Chapter 1
Introduction and Literature Reviews

Let X be a nonempty set and $T : X \to X$ a self-mapping. We say that $x \in X$ is a *fixed point* of T if

$$Tx = x$$

and denote by $F(T)$ the set of all fixed points of T.

The fixed point theory is concerned with finding conditions on the structure that the set X must be endowed as well as on the properties of the operator T, in order to obtain results on;

1. the existence and uniqueness of fixed points;
2. the structure of the fixed point sets;
3. the approximation of fixed points.

The ambient spaces X involved in fixed point theorems cover a variety of spaces: metric space, normed linear space, generalized metric space, uniform space, and linear topological space, while the conditions imposed on the operator T are generally metrical or compactness type conditions.

The theorem which is of fundamental importance in the metrical fixed point theory and will be extended in the wider class of mappings and various spaces is **The Banach contraction principle** which states that *every contraction mapping* T, $d(Tx, Ty) \leq \alpha\, d(x, y)$, *for some* $\alpha \in [0, 1)$, *on a complete metric space* X *has a unique fixed point. Moreover, for each* $x \in X$, *the sequence* $\{T^n x\}$ *converges strongly to this fixed point.*

There are various generalizations of the contraction mapping principle, roughly obtained in two ways:

1) by weakening the contractive properties of the map and, possibly, by simultaneously giving to the space a sufficiently rich structure, in order to compensate the relaxation of the contractiveness assumptions;

2) by extending the structure of the ambient space.

Several fixed point theorems have been also obtained by combining the two ways previously described or by adding supplementary conditions.

Construction of fixed point iteration processes of nonlinear mappings is an important subject in the theory of nonlinear mappings, and finds application in a number of applied areas. Recall that a mapping T on a subset C of a metric space X is said to be *nonexpansive* if $d(Tx, Ty) \leq d(x, y)$, for all $x, y \in C$. Now, fixed point iteration processes for approximating fixed point of nonexpansive mappings in various space have been studied by many mathematicians.

The fixed point theorem for nonexpansive mappings firstly proved by Browder in 1965 which states that every nonexpansive self-mapping on a bounded closed convex subset C of a Hilbert space X has a fixed point. Almost immediately, both Browder

[5] and Göhde [20] proved the same result is true if X belongs to the much wider class of uniformly convex spaces. At the same time, Kirk [30] observed that the assumptions that C has normal structure and X is reflexive are enough to guarantee the fixed point for the nonexpansive mappings.

In 1974, R. E. Bruck [6] proved the existence theorem of common fixed point for any commuting family of nonexpansive self-mappings on a closed convex subset C of a Banach space X. He also showed that the set of common fixed points is a nonexpansive retract of C. In the result of Bruck, C is assumed to have both the fixed point property and the conditional fixed point property for nonexpansive mappings and is either weakly compact or bounded and separable.(C has the *fixed point property for nonexpansive mappings* if every nonexpansive $T : C \to C$ has a fixed point; C has the *conditional fixed point property for nonexpansive mappings* if every nonexpansive $T : C \to C$ satisfies either T has no fixed points in C, or T has a fixed point in every nonempty bounded closed convex $T-$invariant subset of C.)

In 1972, Goebel and Kirk [18] introduced the class of asymptotically nonexpansive mapping which is a natural generalization of a nonexpansive mapping. If X is a metric space and C is a subset of X, a mapping $T : C \to C$ is called *asymptotically nonexpansive* if there exists a sequence $\{k_n\}$ of real numbers with $\lim_{n\to\infty} k_n = 1$ such that

$$d(T^n x, T^n y) \le k_n d(x, y), \text{ for all } x, y \in C \text{ and } n \in \mathbb{N}.$$

They also proved the fixed point theorem for this mapping in a uniformly convex Banach space.

In 2001, T. Domínguez and P. Lorenzo [16] extended the result of Goebel and Kirk into any commuting family $\mathcal{G}$ of asymptotically nonexpansive self-mappings of C. They proved that if C is a weakly compact convex subset of a Banach space X and every asymptotically nonexpansive self-mapping on C satisfies the (ω)-fpp, then the common fixed point set of $\mathcal{G}$ is a nonempty nonexpansive retract of C. (Recall that if X is a Banach space, $C \subset X$ is weakly compact and $T : C \to C$, we say that a nonempty closed convex subset D of C *satisfies property* (ω) *with respect to* T if $\omega_T(x) \subset D$ for every $x \in D$ where

$$\omega_T(x) = \{y \in D : y = w - \lim_k T^{n_k}(x) \text{ for some } n_k \to \infty\}.$$

A mapping $T : C \to C$ is said to satisfy the (ω)-*fixed point property* ((ω)-*fpp*) if T has a fixed point in every nonempty closed convex subset D of C which satisfies (ω)).

In 2008, Kirk and Xu [34] introduced an asymptotic pointwise nonexpansive mapping which is a generalization of an asymptotically nonexpansive mapping. A self-mapping T on a subset C of a metric space X is said to be *asymptotic pointwise nonexpansive* if there exists a sequence of functions $\alpha_n : C \to [0, \infty)$ such that

$$d(T^n x, T^n y) \le \alpha_n(x) d(x, y), \text{ for all } x, y \in C \text{ and } n \in \mathbb{N},$$

and $\limsup_{n\to\infty} \alpha_n(x) \le 1$, for every $x \in C$. They proved that if X is a uniformly convex Banach space and C is a nonempty bounded closed convex subset of X, then an asymptotic pointwise nonexpansive $T : C \to C$ has a fixed point.

In 2009, Hussain and Khamsi [24] discussed the existence of fixed points of asymptotic pointwise nonexpansive mappings in a uniform convex metric space, specifically to the so called CAT(0) space.

While the fixed point theorem of each nonlinear mapping have been studied, the approximation of their fixed points was also developed by many mathematicians.

In 1953 Mann [39] introduced the iteration scheme $\{x_n\}$ by starting at $x_1 \in C$ and for $n \geq 1$,

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \tag{1.0.1}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$, C is a compact convex subset of a Banach space and T is a continuous mapping from C into itself. This iteration is called *the Mann iteration process.* He proved that, under some appropriate conditions, the sequence defined by (1.0.1) converges weakly to a fixed point of T.

In 1967, Halpern [23] defined the iteration, starting from any $u, x_1 \in B$ and for $n \geq 1$,

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) T x_n, \tag{1.0.2}$$

where $\{\alpha_n\}$ is a real sequence in $[0, 1]$, to approximate the fixed point of a nonexpansive mapping T from the unit ball B of a real Hilbert space into itself. This iteration is called *the Halpern iteration process.* He gave the sufficient condition to insure the strong convergence of the sequence defined by (1.0.2) to a fixed point of T.

In 1974, Ishikawa [25] generalized the Mann iteration process as the following: $x_1 \in C$ and for $n \geq 1$,

$$x_{n+1} = \alpha_n T[\beta_n T x_n + (1 - \beta_n) x_n] + (1 - \alpha_n) x_n, \tag{1.0.3}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0, 1]$ and T is a lipschitzian pseudo-contractive map from a compact convex subset C of a Hilbert space into itself. This iteration is called *the Ishikawa iteration process.* He proved that, under some appropriate conditions, the sequence $\{x_n\}$ defined by (1.0.3) converges weakly to a fixed point of T.

In 1991, J. Schu [46] modified the Mann iteration process for an asymptotically nonexpansive mapping as follows:

$$x_{n+1} = \alpha_n T^n x_n + (1 - \alpha_n) x_n, \ \ n \geq 1, \tag{1.0.4}$$

where $\{\alpha_n\}$ is a real sequence in $[0, 1]$ and T is an asymptotically nonexpansive self-mapping of a bounded closed convex subset C of a uniformly convex Banach space X. He proved, under some certain conditions, that the sequence $\{x_n\}$ defined by (1.0.4) converges weakly to some fixed point of T when X satisfies Opial's condition. And it converges strongly if assume in addition that T^m is compact for some $m \in \mathbb{N}$.

In 1994, Tan and Xu [51] modified the Ishikawa iteration process for an asymptotically nonexpansive mapping as follows:

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T^n [\beta_n T^n x_n + (1 - \beta_n) x_n], \ n \geq 1, \tag{1.0.5}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $(0, 1)$ and T is an asymptotically nonexpansive self-mapping on a subset C of X. They proved that if C is bounded closed convex, X is uniformly convex Banach space satisfying Opial's condition and the sequences $\{\alpha_n\}$,

$\{\beta_n\}$ satisfy certain conditions, then the sequence $\{x_n\}$ defined by (1.0.5) converges weakly to a fixed point of T.

In 2001, Khan and Takahashi [29] defined the iteration as follows:

$$x_1 \in C,\ \ x_{n+1} = (1-\alpha_n)x_n + \alpha_n S^n((1-\beta_n)x_n + \beta_n T^n x_n),\ n \geq 1, \tag{1.0.6}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0,1]$, S and T are asymptotically nonexpansive self-mappings on a subset C of X. They proved that if C is bounded closed convex, X is uniformly convex Banach space satisfying Opial's condition and the sequences $\{\alpha_n\}$, $\{\beta_n\}$ satisfy certain conditions, then the sequence $\{x_n\}$ defined by (1.0.6) converges weakly to a common fixed point of S and T. They also proved that the sequence $\{x_n\}$ converges strongly to a common fixed point of S and T if C is compact convex. In the latter case, no Opial's condition is assumed for the uniformly convex space X.

In 2011, Kozlowski [35] generalized the Mann iteration process for an asymptotic pointwise nonexpansive mapping $T : C \to C$ by

$$x_{k+1} = (1-t_k)x_k + t_k T^{n_k}(x_k), \tag{1.0.7}$$

and also generalized the Ishikawa iteration process for the mapping T as follows:

$$x_{k+1} = (1-t_k)x_k + t_k T^{n_k}\left((1-s_k)x_k + s_k T^{n_k}(x_k)\right), \tag{1.0.8}$$

for all $k \in \mathbb{N}$, where $\{s_k\}$, $\{t_k\}$ are sequences in (0,1) and $\{n_k\}$ is an increasing sequence of natural numbers for which $\limsup\limits_{k\to\infty} a_{n_k}(x_k) = 1$ and $a_n(x) = \max\{\alpha_n(x), 1\}$. The sequences $\{x_n\}$ defined by (1.0.7) and (1.0.8) are called *the generalized Mann iteration process* and *the generalized Ishikawa iteration process,* respectively. He proved that, under some reasonable assumptions, the generalized Mann and Ishikawa processes converge weakly to a fixed point of T if C is a bounded closed convex subset of a uniformly convex Banach space X which satisfies Opial's condition. He also showed that both generalized Mann and Ishikawa processes converge strongly to a fixed point of T provided T^m is a compact mapping for an $m \in \mathbb{N}$.

In 2008, Khan, Domlo and Uddin [28] introduced the iterative scheme defined as follows:

$$\begin{aligned}
x_{n+1} &= (1-\alpha_{kn})x_n + \alpha_{kn}T_k^n y_{(k-1)n},\\
y_{(k-1)n} &= (1-\alpha_{(k-1)n})x_n + \alpha_{(k-1)n}T_{k-1}^n y_{(k-2)n},\\
y_{(k-2)n} &= (1-\alpha_{(k-2)n})x_n + \alpha_{(k-2)n}T_{k-2}^n y_{(k-3)n},\\
&\ \vdots\\
y_{2n} &= (1-\alpha_{2n})x_n + \alpha_{2n}T_2^n y_{1n},\\
y_{1n} &= (1-\alpha_{1n})x_n + \alpha_{1n}T_1^n y_{0n},
\end{aligned}$$

where $y_{0n} = x_n$, for all $n \in \mathbb{N}$, $\{\alpha_{in}\}$ are sequences in $[0,1]$ for all $i = 1,2,...,k$ and $T_1, T_2, ..., T_k$ are $(L-\gamma)$ uniform Lipschitz and asymptotically quasi-nonexpansive self-mappings on a closed convex subset C of a uniformly convex Banach space X. They established a necessary and sufficient condition for convergence of the iteration scheme to a common fixed point of $T_1, T_2, ..., T_k$. They also showed that, under appropriate conditions, the sequence $\{x_n\}$ defined above converges weakly to a common fixed point of $T_1, T_2, ..., T_k$ if assume in addition that each $I - T_i$, $i = 1, 2, ..., k$, is demiclosed at 0 and it converges strongly provided T_i^m is semi-compact for some positive integer m and some $1 \leq i \leq k$. (A self-mapping T on a subset C of a Banach space X is said to be *asymptotically quasi-nonexpansive* if there exists a sequence $\{u_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} u_n = 0$ and $\|T^n x - p\| \leq (1 + u_n)\|x - p\|,$ for all $x \in C$, $p \in F(T)$ and $n \in \mathbb{N}$, whenever $F(T) \neq \emptyset$, and T is called $(L - \gamma)$ *uniform Lipschitz* if there are constants $L > 0$ and $\gamma > 0$ such that $\|T^n x - T^n y\| \leq L\|x - y\|^\gamma,$ for all $x, y \in C$, and $n \in \mathbb{N}$).

The following implications hold.

$$\begin{array}{ccc}
T \text{ is nonexpansive} & & \\
\Downarrow & & \\
T \text{ is asymptotically nonexpansive} & \Rightarrow & T \text{ is asymptotically quasi-nonexpansive} \\
\Downarrow & & \\
T \text{ is asymptotic pointwise nonexpansive} & &
\end{array}$$

Base on the results in the literature, we found that:

1) there is no any results in metric or Banach spaces concerning to the existence of common fixed points for a family of asymptotic pointwise nonexpansive mappings,

2) there is no any results in metric or Banach spaces concerning to the convergence of an iteration process for a family of asymptotic pointwise nonexpansive mappings.

The purpose of this thesis is third fold. First, we study and prove a fixed point theorem for any family of commuting asymptotic pointwise nonexpansive mappings in uniformly convex Banach spaces and CAT(0) spaces. We also prove the structure of its common fixed point set. Second, we study and construct new iteration processes for finding a common fixed point of a finite family of asymptotic pointwise nonexpansive mappings in both two spaces. Finally, we study and establish some necessary and sufficient conditions for weak and strong convergence and for $\triangle$ and strong convergence to a common fixed point of a finite family of asymptotic pointwise nonexpansive mappings in uniformly convex Banach spaces and in CAT(0) spaces, respectively.

This thesis is divided into 5 chapters. Chapter 1 is an introduction and literature reviews. Chapter 2 deals with the basic definitions and some useful properties that will be used in the next chapters. Chapters 3 and 4 are the main results of this thesis and the conclusion is in Chapter 5.