Chapter 3
Results on uniformly convex Banach Spaces

3.1 A Common Fixed Point Theorem

The existence of fixed points for asymptotic pointwise nonexpansive mappings in uniformly convex Banach spaces was proved by Kirk and Xu [34] as the following theorem.

Theorem 3.1.1. *Assume X is a uniformly convex Banach space and C is a bounded closed convex subset of X. Then every asymptotic pointwise nonexpansive mapping $T : C \to C$ has a fixed point. Moreover, the set of fixed points of T is closed and convex.*

The following result extends the existence theorem of Kirk and Xu.

Theorem 3.1.2. *Let X be a uniformly convex Banach space and C be a nonempty bounded closed convex subset of X. Then every commuting family $\mathcal{S}$ of asymptotic pointwise nonexpansive mappings on C has a nonempty closed convex common fixed point set.*

Proof. Let $T_1, T_2, ..., T_n \in \mathcal{S}$. By Theorem 3.1.1, $F(T_1)$ is a nonempty closed and convex subset of C. We assume that $A := \bigcap_{j=1}^{k-1} F(T_j)$ is nonempty closed and convex for some $k \in \mathbb{N}$ with $1 < k \leq n$. For $x \in A$ and $j \in \mathbb{N}$ with $1 \leq j < k$, we have

$$T_k(x) = T_k \circ T_j(x) = T_j \circ T_k(x).$$

Thus $T_k(x)$ is a fixed point of T_j, which implies that $T_k(x) \in A$, therefore A is invariant under T_k. Again, by Theorem 3.1.1, T_k has a fixed point in A, i.e.,

$$\bigcap_{j=1}^{k} F(T_j) = F(T_k) \bigcap A \neq \emptyset.$$

Also, the set is closed and convex. By induction, $\bigcap_{j=1}^{n} F(T_j) \neq \emptyset$. This shows that the set $\{F(T) : T \in \mathcal{S}\}$ has the finite intersection property. We note that C is weakly compact because X is reflexive. Since $F(T)$ is weakly closed for every $T \in \mathcal{S}$, we have $\bigcap_{T \in \mathcal{S}} F(T) \neq \emptyset$. This completes the proof. □

3.2 Weak Convergence and Strong Convergence Theorems

Let C be a nonempty subset of a metric space (X, d). We shall denote by $\mathcal{T}(C)$ the class of all asymptotic pointwise nonexpansive mappings from C into C. Let $T_1, ..., T_m \in \mathcal{T}(C)$. Then there exists, for each $i,\ i = 1, 2, ..., m$, a sequence of mappings $\alpha_{in} : C \to [0, \infty)$ such that for all $x, y \in C$, $n \in \mathbb{N}$ and $i = 1, 2, ..., m$,

$$d(T_i^n x, T_i^n y) \leq \alpha_{in}(x)d(x, y) \text{ and } \limsup_{n\to\infty} \alpha_{in}(x) \leq 1. \tag{3.2.1}$$

For each $n \in \mathbb{N}$ and $x \in C$, let $\alpha_n(x) = \max_{1\leq i\leq m} \alpha_{in}(x)$. Then we get that for all $x, y \in C$, $n \in \mathbb{N}$ and $i = 1, 2, ..., m$,

$$d(T_i^n x, T_i^n y) \leq \alpha_n(x)d(x, y) \text{ and } \limsup_{n\to\infty} \alpha_n(x) \leq 1. \tag{3.2.2}$$

Let $a_n(x) = \max\{\alpha_n(x), 1\}$. So we have that

$$d(T_i^n x, T_i^n y) \leq a_n(x)d(x, y), \ \lim_{n\to\infty} a_n(x) = 1 \text{ and } a_n(x) \geq 1, \tag{3.2.3}$$

for all $x, y \in C,\ n \in \mathbb{N},$ and $i = 1, 2, ..., m$. Define $b_n(x) = a_n(x) - 1$, then for each $x \in C$, we have $\lim_{n\to\infty} b_n(x) = 0$.

Definition 3.2.1. Define $\mathcal{T}_r(C)$ as a class of all mappings T in the class $\mathcal{T}(C)$ such that

$$\sum_{n=1}^{\infty} b_n(x) < \infty, \text{ and} \tag{3.2.4}$$

$$a_n \text{ is a bounded function for every } n \in \mathbb{N}. \tag{3.2.5}$$

Let $T_1, ..., T_m \in \mathcal{T}_r(C)$ and let $t \in (0, 1)$ and $\{n_k\}$ be an increasing sequence of natural numbers. Let $x_1 \in C$ and define a sequence $\{x_k\}$ in C as:

$$\begin{aligned} x_{k+1} &= (1-t)x_k + tT_m^{n_k} y_{(m-1)k}, \\ y_{(m-1)k} &= (1-t)x_k + tT_{m-1}^{n_k} y_{(m-2)k}, \\ y_{(m-2)k} &= (1-t)x_k + tT_{m-2}^{n_k} y_{(m-3)k}, \\ &\vdots \\ y_{2k} &= (1-t)x_k + tT_2^{n_k} y_{1k}, \\ y_{1k} &= (1-t)x_k + tT_1^{n_k} y_{0k}, \\ y_{0k} &= x_k, \ \ k \in \mathbb{N}. \end{aligned} \tag{3.2.6}$$

We say that the sequence $\{x_k\}$ in (3.2.6) is *well-defined* if $\limsup_{k\to\infty} a_{n_k}(x_k) = 1$.

As in [35], we observe that $\lim_{k\to\infty} a_k(x) = 1$ for every $x \in C$. Hence we can always choose a subsequence $\{a_{n_k}\}$ which makes $\{x_k\}$ well-defined.

One choice of $\{a_{n_k}\}$ that we can choose is as the following:
Starting by arbitrary $x_1 \in C$ and letting $\epsilon = \frac{1}{2^1}$, there exists a positive integer n_1 such that

$$|\, a_n(x_1) - 1| < \frac{1}{2^1} \text{ for all } n \geq n_1$$

since $\lim_{n\to\infty} a_n(x_1) = 1$. For $x_2 \in C$ and $\epsilon = \frac{1}{2^2}$, there exists an integer $n_2 > n_1$ such that

$$|\, a_n(x_2) - 1| < \frac{1}{2^2} \text{ for all } n \geq n_2$$

since $\lim_{n\to\infty} a_n(x_2) = 1$. Repeting the same argument, we have that for $x_k \in C$ and $\epsilon = \frac{1}{2^k}$, there exists an integer $n_k > n_{k-1} > ... > n_2 > n_1$ such that

$$|\, a_n(x_k) - 1| < \frac{1}{2^k} \text{ for all } n \geq n_k$$

since $\lim_{n\to\infty} a_n(x_k) = 1$. This implies that $\limsup_{k\to\infty} a_{n_k}(x_k) = 1$.

Before proving the main convergence theorems we give the following definitions and some useful lemmas.

Definition 3.2.2. ([7]) A strictly increasing sequence $\{n_i\} \subset \mathbb{N}$ is called *quasi-periodic* if the sequence $\{n_{i+1} - n_i\}$ is bounded, or equivalently if there exists a number $q \in \mathbb{N}$ such that any block of q consecutive natural numbers must contain a term of the sequence $\{n_i\}$. The smallest of such numbers q will be called a *quasi-period* of $\{n_i\}$.

Example 3.2.3. Quasi-periodic sequences.

(1) the sequence $\{1, 3, 5, 7, ..., 2n + 1, ...\}$ is quasi-periodic with quasi-period 2,

(2) the sequence $\{3, 6, 9, 12, ..., 3n, ...\}$ is quasi-periodic with quasi-period 3,

(3) the sequence $\{1, 4, 9, 16, ..., n^2, ...\}$ is not quasi-periodic.

Lemma 3.2.4. *([48]) Let $\{a_n\}$ and $\{u_n\}$ be sequences of nonnegative real numbers satisfy:*

$$a_{n+1} \leq (1 + u_n)a_n, \text{ for all } n \in \mathbb{N}, \text{ and } \sum_{n=1}^{\infty} u_n < \infty.$$

Then $\lim_{n\to\infty} a_n$ exists and if $\liminf_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 3.2.5. *([7]) Suppose $\{r_k\}$ is a bounded sequence of real numbers and $\{d_{k,n}\}$ is a doubly-index sequence of real numbers which satisfy:*

$$\limsup_{k\to\infty} \limsup_{n\to\infty} d_{k,n} \leq 0, \quad \text{and} \quad r_{k+n} \leq r_k + d_{k,n}$$

for each $k, n \in \mathbb{N}$. Then $\{r_k\}$ converges to an $r \in \mathbb{R}$.

Lemma 3.2.6. *([46, 53]) Let X be a uniformly convex Banach space and let $\{t_n\}$ be a sequence in $[a,b]$ for some $a,b \in (0,1)$. Suppose that $\{u_n\}$ and $\{v_n\}$ are sequences in X which satisfy:*

(i) $\limsup_{n\to\infty} \|u_n\| \le r$;

(ii) $\limsup_{n\to\infty} \|v_n\| \le r$;

(iii) $\lim_{n\to\infty} \|t_n u_n + (1-t_n)v_n\| = r$,

for some $r \ge 0$. Then $\lim_{n\to\infty} \|u_n - v_n\| = 0$.

Lemma 3.2.7. *([47]) Let X be a Banach space which satisfies Opial's condition and $\{x_n\}$ be a sequence in X. Let $u, v \in X$ be such that $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exist. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which converge weakly to u and v, respectively, then $u = v$.*

The following two lemmas are proved by Kolowski ([35]).

Lemma 3.2.8. *Let X be a uniformly convex Banach space, C be a nonempty closed convex subset of X and let $T \in \mathcal{T}_r(C.)$ If $\lim_{n\to\infty} \|T(x_n) - x_n\| = 0$ then for any $m \in \mathbb{N}$, $\lim_{n\to\infty} \|T^m(x_n) - x_n\| = 0$.*

Lemma 3.2.9. *Let X be a uniformly convex Banach space with the Opial property and let C be a nonempty closed convex subset of X. Let $T \in \mathcal{T}_r(C)$ and let $\omega \in X$, $\{x_n\} \subset X$, be such that $x_n \rightharpoonup \omega$, and $\lim_{n\to\infty} \|T(x_n) - x_n\| = 0$. Then $\omega \in F(T)$.*

To prove our main convergence theorems, we need to construct the following lemmas.

Lemma 3.2.10. *Let X be a Banach space, C be a nonempty closed convex subset of X and let $T_1, ..., T_m \in \mathcal{T}_r(C)$. Let $t \in (0,1)$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (3.2.6) is well-defined. Assume that $F = \bigcap_{i=1}^m F(T_i) \neq \emptyset$. Then for each $p \in F$, there are sequences of nonnegative real numbers $\{\gamma_k\}$ and $\{\delta_k\}$ (depending on p) such that $\sum_{k=1}^\infty \gamma_k < \infty$, $\sum_{k=1}^\infty \delta_k < \infty$ and the following statements hold:*

(i) $\|y_{ik} - p\| \le (1+\gamma_k)^i \|x_k - p\|$, *for all* $i = 1, 2, ..., m-1$;

(ii) $\|x_{k+1} - p\| \le (1+\delta_k)\|x_k - p\|$;

(iii) $\lim_{k\to\infty} \|x_k - p\|$ *exists.*

Proof. (i) Let $p \in F$ and $\gamma_k = b_{n_k}(p)$ for all $k \in \mathbb{N}$. Then $\sum_{k=1}^\infty \gamma_k < \infty$. Consider

$$\begin{aligned}\|y_{1k} - p\| &= \|(1-t)x_k + tT_1^{n_k}x_k - p\| \\ &= \|(1-t)(x_k - p) + t(T_1^{n_k}x_k - p)\| \\ &\le (1-t)\|x_k - p\| + t\|T_1^{n_k}x_k - p\| \\ &= (1-t)\|x_k - p\| + t\|T_1^{n_k}x_k - T_1^{n_k}p\|\end{aligned}$$

$$
\begin{aligned}
&\leq (1-t)\|x_k - p\| + t(1 + b_{n_k}(p))\|x_k - p\| \\
&= (1-t)\|x_k - p\| + t(1+\gamma_k)\|x_k - p\| \\
&= (1 + t\gamma_k)\|x_k - p\| \\
&\leq (1+\gamma_k)\|x_k - p\|.
\end{aligned}
$$

Suppose that $\|y_{jk} - p\| \leq (1+\gamma_k)^j \|x_k - p\|$ holds for some $1 \leq j \leq m-2$. Then

$$
\begin{aligned}
\|y_{(j+1)k} - p\| &= \|(1-t)x_k + tT_{j+1}^{n_k} y_{jk} - p\| \\
&= \|(1-t)(x_k - p) + t(T_{j+1}^{n_k} y_{jk} - p)\| \\
&\leq (1-t)\|x_k - p\| + t\|T_{j+1}^{n_k} y_{jk} - p\| \\
&= (1-t)\|x_k - p\| + t\|T_{j+1}^{n_k} y_{jk} - T_{j+1}^{n_k} p\| \\
&\leq (1-t)\|x_k - p\| + t(1+\gamma_k)\|y_{jk} - p\| \\
&\leq (1-t)\|x_k - p\| + t(1+\gamma_k)^{j+1}\|x_k - p\| \\
&= \left[1 - t + t\left(1 + \sum_{r=1}^{j+1} \frac{(j+1)j\cdots(j+2-r)}{r!}\gamma_k^r\right)\right]\|x_k - p\| \\
&\leq \left(1 + \sum_{r=1}^{j+1} \frac{(j+1)j\cdots(j+2-r)}{r!}\gamma_k^r\right)\|x_k - p\| \\
&= (1+\gamma_k)^{j+1}\|x_k - p\|.
\end{aligned}
$$

By mathematical induction, we have

$$
\|y_{ik} - p\| \leq (1+\gamma_k)^i \|x_k - p\|, \text{ for all } i = 1, 2, ..., m-1. \tag{3.2.7}
$$

(ii) By using (3.2.7) we obtain that

$$
\begin{aligned}
\|x_{k+1} - p\| &= \|(1-t)x_k + tT_m^{n_k} y_{(m-1)k} - p\| \\
&= \|(1-t)(x_k - p) + t(T_m^{n_k} y_{(m-1)k} - p)\| \\
&\leq (1-t)\|x_k - p\| + t\|T_m^{n_k} y_{(m-1)k} - p\| \\
&= (1-t)\|x_k - p\| + t\|T_m^{n_k} y_{(m-1)k} - T_m^{n_k} p\| \\
&\leq (1-t)\|x_k - p\| + t(1+\gamma_k)\|y_{(m-1)k} - p\| \\
&\leq (1-t)\|x_k - p\| + t(1+\gamma_k)^m\|x_k - p\| \\
&= (1 - t + t(1+\gamma_k)^m)\|x_k - p\| \\
&= \left[1 - t + t\left(1 + \sum_{r=1}^{m} \frac{m(m-1)\cdots(m-r+1)}{r!}\gamma_k^r\right)\right]\|x_k - p\| \\
&\leq \left(1 + \sum_{r=1}^{m} \frac{m(m-1)\cdots(m-r+1)}{r!}\gamma_k^r\right)\|x_k - p\| \\
&= (1+\delta_k)\|x_k - p\|
\end{aligned}
$$

where $\delta_k = \sum_{r=1}^{m} \frac{m(m-1)\cdots(m-r+1)}{r!}\gamma_k^r$. Since $\sum_{k=1}^{\infty} \gamma_k < \infty$, $\sum_{k=1}^{\infty} \delta_k < \infty$.

(iii) follows directly from part (ii) and Lemma 3.2.4. □

Lemma 3.2.11. *Let X be a uniformly convex Banach space, C be a nonempty closed convex subset of X and $T_1, ..., T_m \in \mathcal{T}_r(C)$. Let $t \in (0,1)$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (3.2.6) is well-defined. Assume that $F = \bigcap_{i=1}^m F(T_i) \neq \emptyset$. Then*

(i) $\lim_{k\to\infty} \|x_k - T_i^{n_k} y_{(i-1)k}\| = 0$, *for all* $i = 1, 2, ..., m$;

(ii) $\lim_{k\to\infty} \|x_k - T_i^{n_k} x_k\| = 0$, *for all* $i = 1, 2, ..., m$;

(iii) If the set $\mathcal{J} = \{k \in \mathbb{N} : n_{k+1} = 1 + n_k\}$ *is quasi-periodic, then* $\lim_{k\to\infty} \|x_k - T_i x_k\| = 0$, *for all* $i = 1, 2, ..., m$.

Proof. (i) Let $p \in F$. By Lemma 3.2.10 (iii), we have $\lim_{k\to\infty} \|x_k - p\|$ exists. Let

$$\lim_{k\to\infty} \|x_k - p\| = c. \tag{3.2.8}$$

By Lemma 3.2.10 (i), we have that

$$\|y_{jk} - p\| \leq (1+\gamma_k)^j \|x_k - p\|, \text{ for all } j = 1, 2, ..., m-1 \tag{3.2.9}$$

So we get

$$\limsup_{k\to\infty} \|y_{jk} - p\| \leq c, \text{ for all } j = 1, 2, ..., m-1. \tag{3.2.10}$$

Note that

$$\begin{aligned}
\|x_{k+1} - p\| &= \|(1-t)x_k + tT_m^{n_k} y_{(m-1)k} - p\| \\
&= \|(1-t)(x_k - p) + t(T_m^{n_k} y_{(m-1)k} - p)\| \\
&\leq (1-t)\|x_k - p\| + t\|T_m^{n_k} y_{(m-1)k} - p\| \\
&= (1-t)\|x_k - p\| + t\|T_m^{n_k} y_{(m-1)k} - T_m^{n_k} p\| \\
&\leq (1-t)\|x_k - p\| + t(1+\gamma_k)\|y_{(m-1)k} - p\| \\
&\vdots \\
&\leq (1-t^{m-j})(1+\gamma_k)^{m-j}\|x_k - p\| \\
&\quad + t^{m-j}(1+\gamma_k)^{m-j}\|y_{jk} - p\|.
\end{aligned}$$

Thus

$$\|x_k - p\| \leq \frac{\|x_k - p\|}{t^{m-j}} - \frac{\|x_{k+1} - p\|}{t^{m-j}(1+\gamma_k)^{m-j}} + \|y_{jk} - p\|.$$

It follows that

$$c \leq \liminf_{k\to\infty} \|y_{jk} - p\|, \text{ for all } j = 1, 2, ..., m-1. \tag{3.2.11}$$

From (3.2.10) and (3.2.11), we have

$$\lim_{k\to\infty} \|y_{jk} - p\| = c, \text{ for all } j = 1, 2, ..., m-1.$$

That is,

$$\lim_{k\to\infty}\|(1-t)(x_k-p)+t(T_j^{n_k}y_{(j-1)k}-p)\|=\lim_{k\to\infty}\|y_{jk}-p\|=c,$$

for all $j=1,2,...,m-1$. We also obtain from (3.2.10) that

$$\limsup_{k\to\infty}\|T_j^{n_k}y_{(j-1)k}-p\|\le c,\ \text{ for each } j=1,2,...,m-1.$$

By Lemma 3.2.6, we get that

$$\lim_{k\to\infty}\|T_j^{n_k}y_{(j-1)k}-x_k\|=0,\ \text{ for each } j=1,2,...,m-1. \tag{3.2.12}$$

For the case $j=m$, we have by Lemma 3.2.10 (i) that

$$\begin{aligned}\|T_m^{n_k}y_{(m-1)k}-p\| &= \|T_m^{n_k}y_{(m-1)k}-T_m^{n_k}p\|\\ &\le (1+\gamma_k)\|y_{(m-1)k}-p\|\\ &\le (1+\gamma_k)^m\|x_k-p\|.\end{aligned}$$

But since $\lim_{k\to\infty}\|x_k-p\|=c$, then

$$\limsup_{k\to\infty}\|T_m^{n_k}y_{(m-1)k}-p\|\le c.$$

Moreover,

$$\lim_{k\to\infty}\|(1-t)(x_k-p)+t(T_m^{n_k}y_{(m-1)k}-p)\|=\lim_{k\to\infty}\|x_{k+1}-p\|=c.$$

Again, by Lemma 3.2.6, we get that

$$\lim_{k\to\infty}\|T_m^{n_k}y_{(m-1)k}-x_k\|=0. \tag{3.2.13}$$

Thus, (3.2.12) and (3.2.13) imply that

$$\lim_{k\to\infty}\|T_i^{n_k}y_{(i-1)k}-x_k\|=0,\ \text{ for each } i=1,2,...,m. \tag{3.2.14}$$

(ii) For $j=1$, we have by part (i) that

$$\lim_{k\to\infty}\|T_1^{n_k}x_k-x_k\|=0. \tag{3.2.15}$$

If $j=2,3,...,m$, then we have

$$\begin{aligned}\|T_j^{n_k}x_k-x_k\| &\le \|T_j^{n_k}x_k-T_j^{n_k}y_{(j-1)k}\|+\|T_j^{n_k}y_{(j-1)k}-x_k\|\\ &\le a_{n_k}(x_k)\|x_k-y_{(j-1)k}\|+\|T_j^{n_k}y_{(j-1)k}-x_k\|\\ &\le a_{n_k}(x_k)t\|x_k-T_{j-1}^{n_k}y_{(j-2)k}\|+\|T_j^{n_k}y_{(j-1)k}-x_k\|.\end{aligned}$$

By part (i) and $\limsup_{k\to\infty} a_{n_k}(x_k) = 1$, we get

$$\limsup_{k\to\infty} \|T_j^{n_k} x_k - x_k\| = 0, \text{ for } j = 2, 3, ..., m. \tag{3.2.16}$$

By (3.2.15) and (3.2.16), we have

$$\lim_{k\to\infty} \|T_j^{n_k} x_k - x_k\| = 0, \text{ for all } j = 1, 2, ..., m, \tag{3.2.17}$$

which completes the prove of (ii). Observe that (3.2.13) and the construction of the sequence $\{x_k\}$ yield

$$\lim_{k\to\infty} \|x_{k+1} - x_k\| = 0. \tag{3.2.18}$$

(iii) We will show that

$$\lim_{k\to\infty} \|T_j x_k - x_k\| = 0, \text{ for all } j = 1, 2, ..., m. \tag{3.2.19}$$

It is enough to prove that $\|T_j x_k - x_k\| \to 0$ as $k \to \infty$ through $\mathcal{J}$. Indeed, let q be a quasi-period of $\mathcal{J}$ and $\varepsilon > 0$ be given. Then there exists $N_1 \in \mathbb{N}$ such that

$$\lim_{k\to\infty} \|T_j x_k - x_k\| < \frac{\varepsilon}{3}, \text{ for all } k \in \mathcal{J} \text{ such that } k \geq N_1. \tag{3.2.20}$$

By the quasi-periodicity of $\mathcal{J}$, for each $l \in \mathbb{N}$, there exists $i_l \in \mathcal{J}$ such that $|l - i_l| \leq q$. Without loss of generality, we can assume that $l \leq i_l \leq l+q$ (the proof for the other case is identical). Let $M = \sup\{a_1(x) : x \in C\}$. Then $M \geq 1$. Since $\lim_{l\to\infty} \|x_{l+1} - x_l\| = 0$ by (3.2.18), there exists $N_2 \in \mathbb{N}$ such that

$$\|x_{l+1} - x_l\| < \frac{\varepsilon}{3qM}, \text{ for all } l \geq N_2. \tag{3.2.21}$$

This implies that for all $l \geq \mathbb{N}_2$,

$$\|x_{i_l} - x_l\| \leq \|x_{i_l} - x_{i_l - 1}\| + ... + \|x_{l+1} - x_l\| \leq q\left(\frac{\varepsilon}{3qM}\right) = \frac{\varepsilon}{3M}. \tag{3.2.22}$$

By the definition of T, we have

$$\|T_j x_{i_l} - T_j x_l\| \leq M\|x_{i_l} - x_l\| \leq M\left(\frac{\varepsilon}{3M}\right) = \frac{\varepsilon}{3}. \tag{3.2.23}$$

Let $N = \max\{N_1, N_2\}$. Then for $l \geq N$, we have from (3.2.20), (3.2.22) and (3.2.23) that

$$\|x_l - T_j x_l\| \leq \|x_l - x_{i_l}\| + \|x_{i_l} - T_j x_{i_l}\| + \|T_j x_{i_l} - T_j x_l\| < \frac{\varepsilon}{3M} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \leq \varepsilon. \tag{3.2.24}$$

To prove that $\|T_j x_k - x_k\| \to 0$ as $k \to \infty$ through $\mathcal{J}$. Since $\mathcal{J} = \{k \in \mathbb{N} : n_{k+1} = n_k + 1\}$ is quasi-periodic, for each $k \in \mathcal{J}$, we have

$$\begin{aligned}\|x_k - T_j x_k\| &\le \|x_k - x_{k+1}\| + \|x_{k+1} - T_j^{n_{k+1}} x_{k+1}\| + \|T_j^{n_{k+1}} x_{k+1} - T_j^{n_{k+1}} x_k\| \\ &\quad + \|T_j^{n_k+1} x_k - T_j x_k\| \\ &\le \|x_k - x_{k+1}\| + \|x_{k+1} - T_j^{n_{k+1}} x_{k+1}\| + a_{n_{k+1}}(x_{k+1})\|x_{k+1} - x_k\| \\ &\quad + a_1(x_k)\|T_j^{n_k} x_k - x_k\|.\end{aligned}$$

This, together with (3.2.17) and (3.2.18), we can obtain that $\|T_j x_k - x_k\| \to 0$ as $k \to \infty$ through $\mathcal{J}$. □

Theorem 3.2.12. *Let X be a uniformly convex Banach space with the Opial property and C be a nonempty closed convex subset of X. Let $T_1, ..., T_m \in \mathcal{T}_r(C)$ be such that $F = \bigcap_{i=1}^m F(T_i) \neq \emptyset$. Let $t \in (0,1)$ and $\{n_k\} \subset \mathbb{N}$ be such that the sequence $\{x_k\}$ in (3.2.6) is well-defined. If the set $\mathcal{J} = \{k : n_{k+1} = 1 + n_k\}$ is quasi-periodic, then the sequence $\{x_k\}$ converges weakly to a common fixed point of the family $\{T_1, T_2, ..., T_m\}$.*

Proof. We have by Lemma 3.2.10 (iii) that $\lim_{n\to\infty} \|x_k - p\|$ exists for every $p \in F$. This implies that the sequence $\{x_n\}$ is bounded. Since the Banach space X is uniformly convex, it is reflexive. By Theorem 2.1.35, $\{x_n\}$ has a weakly convergent subsequence.

Next, we shall prove that $\{x_n\}$ has a unique subsequential limit in F. For this, we suppose that the subsequences $\{x_{m_j}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$ converge weakly to u and v, respectively.

Since we have by Lemma 3.2.11 (iii) that $\lim\limits_{k\to\infty} \|x_k - T_i x_k\| = 0$,

$$\lim_{k\to\infty} \|x_{m_j} - T_i x_{m_j}\| = 0 = \lim_{k\to\infty} \|x_{n_j} - T_i x_{n_j}\|, \text{ for all } i = 1, 2, ..., m.$$

It follows from Lemma 3.2.9 that $u, v \in F(T_i)$ for all $i = 1, 2, ..., m$. So $u, v \in F$. Consequently, $\lim\limits_{n\to\infty} \|x_k - u\|$ and $\lim\limits_{n\to\infty} \|x_k - v\|$ exist. By Lemma 3.2.7, we obtain that $u = v$. This implies that the sequence $\{x_n\}$ itself converges weakly to a common fixed point of the family $\{T_1, T_2, ..., T_m\}$ which completes the proof. □

Theorem 3.2.13. *Let X be a uniformly convex Banach space and C be a nonempty closed convex subset of X. Let $T_1, ..., T_m \in \mathcal{T}_r(C)$ be such that T_i^l is semi-compact for some $i \in \{1, 2, ..., m\}$ and $l \in \mathbb{N}$. Let $t \in (0,1)$ and $\{n_k\} \subset \mathbb{N}$ be such that the sequence $\{x_k\}$ in (3.2.6) is well-defined. Suppose that $F = \bigcap_{i=1}^m F(T_i) \neq \emptyset$ and the set $\mathcal{J} = \{k : n_{k+1} = 1 + n_k\}$ is quasi-periodic, then the sequence $\{x_k\}$ converges strongly to a common fixed point of the family $\{T_1, T_2, ..., T_m\}$.*

Proof. By Lemma 3.2.11 (ii) we have

$$\lim_{k\to\infty} \|x_k - T_i x_k\| = 0, \text{ for } i = 1, 2, ..., m. \tag{3.2.25}$$

Let $i \in \{1, 2, ..., m\}$ be such that T_i^l is semi-compact. Thus, by Lemma 3.2.8,

$$\lim_{k\to\infty} \|x_k - T_i^l x_k\| = 0.$$

We can also find a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ such that $\lim_{j\to\infty} x_{k_j} = p \in C$. Hence, from (3.2.25), we have

$$\|p - T_i p\| = \lim_{j\to\infty} \|x_{k_j} - T_i x_{k_j}\| = 0, \text{ for all } i = 1, 2, ..., m.$$

Thus $p \in F$. But since $\lim_{k\to\infty} \|x_k - p\|$ exists, then the sequence $\{x_k\}$ must itself converges to p. This completes the proof. □

To prove the strong convergence of the sequence $\{x_n\}$ defined by (3.2.6) whenever $\{T_1, ..., T_m\} \subset \mathcal{T}_r(C)$ satisfies Condition (A''), we need to construct some lemmas.

Lemma 3.2.14. *Let X be a Banach space, C be a nonempty closed convex subset of X and let $T_1, ..., T_m \in \mathcal{T}_r(C)$ be such that $F = \bigcap_{i=1}^m F(T_i) \neq \emptyset$. Let $t \in (0,1)$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (3.2.6) is well-defined. Assume that $\sum_{k=1}^\infty \sup_{x\in C} b_{n_k}(x) < \infty$. Then there exists a sequence $\{v_k\}$ in $[0,\infty)$ and a nonnegative real number M such that $\sum_{k=1}^\infty v_k < \infty$ and the following statements hold for all $p \in F$:*

(i) $\|x_{k+1} - p\| \leq (1 + v_k)^m \|x_k - p\|$, *for all* $k \in \mathbb{N}$;

(ii) $\|x_{k+l} - p\| \leq M\|x_k - p\|$, *for all* $k, l \in \mathbb{N}$;

Proof. Let $p \in F$.

To prove (i), we let $v_k = \sup_{x\in C} b_{n_k}(x)$ for all $k \in \mathbb{N}$.
Since $\sum_{k=1}^\infty \sup_{x\in C} b_{n_k}(x) < \infty$, then we get $\sum_{k=1}^\infty v_k < \infty$. Consider

$$\begin{aligned}
\|y_{1k} - p\| &= \|(1-t)x_k + tT_1^{n_k} x_k - p\| \\
&= \|(1-t)(x_k - p) + t(T_1^{n_k} x_k - p)\| \\
&\leq (1-t)\|x_k - p\| + t\|T_1^{n_k} x_k - p\| \\
&= (1-t)\|x_k - p\| + t\|T_1^{n_k} x_k - T_1^{n_k} p\| \\
&\leq (1-t)\|x_k - p\| + t(1 + v_k)\|x_k - p\| \\
&= (1 + tv_k)\|x_k - p\| \\
&\leq (1 + v_k)\|x_k - p\|.
\end{aligned}$$

Suppose that $\|y_{jk} - p\| \leq (1 + v_k)^j \|x_k - p\|$ holds for some $j = 1, 2, ..., m-2$. Then

$$\begin{aligned}
\|y_{(j+1)k} - p\| &= \|(1-t)x_k + tT_{j+1}^{n_k} y_{jk} - p\| \\
&= \|(1-t)(x_k - p) + t(T_{j+1}^{n_k} y_{jk} - p)\| \\
&\leq (1-t)\|x_k - p\| + t\|T_{j+1}^{n_k} y_{jk} - p\| \\
&\leq (1-t)\|x_k - p\| + t(1 + v_k)\|y_{jk} - p\| \\
&\leq (1-t)\|x_k - p\| + t(1 + v_k)^{j+1}\|x_k - p\|
\end{aligned}$$

$$= \left[1 - t + t\left(1 + \sum_{r=1}^{j+1} \frac{(j+1)j\cdots(j+2-r)}{r!} v_k^r\right)\right] \|x_k - p\|$$
$$\leq \left(1 + \sum_{r=1}^{j+1} \frac{(j+1)j\cdots(j+2-r)}{r!} \gamma_k^r\right) \|x_k - p\|$$
$$= (1+v_k)^{j+1}\|x_k - p\|.$$

By mathematical induction, we have

$$\|y_{jk} - p\| \leq (1+v_k)^j \|x_k - p\|, \text{ for all } j = 1, 2, ..., m-1. \tag{3.2.26}$$

This implies that

$$\begin{aligned}
\|x_{k+1} - p\| &= \|(1-t)x_k + tT_m^{n_k} y_{(m-1)k} - p\| \\
&= \|(1-t)(x_k - p) + t(T_m^{n_k} y_{(m-1)k} - p)\| \\
&\leq (1-t)\|x_k - p\| + t\|T_m^{n_k} y_{(m-1)k} - p\| \\
&\leq (1-t)\|x_k - p\| + t(1+v_k)\|y_{(m-1)k} - p\| \\
&\leq (1-t)\|x_k - p\| + t(1+v_k)^m\|x_k - p\| \\
&= [1 - t + t(1+v_k)^m]\,\|x_k - p\| \\
&= \left[1 - t + t\left(1 + \sum_{r=1}^{m} \frac{m(m-1)\cdots(m-r+1)}{r!}\, v_k^r\right)\right] \|x_k - p\| \\
&\leq \left(1 + \sum_{r=1}^{m} \frac{m(m-1)\cdots(m-r+1)}{r!}\, v_k^r\right) \|x_k - p\| \\
&= (1+v_k)^m\|x_k - p\|
\end{aligned}$$

which completes the proof of (i).

(ii) We observe that $(1+\alpha)^n \leq e^{n\alpha}$ holds for all $n \in \mathbb{N}$ and $\alpha \geq 0$. Thus, by part (i), for $k, l \in \mathbb{N}$, we have

$$\begin{aligned}
\|x_{k+l} - p\| &\leq (1 + v_{k+l-1})^m \|x_{k+l-1} - p\| \\
&\leq \exp\{m v_{k+l-1}\}\|x_{k+l-1} - p\| \\
&\leq \exp\{m \sum_{i=1}^{k+l-1} v_i\}\|x_k - p\| \\
&\leq \exp\{m \sum_{i=1}^{\infty} v_i\}\|x_k - p\|.
\end{aligned}$$

By setting $M = m\sum_{i=1}^{\infty} v_i$, we obtain (ii). □

Theorem 3.2.15. *Let X be a Banach space, C be a nonempty closed convex subset of X and $T_1, ..., T_m \in \mathcal{T}_r(C)$ be such that $F = \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Let $t \in (0,1)$ and $\{n_k\} \subset \mathbb{N}$*

be such that $\{x_k\}$ in (3.2.6) is well-defined. Assume that $\sum_{k=1}^{\infty} \sup_{x\in C} b_{n_k}(x) < \infty$. Then $\{x_k\}$ converges strongly to a point in F if and only if $\liminf_{k\to\infty} d(x_k, F) = 0$, where $d(x, F) = \inf_{p\in F} d(x, p)$.

Proof. The necessity is obvious. Now, we prove the sufficiency.

Assume that $\liminf_{k\to\infty} d(x_k, F) = 0$. We will show that the sequence $\{x_k\}$ converges strongly to a point in F.

Let $p \in F$, by Lemma 3.2.14 (i), we have

$$\|x_{k+1} - p\| \le (1 + v_k)^m \|x_k - p\|, \text{ for all } k \in \mathbb{N}.$$

This implies that

$$d(x_{k+1}, F) \le (1 + v_k)^m d(x_k, F) = \left(1 + \sum_{r=1}^{m} \frac{m(m-1)\cdots(m-r+1)}{r!} v_k^r\right) d(x_k, F).$$

Since $\sum_{k=1}^{\infty} v_k < \infty$, $\sum_{k=1}^{\infty}\sum_{r=1}^{m} \frac{m(m-1)\cdots(m-r+1)}{r!} v_k^r < \infty$. By Lemma 3.2.4, we get that $\lim_{k\to\infty} d(x_k, F) = 0$.

Next, we will show that $\{x_k\}$ is Cauchy. From Lemma 3.2.14 (ii), there exists $M > 0$ such that

$$\|x_{k+l} - p\| \le M\|x_k - p\|, \text{ for all } k, l \in \mathbb{N}. \tag{3.2.27}$$

Since $\lim_{k\to\infty} d(x_k, F) = 0$, then for each $\epsilon > 0$, there exists $k_1 \in \mathbb{N}$ such that

$$d(x_k, F) < \frac{\epsilon}{2M}, \quad \text{for all } k \ge k_1.$$

Hence, there exists $z_1 \in F$ such that

$$d(x_{k_1}, z_1) \le \frac{\epsilon}{2M}. \tag{3.2.28}$$

By (3.2.27) and (3.2.28), we have that for $k \ge k_1$,

$$\begin{aligned}
\|x_{k+l} - x_k\| &\le \|x_{k+l} - z_1\| + \|x_k - z_1\| \\
&\le M\|x_{k_1} - z_1\| + M\|x_{k_1} - z_1\| \\
&< 2M(\frac{\epsilon}{2M}) \\
&= \epsilon.
\end{aligned}$$

This shows that $\{x_k\}$ is Cauchy and so converges to some $\omega \in C$. We next show that $\omega \in F$. Let $L = \sup\{a_1(x) : x \in C\}$. For $\epsilon > 0$, there exists $k_2 \in \mathbb{N}$ such that

$$\|x_k - \omega\| < \frac{\epsilon}{2(1 + L)}, \text{ for all } k \ge k_2. \tag{3.2.29}$$

Since $\lim_{k\to\infty} d(x_k, F) = 0$, there exists $k_3 \ge k_2$ such that

$$d(x_k, F) < \frac{\epsilon}{2(1 + L)}, \text{ for all } k \ge k_3.$$

Thus, there exists $z_2 \in F$ such that

$$\|x_{k_3} - z_2\| < \frac{\epsilon}{2(1+L)}. \tag{3.2.30}$$

By (3.2.29) and (3.2.30), for each $i = 1, 2, ..., m$, we have

$$\begin{aligned}\|T_i\omega - \omega\| &\leq \|T_i\omega - T_ix_{k_3}\| + \|T_ix_{k_3} - z_2\| + \|z_2 - x_{k_3}\| + \|x_{k_3} - \omega\| \\ &\leq L\|x_{k_3} - \omega\| + L\|x_{k_3} - z_2\| + \|x_{k_3} - z_2\| + \|x_{k_3} - \omega\| \\ &\leq (1+L)\|x_{k_3} - \omega\| + (1+L)\|x_{k_3} - z_2\| \\ &< (1+L)\frac{\epsilon}{2(1+L)} + (1+L)\frac{\epsilon}{2(1+L)} \\ &= \epsilon.\end{aligned}$$

Since ϵ is arbitrary, $T_i\omega = \omega$ for all $i = 1, 2, ..., m$. Thus $\omega \in F$. This completes the proof. □

The next corollary follows immediately from Theorem 3.2.15

Corollary 3.2.16. *Let X be a Banach space, C be a nonempty closed and convex subset of X and $T_1, ..., T_m \in \mathcal{T}_r(C)$. Let $t \in (0, 1)$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (3.2.6) is well-defined. Assume that $F = \bigcap_{i=1}^m F(T_i) \neq \emptyset$ and $\sum_{k=1}^\infty \sup_{x\in C} b_{n_k}(x) < \infty$. Then the sequence $\{x_k\}$ converges strongly to a point in $p \in F$ if and only if there exists a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ which converges to p.*

Theorem 3.2.17. *Let X be a uniformly convex Banach space and C be a nonempty closed convex subset of X. Let $\{T_1, ..., T_m\} \subset \mathcal{T}_r(C)$ be satisfy Condition (A''). Let $t \in (0, 1)$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (3.2.6) is well-defined. Suppose that $\sum_{k=1}^\infty \sup_{x\in C} b_{n_k}(x) < \infty$, $F = \bigcap_{i=1}^m F(T_i) \neq \emptyset$ and the set $\mathcal{J} = \{k \in \mathbb{N} : n_{k+1} = 1+n_k\}$ is quasi-periodic. Then $\{x_k\}$ converges strongly to a common fixed point of the family $\{T_1, T_2, ..., T_m\}$.*

Proof. By Lemma 3.2.11 (iii), $\lim_{k\to\infty} \|x_k - T_ix_k\| = 0$, for all $i = 1, 2, ..., m$. By using Condition (A''), there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for $r \in (0, \infty)$ such that

$$\lim_{k\to\infty} f(d(x_k, F)) \leq \lim_{k\to\infty} \|x_k - T_jx_k\| = 0 \text{ for some } j = 1, ..., m.$$

This implies that $\lim_{k\to\infty} d(x_k, F) = 0$. The conclusion follows from Theorem 3.2.15. □