# Chapter 4
# The related results for CAT(0) Spaces

## 4.1 Common Fixed Point Theorems

In this section we ensure the existence of common fixed points for a family of asymptotic pointwise nonexpansive mappings in a CAT(0) space. Before proving the theorem some definitions have to be explained.

**Definition 4.1.1.** Let $M$ be a metric space and $\mathcal{F}$ be a family of subsets of $M$. Then we say that $\mathcal{F}$ defines a *convexity structure* on $M$ if it contains the closed balls and is stable by intersection.

**Definition 4.1.2.** ([24]) Let $\mathcal{F}$ be a convexity structure on $M$. We will say that $\mathcal{F}$ is *compact* if any family $\{A_\alpha\}_{\alpha\in\Gamma}$ of elements of $\mathcal{F}$ has a nonempty intersection provided $\bigcap_{\alpha\in F} A_\alpha \neq \emptyset$ for any finite subset $F \subset \Gamma$.

Let $X$ be a complete CAT(0) space. We denote by $\mathcal{C}(X)$ the family of all closed convex subsets of $X$. Then $\mathcal{C}(X)$ is a compact convexity structure on $X$ (see e.g., [24]).

The existence of fixed points of just one asymptotic pointwise nonexpansive mapping in CAT(0) space was proved by Hussain-Khamsi [24] as the following theorem.

**Theorem 4.1.3.** *Let $X$ be a complete CAT(0) space and $C$ be a nonempty bounded closed convex subset of $X$. Suppose that $T : C \to C$ is an asymptotic pointwise nonexpansive mapping. Then $F(T)$ is nonempty closed and convex.*

The following theorem is one of our main existence theorem. This theorem is an extension of Theorem 4.1.3.

**Theorem 4.1.4.** *Let $X$ be a complete CAT(0) space, $C$ be a nonempty bounded closed convex subset of $X$. Then for any commuting family $\mathcal{S}$ of asymptotic pointwise nonexpansive mappings on $C$, the set $\mathcal{F}(\mathcal{S})$ of common fixed points of $\mathcal{S}$ is a nonempty nonexpansive retract of $C$.*

*Proof.* Let $\mathcal{T}$ be the family of all finite intersections of the fixed point sets of mappings in the commutative family $\mathcal{S}$. We first show that $\mathcal{T}$ has the finite intersection property. Let $T_1, T_2, ..., T_n \in \mathcal{S}$. By Theorem 4.1.3, $F(T_1)$ is a nonempty closed and convex subset of $C$. We assume that $A := \bigcap_{j=1}^{k-1} F(T_j)$ is nonempty closed and convex for some $k \in \mathbb{N}$ with $1 < k \leq n$. For $x \in A$ and $j \in \mathbb{N}$ with $1 \leq j < k$, we have

$$T_k(x) = T_k \circ T_j(x) = T_j \circ T_k(x).$$

Thus $T_k(x)$ is a fixed point of $T_j$, which implies that $T_k(x) \in A$, therefore $A$ is invariant under $T_k$. Again, by Theorem 4.1.3, $T_k$ has a fixed point in $A$, i.e.,

$$\bigcap_{j=1}^{k} F(T_j) = F(T_k) \bigcap A \neq \emptyset.$$

By induction, $\bigcap_{j=1}^{n} F(T_j) \neq \emptyset$. Hence $\mathcal{T}$ has the finite intersection property since $\mathcal{C}(X)$ is compact,

$$\mathcal{F}(\mathcal{S}) = \bigcap_{T \in \mathcal{T}} T \neq \emptyset.$$

Obviously, the set is closed and convex. By Lemma 2.2.11 we obtain that the projection map $x \mapsto P(x)$ is also a nonexpansive retraction from $C$ onto $F(T.)$ This implies the desired conclusion. $\square$

As a consequence of Theorem 4.1.4, we obtain an analog of Bruck's theorem ([6]).

**Corollary 4.1.5.** *Let $X$ be a complete CAT(0) space, $C$ be a nonempty bounded closed convex subset of $X$. Then for any commuting family $\mathcal{S}$ of nonexpansive mappings on $C$, the set $\mathcal{F}(\mathcal{S})$ of common fixed points of $\mathcal{S}$ is a nonempty nonexpansive retract of $C$.*

## 4.2 $\triangle$-Convergence and Strong Convergence Theorems

Let $X$ be a complete CAT(0) space and $C$ be a closed convex subset of $X$. Let $T_1, ..., T_m \in \mathcal{T}_r(C)$ and let $t \in (0, 1)$ and $\{n_k\} \subset \mathbb{N}$ be an increasing sequence of natural numbers. Let $x_1 \in C$ and define a sequence $\{x_k\}$ in $C$ as:

$$\begin{aligned} x_{k+1} &= (1-t)x_k \oplus tT_m^{n_k} y_{(m-1)k}, \qquad (4.2.1)\\ y_{(m-1)k} &= (1-t)x_k \oplus tT_{m-1}^{n_k} y_{(m-2)k},\\ y_{(m-2)k} &= (1-t)x_k \oplus tT_{m-2}^{n_k} y_{(m-3)k},\\ &\vdots\\ y_{2k} &= (1-t)x_k \oplus tT_2^{n_k} y_{1k},\\ y_{1k} &= (1-t)x_k \oplus tT_1^{n_k} y_{0k},\\ y_{0k} &= x_k,\ k \in \mathbb{N}. \end{aligned}$$

We say that the sequence $\{x_k\}$ in (4.2.1) is *well-defined* if $\limsup_{k\to\infty} a_{n_k}(x_k) = 1$.

As in [35], we can choose a subsequence $\{a_{n_k}\}$ which makes the sequence $\{x_k\}$ well-defined since $\lim_{k\to\infty} a_k(x) = 1$ for every $x \in C$.

We may observe that the same method used in proving the results in uniformly convex Banach spaces can be used to obtain the analogous results for CAT(0) spaces by replacing the norm with the distance. For completeness of the thesis we write them in details.

**Lemma 4.2.1.** *Let $X$ be a complete CAT(0) space, $C$ be a nonempty closed convex subset of $X$ and let $T_1, ..., T_m \in \mathcal{T}_r(C)$. Let $t \in (0,1)$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (4.2.1) is well-defined. Assume that $F = \bigcap_{i=1}^m F(T_i) \neq \emptyset$. Then for each $p \in F$, there are sequences of nonnegative real numbers $\{\gamma_k\}$ and $\{\delta_k\}$ (depending on $p$) such that $\sum_{k=1}^\infty \gamma_k < \infty$, $\sum_{k=1}^\infty \delta_k < \infty$ and the following statements hold:*

*(i)* $d(y_{ik}, p) \leq (1+\gamma_k)^i d(x_k, p)$, *for all* $i = 1, 2, ..., m-1$;

*(ii)* $d(x_{k+1}, p) \leq (1+\delta_k) d(x_k, p)$;

*(iii)* $\lim_{k\to\infty} d(x_k, p)$ *exists.*

*Proof.* (i) Let $p \in F$ and $\gamma_k = b_{n_k}(p)$ for all $k \in \mathbb{N}$. Then $\sum_{k=1}^\infty \gamma_k < \infty$. Consider

$$\begin{aligned} d(y_{1k}, p) &= d((1-t)x_k \oplus tT_1^{n_k} x_k, p) \\ &\leq (1-t)d(x_k, p) + td(T_1^{n_k} x_k, p) \\ &\leq (1-t)d(x_k, p) + t(1 + b_{n_k}(p))d(x_k, p) \\ &= (1-t)d(x_k, p) + t(1+\gamma_k)d(x_k, p) \\ &\leq (1+\gamma_k)d(x_k, p). \end{aligned}$$

Suppose that $d(y_{jk}, p) \leq (1+\gamma_k)^j d(x_k, p)$ holds for some $j = 1, 2, ..., m-2$. Then

$$\begin{aligned} d(y_{(j+1)k}, p) &= d((1-t)x_k \oplus tT_{j+1}^{n_k} y_{jk}, p) \\ &\leq (1-t)d(x_k, p) + td(T_{j+1}^{n_k} y_{jk}, p) \\ &\leq (1-t)d(x_k, p) + t(1+\gamma_k)d(y_{jk}, p) \\ &\leq (1-t)d(x_k, p) + t(1+\gamma_k)^{j+1} d(x_k, p) \\ &= \left[1 - t + t\left(1 + \sum_{r=1}^{j+1} \frac{(j+1)j\cdots(j+2-r)}{r!}\gamma_k^r\right)\right] d(x_k, p) \\ &\leq \left(1 + \sum_{r=1}^{j+1} \frac{(j+1)j\cdots(j+2-r)}{r!}\gamma_k^r\right) d(x_k, p) \\ &= (1+\gamma_k)^{j+1} d(x_k, p). \end{aligned}$$

By mathematical induction, we have

$$d(y_{ik}, p) \leq (1+\gamma_k)^i d(x_k, p), \text{ for all } i = 1, 2, ..., m-1. \tag{4.2.2}$$

(ii) By using (4.2.2) we obtain that

$$\begin{aligned} d(x_{k+1}, p) &= d((1-t)x_k \oplus tT_m^{n_k} y_{(m-1)k}, p) \\ &\leq (1-t)d(x_k, p) + td(T_m^{n_k} y_{(m-1)k}, p) \\ &\leq (1-t)d(x_k, p) + t(1+\gamma_k)d(y_{(m-1)k}, p) \\ &\leq (1-t)d(x_k, p) + t(1+\gamma_k)^m d(x_k, p) \\ &= \left[1 - t + t\left(1 + \sum_{r=1}^{m} \frac{m(m-1)\cdots(m-r+1)}{r!}\gamma_k^r\right)\right] d(x_k, p) \\ &\leq \left(1 + \sum_{r=1}^{m} \frac{m(m-1)\cdots(m-r+1)}{r!}\gamma_k^r\right) d(x_k, p) \\ &= (1+\delta_k)d(x_k, p), \end{aligned}$$

where $\delta_k = \sum_{r=1}^{m} \frac{m(m-1)\cdots(m-r+1)}{r!} \gamma_k^r$. Since $\sum_{k=1}^{\infty} \gamma_k < \infty$, then $\sum_{k=1}^{\infty} \delta_k < \infty$.

(iii) follows from part (ii) and Lemma 3.2.4. $\square$

**Lemma 4.2.2.** *Let $X$ be a complete CAT(0) space, $C$ be a nonempty closed convex subset of $X$ and $T_1, ..., T_m \in \mathcal{T}_r(C)$. Let $t \in (0,1)$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (4.2.1) is well-defined. Assume that $F = \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Then*

*(i)* $\lim_{k\to\infty} d(x_k, T_i^{n_k} y_{(i-1)k}) = 0$, *for all* $i = 1, 2, ..., m$;

*(ii)* $\lim_{k\to\infty} d(x_k, T_i^{n_k} x_k) = 0$, *for all* $i = 1, 2, ..., m$,

*(iii) If the set* $\mathcal{J} = \{k \in \mathbb{N} : n_{k+1} = 1 + n_k\}$ *is quasi-periodic, then* $\lim_{k\to\infty} d(x_k, T_i x_k) = 0$, *for all* $i = 1, 2, ..., m$.

*Proof.* Let $p \in F$ By Lemma 4.2.1 (iii), we get $\lim_{k\to\infty} d(x_k, p)$ exists. Let

$$\lim_{k\to\infty} d(x_k, p) = c. \tag{4.2.3}$$

By Lemma 4.2.1 (i), we have

$$d(y_{jk}, p) \leq (1+\gamma_k)^j d(x_k, p), \text{ for all } j = 1, 2, ..., m-1 \tag{4.2.4}$$

By taking on both sides the lim sup as $k \to \infty$, we get that

$$\limsup_{k\to\infty} d(y_{jk}, p) \leq c, \text{ for } j = 1, 2, ..., m-1. \tag{4.2.5}$$

Since

$$\begin{aligned} d(x_{k+1}, p) &= d((1-t)x_k \oplus tT_m^{n_k} y_{(m-1)k}, p) \\ &\leq (1-t)d(x_k, p) + td(T_m^{n_k} y_{(m-1)k}, p) \\ &\leq (1-t)d(x_k, p) + t(1+\gamma_k)d(y_{(m-1)k}, p) \\ &\ \ \vdots \\ &\leq (1-t^{m-j})(1+\gamma_k)^{m-j} d(x_k, p) \\ &\quad + t^{m-j}(1+\gamma_k)^{m-j} d(y_{jk}, p). \end{aligned}$$

Then

$$d(x_k, p) \leq \frac{d(x_k, p)}{t^{m-j}} - \frac{d(x_{k+1}, p)}{t^{m-j}(1+\gamma_k)^{m-j}} + d(y_{jk}, p).$$

It follows that

$$c \leq \liminf_{k\to\infty} d(y_{jk}, p), \text{ for } j = 1, 2, ..., m-1. \tag{4.2.6}$$

From (4.2.5) and (4.2.6), we have

$$\lim_{k\to\infty} d(y_{jk}, p) = c, \text{ for each } j = 1, 2, ..., m-1.$$

That is,

$$\lim_{k\to\infty} d((1-t)x_k \oplus tT_j^{n_k} y_{(j-1)k}, p) = \lim_{k\to\infty} (y_{jk}, p) = c,$$

for all $j = 1, 2, ..., m - 1$. We also obtain from (4.2.5) that

$$\limsup_{k\to\infty} d(T_j^{n_k} y_{(j-1)k}, p) \le c, \text{ for each } j = 1, 2, ..., m - 1.$$

By Lemma 2.2.14, we get that

$$\lim_{k\to\infty} d(T_j^{n_k} y_{(j-1)k}, x_k) = 0, \text{ for each } j = 1, 2, ..., m - 1. \tag{4.2.7}$$

For the case $j = m$, we have by Lemma 4.2.1(i) that

$$\begin{aligned} d(T_m^{n_k} y_{(m-1)k}, p) &\le (1 + \gamma_k) d(y_{(m-1)k}, p) \\ &\le (1 + \gamma_k)^m d(x_k, p). \end{aligned}$$

Since $\lim_{k\to\infty} d(x_k, p) = c$,

$$\limsup_{k\to\infty} d(T_m^{n_k} y_{(m-1)k}, p) \le c.$$

Moreover,

$$\lim_{k\to\infty} d((1 - t)x_k \oplus tT_m^{n_k} y_{(m-1)k}, p) = \lim_{k\to\infty} d(x_{k+1}, p) = c.$$

Again, by Lemma 2.2.14, we get that

$$\lim_{k\to\infty} d(T_m^{n_k} y_{(m-1)k}, x_k) = 0. \tag{4.2.8}$$

Thus, (4.2.7) and (4.2.8) imply that

$$\lim_{k\to\infty} d(T_i^{n_k} y_{(i-1)k}, x_k) = 0, \text{ for each } i = 1, 2, ..., m. \tag{4.2.9}$$

(ii) For $j = 1$, we have by part (i) that

$$\lim_{k\to\infty} d(T_1^{n_k} x_k, x_k) = 0. \tag{4.2.10}$$

If $j = 2, 3, ..., m$, then we have

$$\begin{aligned} d(T_j^{n_k} x_k, x_k) &\le d(T_j^{n_k} x_k, T_j^{n_k} y_{(j-1)k}) + d(T_j^{n_k} y_{(j-1)k}, x_k) \\ &\le a_{n_k}(x_k) d(x_k, y_{(j-1)k}) + d(T_j^{n_k} y_{(j-1)k}, x_k) \\ &\le a_{n_k}(x_k) t d(x_k, T_{j-1}^{n_k} y_{(j-2)k}) + d(T_j^{n_k} y_{(j-1)k}, x_k). \end{aligned}$$

By part (i) and $\limsup_{k\to\infty} a_{n_k}(x_k) = 1$, we get

$$\limsup_{k\to\infty} d(T_j^{n_k} x_k, x_k) = 0 \text{ for } j = 2, 3, ..., m. \tag{4.2.11}$$

By (4.2.10) and (4.2.11), we have

$$\lim_{k\to\infty} d(T_j^{n_k} x_k, x_k) = 0, \text{ for all } j = 1, 2, ..., m, \tag{4.2.12}$$

which completes the prove of (ii).

Observe that (4.2.8) and the construction of the sequence $\{x_k\}$ yield

$$\lim_{k\to\infty} d(x_{k+1}, x_k) = 0. \tag{4.2.13}$$

(iii) We will show that

$$\lim_{k\to\infty} d(T_j x_k, x_k) = 0 \text{ for all } j = 1, 2, ..., m. \tag{4.2.14}$$

It is enough to prove that $d(T_j x_k, x_k) \to 0$ as $k \to \infty$ through $\mathcal{J}$. Indeed, let $q$ be a quasi-period of $\mathcal{J}$ and $\varepsilon > 0$ be given. Then there exists $N_1 \in \mathbb{N}$ such that

$$\lim_{k\to\infty} d(T_j x_k, x_k) < \frac{\varepsilon}{3}, \text{ for all } k \in \mathcal{J} \text{ such that } k \geq N_1. \tag{4.2.15}$$

By the quasi-periodicity of $\mathcal{J}$, for each $l \in \mathbb{N}$ there exists $i_l \in \mathcal{J}$ such that $|l - i_l| \leq q$. Without loss of generality, we can assume that $l \leq i_l \leq l+q$ (the proof for the other case is identical). Let $M = \sup\{a_1(x) : x \in C\}$. Then $M \geq 1$. Since $\lim_{l\to\infty} d(x_{l+1}, x_l) = 0$ by (4.2.13), there exists $N_2 \in \mathbb{N}$ such that

$$d(x_{l+1}, x_l) < \frac{\varepsilon}{3qM}, \text{ for all } l \geq N_2. \tag{4.2.16}$$

This implies that for all $l \geq \mathbb{N}_2$,

$$d(x_{i_l}, x_l) \leq d(x_{i_l}, x_{i_l - 1}) + ... + d(x_{l+1}, x_l) \leq q\left(\frac{\varepsilon}{3qM}\right) = \frac{\varepsilon}{3M}. \tag{4.2.17}$$

By the definition of $T$, we have

$$d(T_j x_{i_l}, T_j x_l) \leq M d(x_{i_l}, x_l) \leq M\left(\frac{\varepsilon}{3M}\right) = \frac{\varepsilon}{3}. \tag{4.2.18}$$

Let $N = \max\{N_1, N_2\}$. Then for $l \geq N$, we have from (4.2.15), (4.2.17) and (4.2.18) that

$$d(x_l, T_j x_l) \leq d(x_l, x_{i_l}) + d(x_{i_l}, T_j x_{i_l}) + d(T_j x_{i_l}, T_j x_l) < \frac{\varepsilon}{3M} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \leq \varepsilon.$$

To prove that $d(T_j x_k, x_k) \to 0$ as $k \to \infty$ through $\mathcal{J}$. Since $\mathcal{J} = \{k \in \mathbb{N} : n_{k+1} = n_k + 1\}$ is quasi-periodic then for each $k \in \mathcal{J}$, we have

$$\begin{aligned}
d(x_k, T_j x_k) \leq\ & d(x_k, x_{k+1}) + d(x_{k+1}, T_j^{n_{k+1}} x_{k+1}) + d(T_j^{n_{k+1}} x_{k+1}, T_j^{n_{k+1}} x_k) \\
& + d(T_j^{n_k+1} x_k, T_j x_k) \\
\leq\ & d(x_k, x_{k+1}) + d(x_{k+1}, T_j^{n_{k+1}} x_{k+1}) + a_{n_{k+1}}(x_{k+1}) d(x_{k+1}, x_k) \\
& + a_1(x_k) d(T_j^{n_k} x_k, x_k).
\end{aligned}$$

This, together with (4.2.12) and (4.2.13), we can obtain that $d(T_j x_k, x_k) \to 0$ as $k \to \infty$ through $\mathcal{J}$. □

For approximating fixed points in this space, we are interested in proving the $\triangle-$convergence and strong convergence of the sequence defined in (4.2.1). For the $\triangle-$convergence, its proof is different to that of weakly convergence in Banach spaces. More lemmas are needed to apply. All of them can be found in [35] (see also [24]).

**Lemma 4.2.3.** *Let $X$ be a complete CAT(0) space, $C$ be a nonempty closed convex subset of $X$ and let $T \in \mathcal{T}_r(C)$. If $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, then $\lim_{n\to\infty} d(x_n, T^l x_n) = 0$ for every $l \in \mathbb{N}$.*

The following lemma is the demiclosed principal for asymptotic pointwise nonexpansive mapping in CAT(0) spaces.

**Lemma 4.2.4.** *Let $X$ be a complete CAT(0) space, $C$ be a nonempty closed convex subset of $X$ and let $T \in \mathcal{T}_r(C)$. Suppose $\{x_n\}$ is a bounded sequence in $C$ such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and $\Delta - \lim_n x_n = w$. Then $Tw = w$.*

**Lemma 4.2.5.** *Let $X$ be a complete CAT(0) space, $C$ be a nonempty closed convex subset of $X$ and let $T : C \to C$ be an asymptotic pointwise nonexpansive mapping. Suppose $\{x_n\}$ is a bounded sequence in $C$ such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and $\{d(x_n, v)\}$ converges for each $v \in F(T)$, then $\omega_w(x_n) \subset F(T)$. Here $\omega_w(x_n) = \cup A(\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. Moreover, $\omega_w(x_n)$ consists of exactly one point.*

*Proof.* Let $u \in \omega_w(x_n)$. Then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 2.2.13 (i) and (ii), there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_n v_n = v \in C$. Since $\lim_{n\to\infty} d(x_n, Tx_n) = 0$,

$$\lim_{n\to\infty} d(v_n, Tv_n) = 0.$$

By Lemma 4.2.4, we have $v \in F(T)$. We also have $u = v$ by Lemma 2.2.13 (iii). This shows that $\omega_w(x_n) \subset F(T)$.

Finally we will show that $\omega_w(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and let $A(\{x_n\}) = \{x\}$. Since $u \in \omega_w(x_n) \subset F(T)$, $\{d(x_n, u)\}$ converges. Again, by Lemma 2.2.13 (iii), $x = u$. This completes the proof. □

Now, we are ready to prove our $\Delta-$convergence and strong convergence theorems.

**Theorem 4.2.6.** *Let $X$ be a complete CAT(0) space, $C$ be a nonempty closed convex subset of $X$ and $T_1, ..., T_m \in \mathcal{T}_r(C)$. Let $t \in (0, 1)$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (4.2.1) is well-defined. Suppose that $F = \bigcap_{i=1}^m F(T_i) \neq \emptyset$ and the set $\mathcal{J} = \{k \in \mathbb{N} : n_{k+1} = 1 + n_k\}$ is quasi-periodic. Then $\{x_k\}$ $\Delta-$converges to a common fixed point of the family $\{T_1, T_2, ..., T_m\}$.*

*Proof.* Let $p \in F$. By Lemma 4.2.1 (iii), we have $\lim_{k\to\infty} d(x_k, p)$ exists and hence $\{x_k\}$ is bounded. Since we have from Lemma 4.2.2 (iii) that $\lim_{k\to\infty} d(x_k, T_j x_k) = 0$ for all $j = 1, 2, ..., m$, it follows from Lemma 4.2.5 that $\omega_w(x_k) \subset F(T_j)$ for all $j = 1, 2, ..., m$.

And thus $\omega_w(x_k) \subset \bigcap_{j=1}^m F(T_j) = F$. Since $\omega_w(x_k)$ consists of exactly one point, $\{x_k\}$ $\Delta-$converges to an element of $F$ as desired. $\square$

**Theorem 4.2.7.** *Let $X$ be a complete CAT(0) space, $C$ be a nonempty closed convex subset of $X$ and $T_1, ..., T_m \in \mathcal{T}_r(C)$. Assume that $T_i^l$ is semi-compact for some $i \in \{1, 2, ..., m\}$ and $l \in \mathbb{N}$. Let $t \in (0, 1)$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (4.2.1) is well-defined. Suppose that $F = \bigcap_{i=1}^m F(T_i) \neq \emptyset$ and the set $\mathcal{J} = \{k \in \mathbb{N} : n_{k+1} = 1 + n_k\}$ is quasi-periodic. Then $\{x_k\}$ converges strongly to a common fixed point of the family $\{T_1, T_2, ..., T_m\}$.*

*Proof.* By Lemma 4.2.2 (iii), we have

$$\lim_{k\to\infty} d(x_k, T_i x_k) = 0, \text{ for } i = 1, 2, ..., m. \tag{4.2.19}$$

Let $i \in \{1, 2, ..., m\}$ be such that $T_i^l$ is semi-compact. Thus, by Lemma 4.2.3,

$$\lim_{k\to\infty} d(x_k, T_i^l x_k) = 0.$$

We can also find a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ such that $\lim_{j\to\infty} x_{k_j} = p \in C$. Hence, from (4.2.19), we have

$$d(p, T_i p) = \lim_{j\to\infty} d(x_{k_j}, T_i x_{k_j}) = 0, \text{ for all } i = 1, 2, ..., m.$$

Thus $p \in F$ and but since $\lim_{k\to\infty} d(x_k, q)$ exists, $\{x_k\}$ must itself converges to $p$ which completes the proof. $\square$

Next is the lemma we constructed for proving the sufficient condition for strong convergence of the sequence $\{x_n\}$ defined by (4.2.1) to a common fixed point of the family $\{T_1, T_2, ..., T_m\}$.

**Lemma 4.2.8.** *Let $X$ be a complete CAT(0) space, $C$ be a nonempty closed convex subset of $X$ and $T_1, ..., T_m \in \mathcal{T}_r(C)$ be such that $\sum_{k=1}^\infty \sup_{x\in C} b_{n_k}(x) < \infty$. Let $t \in (0, 1)$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (4.2.1) is well-defined. Assume that $F = \bigcap_{i=1}^m F(T_i) \neq \emptyset$. Then*

*(i) there exists a sequence $\{v_k\}$ in $[0, \infty)$ such that $\sum_{k=1}^\infty v_k < \infty$ and $d(x_{k+1}, p) \leq (1 + v_k)^m d(x_k, p)$, for all $p \in F$ and all $k \in \mathbb{N}$,*

*(ii) there exists a constant $M > 0$ such that $d(x_{k+l}, p) \leq M d(x_k, p)$, for all $p \in F$ and $k, l \in \mathbb{N}$.*

*Proof.* Let $p \in F$.

(i) Let $v_k = \sup_{x\in C} b_{n_k}(x)$ for all $k \in \mathbb{N}$. Since $\sum_{k=1}^\infty \sup_{x\in C} b_{n_k}(x) < \infty$, we have $\sum_{k=1}^\infty v_k < \infty$. By Lemma 2.2.10 (i), we have

$$\begin{aligned}
d(y_{1k},p) &= d((1-t)x_k \oplus tT_1^{n_k}x_k, p)\\
&\le (1-t)d(x_k,p) + td(T_1^{n_k}x_k,p)\\
&\le (1-t)d(x_k,p) + t(1+b_{n_k}(p))d(x_k,p)\\
&\le (1+b_{n_k}(p))d(x_k,p)\\
&\le (1+v_k)d(x_k,p).
\end{aligned}$$

Suppose that $d(y_{jk},p) \le (1+v_k)^j d(x_k,p)$ holds for some $j = 1,2,...,m-2$. Then

$$\begin{aligned}
d(y_{(j+1)k},p) &= d((1-t)x_k \oplus tT_{j+1}^{n_k}y_{jk}, p)\\
&\le (1-t)d(x_k,p) + td(T_{j+1}^{n_k}y_{jk},p)\\
&\le (1-t)d(x_k,p) + t(1+b_{n_k}(p))d(y_{jk},p)\\
&\le (1-t)d(x_k,p) + t(1+v_k)d(y_{jk},p)\\
&\le (1-t)d(x_k,p) + t(1+v_k)^{j+1}d(x_k,p)\\
&= \left[1-t+t\left(1+\sum_{r=1}^{j+1}\frac{(j+1)j\cdots(j+2-r)}{r!}v_k^r\right)\right]d(x_k,p)\\
&\le \left[1+\sum_{r=1}^{j+1}\frac{(j+1)j\cdots(j+2-r)}{r!}v_k^r\right]d(x_k,p)\\
&= (1+v_k)^{j+1}d(x_k,p).
\end{aligned}$$

By mathematical induction, we have

$$d(y_{ik},p) \le (1+v_k)^i d(x_k,p), \text{ for all } i = 1,2,...,m-1. \tag{4.2.20}$$

This implies that

$$\begin{aligned}
d(x_{k+1},p) &= d((1-t)x_k \oplus tT_m^{n_k}y_{(m-1)k}, p)\\
&\le (1-t)d(x_k,p) + td(T_m^{n_k}y_{(m-1)k},p)\\
&\le (1-t)d(x_k,p) + t(1+b_{n_k}(p))d(y_{(m-1)k},p)\\
&\le (1-t)d(x_k,p) + t(1+v_k)(1+v_k)^{m-1}d(x_k,p)\\
&\le (1-t)d(x_k,p) + t(1+v_k)^m d(x_k,p)\\
&= \left[1-t+t\left(1+\sum_{r=1}^{m}\frac{m(m-1)\cdots(m-r+1)}{r!}v_k^r\right)\right]d(x_k,p)\\
&\le \left[1+\sum_{r=1}^{m}\frac{m(m-1)\cdots(m-r+1)}{r!}v_k^r\right]d(x_k,p)\\
&= (1+v_k)^m d(x_k,p)
\end{aligned}$$

which completes the proof of part (i).

(ii) We observe that $(1+\alpha)^n \le e^{n\alpha}$ holds for all $n \in \mathbb{N}$ and $\alpha \ge 0$.

It follows from part (i) that for $k, l \in \mathbb{N}$,

$$\begin{aligned} d(x_{k+l}, p) &\le (1+v_{k+l-1})^m d(x_{k+l-1}, p) \\ &\le \exp\{m v_{k+l-1}\} d(x_{k+l-1}, p) \le \cdots \le \exp\left\{ m \sum_{i=1}^{k+l-1} v_i \right\} d(x_k, p) \\ &\le \exp\left\{ m \sum_{i=1}^{\infty} v_i \right\} d(x_k, p). \end{aligned}$$

The proof is complete by setting $M = \exp\{m \sum_{i=1}^{\infty} v_i\}$. $\square$

**Theorem 4.2.9.** *Let $X$ be a complete CAT(0) space, $C$ be a nonempty closed convex subset of $X$ and $T_1, ..., T_m \in \mathcal{T}_r(C)$ be such that $\sum_{k=1}^{\infty} \sup_{x \in C} b_{n_k}(x) < \infty$. Let $t \in (0,1)$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (4.2.1) is well-defined. Assume that $F = \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Then $\{x_k\}$ converges strongly to some point in $F$ if and only if $\liminf_{k\to\infty} d(x_k, F) = 0$, where $d(x, F) = \inf_{p \in F} d(x, p)$.*

*Proof.* The necessity is obvious. Now, we prove the sufficiency. From Lemma 4.2.8 (i), we have

$$d(x_{k+1}, p) \le (1+v_k)^m d(x_k, p), \text{ for all } p \in F \text{ and all } k \in \mathbb{N}.$$

This implies that

$$d(x_{k+1}, F) \le (1+v_k)^m d(x_k, p) = \left( 1 + \sum_{r=1}^{m} \frac{m(m-1)\cdots(m-r+1)}{r!} v_k^r \right) d(x_k, F).$$

Since $\sum_{k=1}^{\infty} v_k < \infty$, $\sum_{k=1}^{\infty} \sum_{r=1}^{m} \frac{m(m-1)\cdots(m-r+1)}{r!} v_k^r < \infty$. By Lemma 3.2.4, we get $\lim_{k\to\infty} d(x_k, F) = 0$. Next, we show that $\{x_k\}$ is a Cauchy sequence. By Lemma 4.2.8 (ii), there exists an $M > 0$ such that

$$d(x_{k+l}, p) \le M d(x_k, p), \text{ for all } p \in F \text{ and } k, l \in \mathbb{N}. \tag{4.2.21}$$

Since $\lim_{k\to\infty} d(x_k, F) = 0$, for each $\epsilon > 0$, there exists $k_1 \in \mathbb{N}$ such that

$$d(x_k, F) < \frac{\epsilon}{2M}, \text{ for all } k \ge k_1.$$

Hence there exists $z_1 \in F$ such that

$$d(x_{k_1}, z_1) < \frac{\epsilon}{2M}. \tag{4.2.22}$$

By (4.2.21) and (4.2.22) for $k \ge k_1$, we have

$$d(x_{k+l}, x_k) \le d(x_{k+l}, z_1) + d(x_k, z_1) \le M d(x_{k_1}, z_1) + M d(x_{k_1}, z_1) < 2M \left( \frac{\epsilon}{2M} \right) = \epsilon.$$

This shows that $\{x_k\}$ is a Cauchy sequence and so converges to some $p \in C$. Actually, $p \in C$ because $\{x_k\} \subset C$ and $C$ is a closed subset of $X$. Next we show that $p \in F$. Since $F(T_i)$ is a closed subset in $C$ for all $i = 1, 2, ..., m$, so is $F = \cap_{i=1}^{m} F(T_i)$. From the continuity of $d(x, F)$ with $d(x_k, F) \to 0$ and $x_k \to p$ as $k \to \infty$, we get $d(p, F) = 0$ and thus $p \in F$. Therefore, the proof is complete. $\square$

As an immediate consequence of Theorem 4.2.9, we obtain the following corollary.

**Corollary 4.2.10.** *Let $X$ be a complete CAT(0) space, $C$ be a nonempty closed convex subset of $X$ and $T_1, ..., T_m \in \mathcal{T}_r(C)$ be such that $\sum_{k=1}^{\infty} \sup_{x \in C} b_{n_k}(x) < \infty$. Let $t \in (0,1)$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (4.2.1) is well-defined. Assume that $F = \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Then $\{x_k\}$ converges strongly to a point $p \in F$ if and only if there exists a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ which converges to $p$.*

**Theorem 4.2.11.** *Let $X$ be a complete CAT(0) space and $C$ be a nonempty closed convex subset of $X$. Let $\{T_1, ..., T_m\} \subset \mathcal{T}_r(C)$ be satisfy Condition $(A'')$. Assume that $F = \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$ and $\sum_{k=1}^{\infty} \sup_{x \in C} b_{n_k}(x) < \infty$. Let $t \in (0,1)$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (4.2.1) is well-defined. If the set $\mathcal{J} = \{k \in \mathbb{N} : n_{k+1} = 1 + n_k\}$ is quasi-periodic, then $\{x_k\}$ converges strongly to a common fixed point of the family $\{T_1, T_2, ..., T_m\}$.*

*Proof.* By Lemma 4.2.2 (iii), $\lim_{k\to\infty} d(x_k, T_i x_k) = 0$, for all $i = 1, 2, ..., m$. By using Condition $(A'')$, there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for $r \in (0, \infty)$ such that

$$\lim_{k\to\infty} f(d(x_k, F)) \leq \lim_{k\to\infty} d(x_k, T_j x_k) = 0 \text{ for some } j = 1, ..., m.$$

This implies that $\lim_{k\to\infty} d(x_k, F) = 0$. The conclusion follows from Theorem 4.2.9. $\square$