Chapter 2
Basic Concepts and Preliminaries

2.1 Metric Spaces, Banach Spaces and Hilbert Spaces

Our purpose in this section is to talk about the basic definitions and elementary properties of metric spaces, Hilbert spaces, and Banach spaces. Indeed a Banach space is a normed space equipped with a function, norm, defined on it. The norm on the space is used to define the convergence of the sequences and the other structures on the spaces. A Hilbert space is a normed space which an inner product defined on it.

Definition 2.1.1. ([36]) A *metric space* is a pair (X, d), where X is a set and d is a metric on X(or *distance function on* X), that is, a real valued function defined on $X \times X$ such that for all $x, y, z \in X$ we have:

(1) d is real valued, finite and nonnegative,

(2) $d(x, y) = 0$ if and only if $x = y$,

(3) $d(x, y) = d(y, x)$ (Symmetry),

(4) $d(x, y) \leq d(x, z) + d(z, y)$ (Triangle inequality).

The elements of X are called the *point* of the metric (X, d).

Example 2.1.2. (see e.g., [36])

(1) The real line $\mathbb{R}$ with $d(x, y) := |x - y|$ for all $x, y \in \mathbb{R}$ is a metric space. The metric d is called the *usual metric* for $\mathbb{R}$.;

(2) The Euclidian space $\mathbb{R}^n$ with

$$d(x, y) := \{\sum_{i=1}^{n}(x_i - y_i)^2\}^{1/2},$$

$x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ is a metric space. The metric d is called the *usual metric* for $\mathbb{R}^n$. The following two mappings:

$$\delta(x, y) = \sum_{i=1}^{n} |\, x_i - y_i|, \quad x, y \in \mathbb{R}^n,$$

$$\rho(x, y) = \max_{1 \leq i \leq n} |\, x_i - y_i|, \quad x, y \in \mathbb{R}^n,$$

are also metrics on $\mathbb{R}^n$.

(3) The space of all continuous complex-valued functions on the closed interval $[a, b]$ with

$$d(f, g) := \max_{x\in[a,b]} |f(x) - g(x)|$$

is a metric space. The metric d is called the *Chebyshev metric*. This metric space usually denoted by $C[a, b]$.

Definition 2.1.3. ([36]) A sequence $\{x_n\}$ in a metric space $X = (X, d)$ is said to be *convergent* if there is an $x \in X$ such that

$$\lim_{n\to\infty} d(x_n, x) = 0$$

x is called the *limit* of $\{x_n\}$ and we write

$$\lim_{n\to\infty} x_n = x \ \text{ or, simple, } x_n \to x.$$

In this case, we say that $\{x_n\}$ *converges to* x. If $\{x_n\}$ is not convergent, it is said to be *divergent.*

For any sequence $\{x_n\}$ in X, we can consider a subsequence $\{x_{n_k}\} = \{x_{n_1}, x_{n_2}, x_{n_3}, ...\}$ of $\{x_n\}$, where $n_1 < n_2 < n_3 < ....$

Proposition 2.1.4. *([40]) If a sequence in a metric space converges to a limit, then every subsequence of that sequence converges to that same limit.*

Definition 2.1.5. ([36]) A sequence $\{x_n\}$ in a metric space $X = (X, d)$ is said to be *Cauchy* if for every $\varepsilon > 0$, there is an $N(\varepsilon) \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$ for every $m, n \geq N(\varepsilon)$.

Theorem 2.1.6. *([40]) Every convergent sequence in a metric space is Cauchy.*

But the converse need not be true.

Definition 2.1.7. ([40]) If every Cauchy sequence in a metric space (X, d) converges then the metric space X is said to be *complete*.

Proposition 2.1.8. *([40]) If a Cauchy sequence in a metric space has a convergent subsequence, then the entire sequence converges to the limit of the subsequence.*

The *diameter* of a nonempty subset C in a metric space (X, d) is defined by

$$diam(C) := \sup\{d(x, y) : x, y \in C\}.$$

The set C is said to be *bounded* if $diam(C) < \infty$.

Proposition 2.1.9. *([40]) Every Cauchy sequence in a metric space (and hence every convergent sequence) is bounded.*

Definition 2.1.10. ([40]) Let S be a subset of a metric space X. Then S is *compact* if, for each collection $\mathfrak{B}$ of open subsets of X whose union includes S, there is a finite subcollection of $\mathfrak{B}$ whose union includes S. That is, the set S is compact if each open covering of S can be thinned to a finite subcovering.

Definition 2.1.11. ([40]) Let S be a subset of a metric space X. Consider S to be a metric space with the metric inherited from X. Then S has the *finite intersection property* if $\cap\{F : F \in \mathfrak{F}\}$ is nonempty whenever $\mathfrak{F}$ is a collection of closed subsets of S such that each finite subcollection of $\mathfrak{F}$ has nonempty intersection.

In the preceding definition, the set S, not X, should really be considered to be the universal set when taking intersections of empty families $\mathfrak{F}$.

Proposition 2.1.12. *([40]) Let S be a subset of a metric space X. Then the following are equivalent.*

(i) The set S is compact.

(ii) The set S has the finite intersection property.

(iii) Each sequence in S has a convergent subsequence whose limit is in S.

Let (X,d) be a metric space and C be a nonempty subset of X. A mapping $T : C \to C$ is called *semi-compact* if for any sequence $\{x_n\}$ in C such that $\lim\limits_{n\to\infty} d(x_n, Tx_n) = 0$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $q \in C$ such that $\lim\limits_{j\to\infty} x_{n_j} = q$.

Definition 2.1.13. ([1]) A mapping T from a metric space (X,d) into another metric space (Y,ρ) is said to satisfy *Lipschitz condition* on X if there exists a constant $L > 0$ such that

$$\rho(Tx,Ty) \leq Ld(x,y) \text{ for all } x,y \in X.$$

If L is the least number for which Lipschitz condition holds, then L is called *Lipschitz constant.* In this case, we say that T is an *L-Lipschitz mapping* or simply a *Lipschitzian mapping* with Lipschitz constant L.

Definition 2.1.14. ([51]) A self-mapping T on a Hilbert space H is said to be *pseudo-contractive* if $\|Tx - Ty\|^2 \leq \|x-y\|^2 + \|(I-T)x - (I-T)y\|^2$, for all $x,y \in H$).

Definition 2.1.15. ([26]) Let X be a metric space and C be a nonempty subset of X. A family of mapping $\{T_i : i = 1,2,...,m\}$ on C is said to satisfy *Condition (A'')* if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that $d(x, T_jx) \geq f(d(x,F))$, for some $j = 1,2,...,m$ and for all $x \in C$ where $d(x,F) = \inf\left\{d(x,p) : p \in F = \bigcap_{i=1}^{m} F(T_i)\right\}$.

To state fundamental theorems relating to Banach spaces and Hilbert spaces, we begin with the concept of vector spaces ([40]) as follows:

Definition 2.1.16. A *vector space* (or *linear space*) over a field $\mathbb{K}$ is a nonempty set X of elements $x, y, ...$ (called *vectors*) together with two algebraic operations. These operations are called *vector addition* and *multiplication of vectors by scalars,* that is, by elements of $\mathbb{K}$.

Vector addition

(1) $x + y = y + x$;

(2) $x + (y + z) = (x + y) + z$;

(3) there exists in X a unique element, denoted by 0 and called the *zero element*, such that $x + 0 = x$ for every x;

(4) to each element x in X there corresponds a unique element in X, denoted by $-x$ and called the *negative* of x, such that $x + (-x) = 0$.

Multiplication by scalars

(5) $\alpha(x + y) = \alpha x + \alpha y$;

(6) $(\alpha + \beta)x = \alpha x + \beta x$;

(7) $(\alpha\beta)x = \alpha(\beta x)$;

(8) $1x = x$.

Note that the above definition involves a general field $\mathbb{K}$, but in functional analysis, $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$ and the elements of $\mathbb{K}$ are called *scalars*.

Definition 2.1.17. ([40]) Let X be a linear space (or vector space). A *norm* on X is a real-valued function $\|\cdot\|$ on X such that the following conditions are satisfied by all members x and y of X and each scalar α:

(1) $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$,

(2) $\|\alpha x\| = |\alpha|\|x\|$,

(3) $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality)

The ordered pair $(X, \|\cdot\|)$ is called a *normed space* or *normed linear space*.

Example 2.1.18. ([36])

(1) Let p be a real number such that $p \geq 1$. The collection of all sequences $\{\alpha_n\}$ of scalars for which $\sum_{n=1}^{\infty} |\,\alpha_n|^p$ is finite with the norm defined by

$$\|\{\alpha_n\}\|_p := (\sum_{n=1}^{\infty} |\,\alpha_n|^p)^{\frac{1}{p}},$$

is a normed space. The resulting normed space is called ℓ_p.

(2) The collection of all bounded sequences of scalars with the norm defined by

$$\|\{\alpha_n\}\|_\infty := \sup\{|\,\alpha_n| : n \in \mathbb{N}\},$$

is a normed space. The resulting normed space is called ℓ_∞.

(3) The space of all continuous complex-valued functions on the closed interval $[a, b]$ with the norm defined by

$$\|f\|_2 := (\int_b^a |f(t)|^2 dt)^{\frac{1}{2}}$$

is a normed space.

Proposition 2.1.19. *([40]) Let X be a normed space. Then $|\|x\| - \|y\|| \leq \|x - y\|$ whenever $x, y \in X$. Thus, the function $x \mapsto \|x\|$ is continuous from X into $\mathbb{R}$.*

Definition 2.1.20. ([40]) A nonempty subset C of a normed space X is said to be *convex* if $\lambda x + (1 - \lambda)y \in C$ for all $x, y \in C$ and $\lambda \in (0, 1)$.

Definition 2.1.21. ([40]) Let X be normed space. The *metric induced by the norm* of X is the metric d on X defined by the formula

$$d(x, y) := \|x - y\|, \quad \text{for all } x, y \in X.$$

The *norm topology* of X is the topology obtained from this metric.

Definition 2.1.22. ([40]) Let X be normed space. The *closed unit ball* of X is $\{x \in X : \|x\| \leq 1\}$ and is denoted by B_X. The *open unit ball* of X is $\{x \in X : \|x\| < 1\}$. The *unit sphere* of X is $\{x \in X : \|x\| = 1\}$ and is denoted by S_X.

Definition 2.1.23. ([36]) A normed space which is complete under the induced metric is called a *Banach space.*

Example 2.1.24. ([36])

(1) The normed space ℓ_p, where $1 \leq p < \infty$, is a Banach space.

(2) The normed space ℓ_∞ is a Banach space.

(3) The space $C[a, b]$ of all complex-valued functions on the closed interval $[a, b]$ with the norm $\|f(t)\| := \max_{t \in [a,b]} |f(t)|$ is a Banach space.

Definition 2.1.25. ([36]) A *functional* is an operator whose range lies in the real line $\mathbb{R}$ or in the complex plane $\mathbb{C}$.

Definition 2.1.26. ([36]) A *linear functional* is a linear operator with domain in a vector space X and range in the scalar field $\mathbb{F}$ of X; thus,

$$f : dom(f) \to \mathbb{F},$$

where $\mathbb{F} = \mathbb{R}$ if X is a real vector space and $\mathbb{F} = \mathbb{C}$ if X is a complex vector space.

Recall that a subset A of a normed space is *bounded* if there is a nonnegative number M such that $\|x\| \leq M$ for each $x \in A$. For the boundedness of the operator can be defined in this sense, that is,

Let X and Y be a normed spaces. A linear operator T from X into Y is *bounded* if there exists an integer $c > 0$ such that $\|Tx\| \leq c\|x\|$ for all $x \in X$. The collection of all bounded linear operators from X into Y is denoted by $B(X, Y)$. But if the range Y of the linear operators is just a field $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$), we call it specifically as defined follows.

Definition 2.1.27. ([40]) Let X be a normed space. The *(continuous) dual space* of X or *dual* of X or *conjugate space* of X is the normed space $B(X, \mathbb{F})$ of all bounded linear functionals on X with the operator norm defined by

$$\|f\| := \sup\{|f(x)| : x \in X, \|x\| \leq 1\}.$$

This space is denoted by X^*.

Theorem 2.1.28. *([40]) If X is a normed space, then X^* is a Banach space.*

Definition 2.1.29. ([36]) A sequence $\{x_n\}$ in a normed space X is said to be *strongly convergent* (or *convergent in norm*) if there is an $x \in X$ such that

$$\lim_{n\to\infty} \|x_n - x\| = 0.$$

This is written

$$\lim_{n\to\infty} x_n = x$$

or simply

$$x_n \to x.$$

The point x is called the *strong limit* of $\{x_n\}$, and we say that $\{x_n\}$ *converges strongly to x.*

The topology induced by a norm is quite strong in the sense that it has many open sets. In order that each bounded sequence in X has a norm convergent subsequence, it is necessary and sufficient that X has to be finite dimensional. This fact leads us to consider other weaker topologies on normed spaces to search for subsequential extraction principles. To define the weaker topology for a normed space X, we give the definition of weak convergence of a sequence in the space first.

Definition 2.1.30. ([36]) A sequence $\{x_n\}$ in a normed space X is said to be *weakly convergent* if there is an $x \in X$ such that for every $f \in X^*$,

$$\lim_{n\to\infty} f(x_n) = f(x).$$

This is written

$$w - \lim_{n\to\infty} x_n = x$$

or simply

$$x_n \rightharpoonup x.$$

The element x is called the *weak limit* of $\{x_n\}$, and we say that $\{x_n\}$ *converges weakly to x.*

A subset C of X is *weakly closed* if it is closed in the weak topology, that is, if it contains the weak limit of all of its weakly convergent sequences. The weakly open sets are now taken as those sets whose complements are weakly closed. The resulting topology on X is called *the weak topology on X*. Sets which are compact in this topology are said to be *weakly compact*.

For $x \in X$, define the mapping $\varphi : X \to X^{**}$ by

$$\varphi(x) = f_x,$$

where $f_x(g) = g(x)$ for all $g \in X^*$. We can see that φ is linear injective and preserves the norm. This mapping φ is called *the canonical embedding* from X into X^{**}. (Note that X^{**} is the dual space $(X^*)^*$ of X^*.

Definition 2.1.31. ([40]) A normed space X is said to be *reflexive* if the canonical embedding mapping $\varphi : X \to X^{**}$ is surjective. In this case, we write $X \cong X^{**}$ or $X = X^{**}$.

Example 2.1.32. ([40]) For $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have that ℓ_p^* is isometrically isomorphic to ℓ_q and ℓ_q^* is isometrically isomorphic to ℓ_p, it follows that ℓ_p^{**} is isometrically isomorphic to ℓ_p. Thus all ℓ_p, where $1 < p < \infty$, are reflexive.

To see the example of nonreflexive space, we recall the notation of one of the subspaces of ℓ_∞. Let c_0 be the collection of all sequences of scalars which converge to zero. It is well-known that c_0 is a closed subspace of ℓ_∞ which is complete, so c_0 itself is a complete normed space with the same norm inherited from ℓ_∞, i.e., it is a Banach space.

Example 2.1.33. ([40] Since we have c_0^* is isometrically isomorphic to ℓ_1 and ℓ_1^* is isometrically isomorphic to ℓ_∞, thus c_0^{**} is isometrically isomorphic to ℓ_∞. This shows that the Banach space c_0 is not reflexive.

Theorem 2.1.34. *([40]) Every finite-dimensional normed space is reflexive.*

Theorem 2.1.35. *([40]) Suppose that X is a Banach space. Then the following are equivalent.*

(i) X is reflexive.

(ii) The dual space X^* of X is reflexive.

(iii) Every bounded sequence in X has a weakly convergent subsequence.

(iv) The closed unit ball of X is weakly compact.

(v) Whenever $\{C_n\}$ is a sequence of nonempty bounded closed convex sets in X such that $C_{n+1} \subset C_n$ for each $n \in \mathbb{N}$, it follows that $\bigcap_{n=1}^{\infty} C_n \neq \emptyset$.

Theorem 2.1.36. *([40]) The closure and weak closure of a convex subset of a normed space are the same. In particular, a convex subset of a normed space is closed if and only if it is weakly closed.*

Next are the geometric properties of Banach spaces which are quite often used in assuming into the space when we study fixed point theory in Banach spaces.

Definition 2.1.37. ([40]) A normed space X is *rotund* or *strictly convex* if $\|tx_1 + (1-t)x_2\| < 1$ whenever x_1 and x_2 are different points of S_X and $0 < t < 1$.

Definition 2.1.37 actually says that a normed space X is strictly convex if and only if whenever x_1 and x_2 are different points of S_X the open line segment $\{tx_1 + (1-t)x_2 : 0 < t < 1\}$ lies entirely in the interior of B_X. Incidentally, the next argument establishes the another characterization of strictly convexity that allows examining this property only the midpoints of straight line segments rather that the entire segments.

Proposition 2.1.38. *([40]) Suppose that X is a normed space. Then X is strictly convex if and only if $\|\frac{1}{2}(x_1 + x_2)\| < 1$ whenever x_1 and x_2 are different points of S_X.*

Example 2.1.39. ([40]) The scalar field $\mathbb{F}$, viewed as a normed space over $\mathbb{F}$, is strictly convex. More generally, every normed space that is zero- or one-dimensional is strictly convex.

Example 2.1.40. ([40]) Let $e_1 = (1, 0, 0, ..., 0, ...)$ and $e_2 = (0, 1, ..., 0, ...)$. Let $x_1 = e_1 + e_2$ and $x_2 = e_1 - e_2$. Then

$$\|x_1\|_\infty = \|x_2\|_\infty = \|\frac{1}{2}(x_1 + x_2)\|_\infty = 1,$$

this shows that neither c_0 nor ℓ_∞ is strictly convex.

After we see the characterization of strictly convexity of a normed space above, it is natural to survey how much the midpoint of such a segment far away from the boundary of the closed unit ball. If the segment has some minimum positive length, it is reasonable to use this length measures the amount of convexity of the space. It is quite possible for a normed space X to be strictly convex and yet for there to be sequences $\{x_n\}$ and $\{y_n\}$ in S_X such that $\|x_n - y_n\| > \epsilon$ for some $\epsilon > 0$ still satisfying $\sup_{n\in\mathbb{N}} \|\frac{1}{2}(x_1 + x_2)\| = 1$. This leads to the definition of uniformly convex if this does not happen; that is, for every positive ϵ, there is a positive δ depending on ϵ such that $\|\frac{1}{2}(x + y)\| \leq 1 - \delta$ whenever $x, y \in S_X$ and $\|x - y\| \geq \epsilon$. This definition can be stated formally in term of the amount of convexity of the space as follows.

Definition 2.1.41. ([40]) Let X be a normed space. Define a function $\delta_X : [0, 2] \to [0, 1]$ by the formula

$$\delta_X(\epsilon) = \inf\{1 - \|\frac{1}{2}(x + y)\| : x, y \in S_X, \|x - y\| \geq \epsilon\}$$

if $X \neq \{0\}$, and by the formula

$$\delta_X(\epsilon) = \begin{cases} 0 & \text{if } \epsilon = 0 \\ 1 & \text{if } 0 < \epsilon \leq 2. \end{cases}$$

if $X = 0$. Then δ_X is the *modulus of rotundity* or *modulus of convexity* of X. The space X is *uniformly rotund* or *uniformly convex* if $\delta_X(\epsilon) > 0$ whenever $0 < \epsilon \leq 2$.

Notice that if X is a uniformly convex normed space with modulus of convexity δ_X and x and y are different elements of S_X, then $\|\frac{1}{2}(x + y)\| \leq 1 - \delta_X(\|x - y\|) < 1$, which proves the following basic fact of this property.

Proposition 2.1.42. *([40]) Every uniformly convex normed space is strictly convex.*

Example 2.1.43. ([40])

(1) Suppose that $1 < p < \infty$. Then ℓ_p is uniformly convex.

(2) Since neither c_0 nor ℓ_∞ is strictly convex, it follows from Proposition 2.1.42 that they are not uniformly convex.

Let S be a subset of a normed space X. A point $x \in S$ is a *diametral point* of S provided $r(S) := \sup\{\|x - y\| : y \in S\} = diam(S)$. A convex set $K \subset X$ is said to have *normal structure* if for each bounded convex subset H of K which contains more than one point, there is some point $x \in H$ which is not a diametral point of H. Thus sets with normal structure have no convex subsets S which consist entirely of diametral points except singletons; i.e.,

$$diam(S) > 0 \Rightarrow r(S) < diam(S).$$

Next is one of the properties of a Banach space which is usually used in the assumptions to approximate the fixed point of the mappings on that space.

Definition 2.1.44. ([42]) A Banach space is said to have the *Opial's condition* if given whenever $\{x_n\}$ converge weakly to $x \in X$,

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|,$$

for each $y \in X$ with $x \neq y$.

It is well-known that all Hilbert spaces and ℓ_p spaces, where $1 \leq p < \infty$, have this property, while all L_p spaces do not unless $p = 2$.

The last portion of this part is about the special normed spaces which are very important spaces because they have rich properties and have been the most useful spaces in practical applications in general function analysis. The definition of these spaces are the following.

Definition 2.1.45. ([36]) An *inner product space* is a vector space X with an inner product defined on X. A *Hilbert space* is a complete inner product space (complete in the metric defined by the inner product). Here, an *inner product* on X is a mapping of $X \times X$ into the scalar field $\mathbb{F}$ of X; that is, with every pair of vectors x and y, there is an associated a scalar which is written $\langle x, y\rangle$ and is called the *inner product* of x and y, such that for all vectors x, y, z and scalars α we have

(1) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$

(2) $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$

(3) $\langle x, y\rangle = \overline{\langle y, x\rangle}$

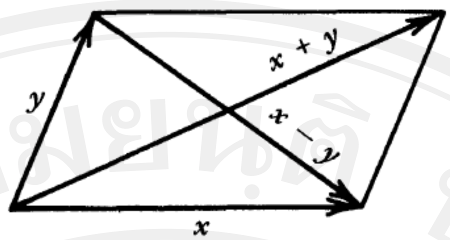
(4) $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ if and only if $x = 0$.

An inner product on X defines a *norm* on X given by

$$\|x\| = \sqrt{\langle x, x\rangle}.$$

Hence inner product spaces are normed spaces and Hilbert spaces are Banach spaces. In fact it can be straightforward calculation that a norm on an inner product space X satisfies the important *Parallelogram equality:*

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2,$$



for all $x, y \in X$ (see Figure 2.1).

Figure 2.1: Parallelogram equality

As above, every inner product space is a normed space but not all normed spaces are inner product spaces. In deed a norm which does not satisfy the Parallelogram equality cannot be obtained from an inner product.

2.2 CAT(0) Spaces

In this section, we focus on the special metric space which has the geometry defined on it. Its important properties are also established.

Definition 2.2.1. ([33]) Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic* from x to y) is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that

(1) $c(0) = x$ and $c(l) = y$,

(2) $d(c(t), c(t')) = |t - t'|$, for all $t, t' \in [0, l]$.

In particular, c is an isometry and $d(x, y) = l$. The image of c is called a *geodesic* (or *metric*) *segment* joining x and y (see Figure 2.2). When it is unique, this geodesic segment is denoted by $[x, y]$.

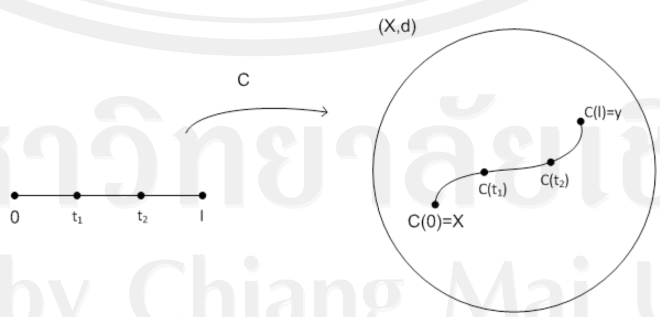


Figure 2.2 : Geodesic path and geodesic segment

Definition 2.2.2. ([33]) A metric space (X, d) is said to be *geodesic space* if every two points of X are joined by a geodesic, and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$.

The way to define a convex subset in a geodesic space is the same as defined in general metric spaces.

Definition 2.2.3. ([33]) A subset Y of a geodesic space (X, d) is said to be *convex* if Y includes every geodesic segment joining any two of its points (see Figure 2.3).

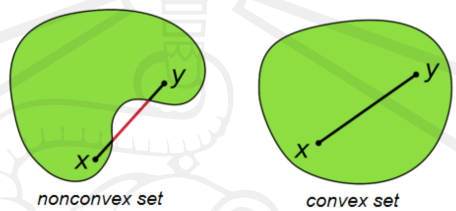


Figure 2.3 : Convex set

Definition 2.2.4. ([33]) A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ in a geodesic space (X, d) consists of three points x_1, x_2, x_3 in X (the *vertices* of $\triangle$) and a geodesic segment between each pair of vertices (the *edges* of $\triangle$), see Figure 2.4.

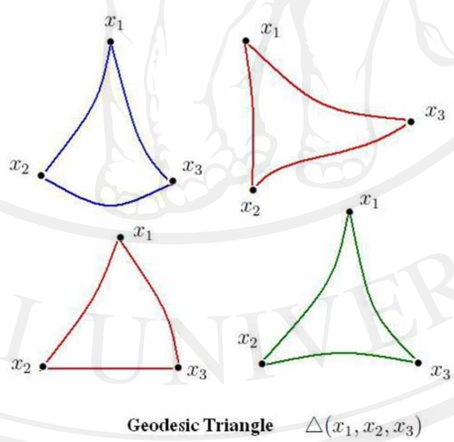


Figure 2.4 : Geodesic triangle

Definition 2.2.5. ([33]) A *comparison triangle* for geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ in the Euclidean plane $\mathbb{R}^2$ such that

$$d_{\mathbb{R}^2}(\overline{x}_i, \overline{x}_j) = d(x_i, x_j),$$

for $i, j \in \{1, 2, 3\}$, (see Figure 2.5).

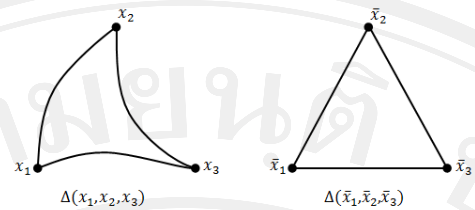


Figure 2.5 : Comparison triangle

We are ready to give the definition of the CAT(0) space as follows.

Definition 2.2.6. ([33]) A geodesic space is said to be a CAT(0) space if all geodesic triangles satisfy the following comparison axiom.

CAT(0) : Let $\triangle$ be a geodesic triangle in X and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the *CAT(0) inequality* if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$,

$$d(x,y) \leq d_{\mathbb{R}^2}(\overline{x},\overline{y}), \text{ see Figure 2.6.}$$

Figure 2.6 : CAT(0) inequality

Lemma 2.2.7. *Let z, x, y be points in a CAT(0) space X and $m[x,y]$ be the midpoint of the segment $[x,y]$, then the CAT(0) inequality implies*

$$d(z, m[x,y])^2 \leq \frac{1}{2}d(z,x)^2 + \frac{1}{2}d(z,y)^2 - \frac{1}{4}d(x,y)^2.$$

This is called the *(CN) inequality* of Bruhat and Tits [8]. In fact a geodesic space is a CAT(0) space if and only if it satisfies the (CN) inequality ([4]).

In 2008, Dhompongsa and Panyanak [15] introduced the following notations and also proved the properties that used quite often for studying the fixed point theory in CAT(0) spaces.

Lemma 2.2.8. *([15]) Let (X,d) be a CAT(0) space. Then*

(i) (X,d) is uniquely geodesic.

(ii) *Let* $x, y \in X$, $x \neq y$ *and* $z, w \in [x, y]$ *such that* $d(x, z) = d(x, w)$. *Then* $z = w$.

(iii) *Let* $x, y \in X$. *For each* $t \in [0, 1]$, *there exists a unique point* $z \in [x, y]$ *such that*

$$d(x, z) = td(x, y) \text{ and } d(y, z) = (1 - t)d(x, y).$$

We use the notation $(1 - t)x \oplus ty$ *for such the point* z, *(see Figure 2.7).*

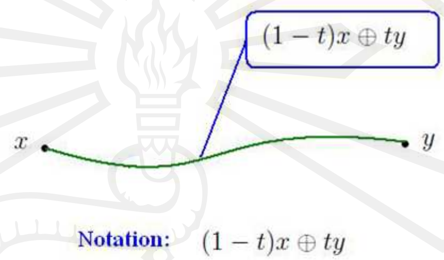


Figure 2.7

By using this notation, It is easy to verify that if $x, y \in X$ such that $x \neq y$ and $s, t \in [0, 1]$, then $(1 - t)x \oplus ty = (1 - s)x \oplus sy$ if and only if $s = t$.

Lemma 2.2.9. *([15]) Let* X *be a CAT(0) space and let* $x, y \in X$ *such that* $x \neq y$. *Then*

(i) $[x, y] = \{(1 - t)x \oplus ty : t \in [0, 1]\}$.

(ii) $d(x, z) + d(z, y) = d(x, y)$ *if and only if* $z \in [x, y]$.

(iii) *The mapping* $f : [0, 1] \to [x, y]$, $f(t) = (1 - t)x \oplus ty$ *is continuous and bijective.*

Lemma 2.2.10. *Let* X *be a CAT(0) space. Then*

(i) *For* $x, y, z \in X$ *and* $t \in [0, 1]$, *we have*

$$d((1 - t)x \oplus ty, z) \leq (1 - t)d(x, z) + td(y, z)$$

(ii) *For* $x, y, z \in X$ *and* $t \in [0, 1]$, *we have*

$$d((1 - t)x \oplus ty, z)^2 \leq (1 - t)d(x, z)^2 + td(y, z)^2 - t(1 - t)d(x, y)^2, \text{ (see figure 2.8).}$$

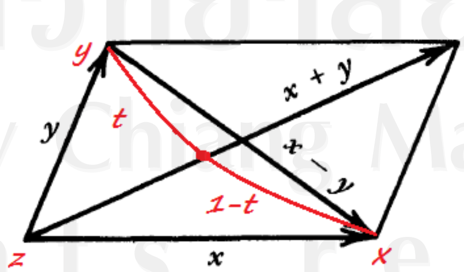


Figure 2.8

Recall that if X is a metric space and K is a nonempty subset of X. A subset H of K is said to be a *retract* of K if there exists a continuous mapping $R : K \to H$ with $R(x) = x$ for all $x \in H$. Any such mapping R is called a *retraction* of K. If R is nonexpansive, then R is called a *nonexpansive retraction* and H is a *nonexpansive retract* of K.

Lemma 2.2.11. *([4]) Let X be a complete CAT(0) space. If C is a nonempty closed convex subset of X, then for every $x \in X$, there exists a unique point $P(x) \in C$ such that $d(x, P(x)) = \inf\{d(x, y) : y \in C\}$. Moreover, the map $x \mapsto P(x)$ is a nonexpansive retraction from X onto C.*

In 1976, Lim [37] introduced a concept of convergence in a general metric space setting which he called $\triangle-$*convergence.* In 2008, Kirk and Panyanak [33] specialized Lim's concept to CAT(0) spaces and showed that many Banach space results which involve weak convergence have precise analogs in this setting.

Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x_n, x).$$

The *asymptotic radius* $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}$$

and the *asymptotic center* $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known (see, e.g., [14]) that in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point.

We now give the definition and some basic properties of $\triangle-$convergence.

Definition 2.2.12. ([33]) A sequence $\{x_n\}$ in X is said to $\triangle-$*converge* to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\triangle - lim_n x_n = x$ and call x the $\triangle - limit$ of $\{x_n\}$.

Lemma 2.2.13. *Let X be a complete CAT(0) space.*

(i) Every bounded sequence in X has a $\triangle-$convergent subsequence ([33]).

(ii) If C is a closed convex subset of X and if $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C([13]).

(iii) If $\{x_n\}$ is a bounded sequence in X with $A(\{x_n\}) = \{x\}$ and $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges, then $x = u$ ([15]).

Lemma 2.2.14. *([27, 41]) Let X be a complete CAT(0) space. Suppose that $\{t_n\}$ is a sequence in $[b, c]$ for some $b, c \in (0, 1)$ and $\{u_n\}$, $\{v_n\}$ are sequences in X such that*

(i) $\limsup_{n\to\infty} d(u_n, w) \le r$,

(ii) $\limsup_{n\to\infty} d(v_n, w) \le r$, *and*

(iii) $\lim_{n\to\infty} d((1 - t_n)u_n \oplus t_n v_n, w) = r$,

for some $r \ge 0$. Then $\lim_{n\to\infty} d(u_n, v_n) = 0$.