3) $\limsup_{n\to\infty} \big(d(w_{n+1}, w_n) - d(z_{n+1}, z_n)\big) \leq 0$,

(C4) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$.

Then $\lim_{n\to\infty} d(w_n, z_n) = 0$.

6.2 Strong Convergence of Modified Halpern Iterations in CAT(0) Spaces

In this section, we prove four kinds of strong convergence theorems for the modified Halpern iterations of nonexpansive mappings in CAT(0) spaces.

Let C be a nonempty closed convex subset of a complete CAT(0) space and $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$.

1) If $\{x_n\}$ is defined by

$$x_{n+1} = \beta_n u \oplus (1 - \beta_n)(\alpha_n x_n \oplus (1 - \alpha_n)Tx_n)$$

where $\{\alpha_n\}$, $\{\beta_n\}$ satisfy the following conditions:

(A1) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,

(A2) $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$, and $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$,

then $\{x_n\}$ converges to a point $z \in F(T)$ which is nearest to u.

2) If $\{x_n\}$ is defined by

$$x_{n+1} = \beta_n x_n \oplus (1 - \beta_n)(\alpha_n u \oplus (1 - \alpha_n)Tx_n)$$

where $\{\alpha_n\}$, $\{\beta_n\}$ satisfy the following conditions:

(B1) $\lim_{n\to\infty} \alpha_n = 0$;

(B2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(B3) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

then $\{x_n\}$ converges to a point $z \in F(T)$ which is nearest to u.

3) If $\{x_n\}$ is defined by

$$x_{n+1} = \lambda_n x_n \oplus (1 - \lambda_n)T(\alpha_n u \oplus (1 - \alpha_n)x_n)$$

where $\{\alpha_n\}$, $\{\lambda_n\}$ satisfy the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$,

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$, and

(C3) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 1$.

then $\{x_n\}$ converges to a point $z \in F(T)$ which is nearest to u.

4) If $\{x_n\}$ is defined by

$$x_{n+1} = \lambda_n(\alpha_n u \oplus (1 - \alpha_n)x_n) \oplus (1 - \lambda_n)Tx_n$$

where $\{\alpha_n\}$, $\{\lambda_n\}$ satisfy the following conditions:

(D1) $\lim_{n\to\infty} \alpha_n = 0$,

(D2) $\sum_{n=1}^{\infty} \alpha_n = \infty$, and

(D3) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 1$.

then $\{x_n\}$ converges to a point $z \in F(T)$ which is nearest to u.

6.3 Strong Convergence of Modified Noor Iterations in CAT(0) Spaces

In 2006, Su and Qin [49] introduced the composite iteration scheme:

$$\begin{cases} w_n &= \delta_n x_n + (1-\delta_n)Tx_n, \\ z_n &= \gamma_n x_n + (1-\gamma_n)Tw_n, \\ y_n &= \beta_n x_n + (1-\beta_n)Tz_n, \\ x_{n+1} &= \alpha_n u + (1-\alpha_n)y_n, \quad \forall n \geq 0 \end{cases} \tag{6.3.1}$$

where $x_0, u \in C$ are an arbitrarily chosen and $\{\alpha_n\}, \{\beta_n\}$ are two sequences in (0,1) and $\{\gamma_n\}, \{\delta_n\}$ are in [0,1]. They proved under certain appropriate assumptions on the sequences $\{\alpha_n\}, \{\beta_n\}$ $\{\gamma_n\}$ and $\{\delta_n\}$ that $\{x_n\}$ converges to a fixed point of T in the framework of a uniformly smooth Banach space.

In this section, we extend Su and Qin's result to a complete CAT(0) space as follows:

Theorem 6.3.1. *Let C be a nonempty closed and convex subset of a complete CAT(0) space X and let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Given a point $u, x_0 \in C$ are arbitrarily chosen and given sequences $\{\alpha_n\}$, $\{\beta_n\}$ in $(0,1)$ and $\{\gamma_n\}$, $\{\delta_n\}$ in $[0,1]$, the following conditions are satisfied:*

(C1) $\sum_{n=0}^{\infty} \alpha_n = \infty, \quad \lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \ \sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty,$

$\sum_{n=0}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$, *and* $\sum_{n=0}^{\infty} |\delta_{n+1} - \delta_n| < \infty$;

(C3) $\beta_n + (1+\beta_n)(1-\gamma_n)(2-\delta_n) \in [0, a)$ *for some* $a \in (0,1)$.

The sequence $\{x_n\}$ is defined iteratively by (5.1.27). Then $\{x_n\}$ converges to a point $z \in F(T)$ which is nearest to u.

Corollary 6.3.2. *Let C be a nonempty closed and convex subset of a complete CAT(0) space X and let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Given a point u, $x_0 \in C$ are arbitrarily chosen and given sequences $\{\alpha_n\}$, $\{\beta_n\}$ in $(0,1)$ and $\{\gamma_n\}$ in $[0,1]$, the following conditions are satisfied:*

(C1) $\sum_{n=0}^{\infty} \alpha_n = \infty, \quad \lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=0}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty, \sum_{n=0}^{\infty}|\beta_{n+1}-\beta_n|<\infty,$

$\sum_{n=0}^{\infty}|\gamma_{n+1}-\gamma_n|<\infty;$

(C3) $\beta_n+(1+\beta_n)(1-\gamma_n)\in[0,a)$ *for some* $a\in(0,1)$.

The sequence $\{x_n\}$ *is defined iteratively by*

$$\begin{cases} z_n &= \gamma_n x_n \oplus (1-\gamma_n)Tx_n, \\ y_n &= \beta_n x_n \oplus (1-\beta_n)Tz_n, \\ x_{n+1} &= \alpha_n u \oplus (1-\alpha_n)y_n, \quad \forall n \geq 0. \end{cases} \tag{6.3.2}$$

Then $\{x_n\}$ *converges to a point* $z\in F(T)$ *which is nearest to* u.

Corollary 6.3.3. *Let* C *be a nonempty closed and convex subset of a complete CAT(0) space* X *and let* $T:C\to C$ *be a nonexpansive mapping such that* $F(T)\neq\emptyset$. *Given a point* u, $x_0\in C$ *are arbitrarily chosen and given sequences* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *in* $(0,1)$, *the following conditions are satisfied:*

(C1) $\sum_{n=0}^{\infty}\alpha_n=\infty, \quad \lim_{n\to\infty}\alpha_n=0;$

(C2) $\sum_{n=0}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty, \sum_{n=0}^{\infty}|\beta_{n+1}-\beta_n|<\infty;$

(C3) $\beta_n\in[0,a)$ *for some* $a\in(0,1)$.

The sequence $\{x_n\}$ *is defined iteratively by*

$$\begin{cases} y_n &= \beta_n x_n \oplus (1-\beta_n)Tx_n, \\ x_{n+1} &= \alpha_n u \oplus (1-\alpha_n)y_n, \quad \forall n \geq 0. \end{cases} \tag{6.3.3}$$

Then $\{x_n\}$ *converges to a point* $z\in F(T)$ *which is nearest to* u.