# Chapter 5
# Conclusion

In this chapter, we conclude all main results obtained in this thesis. It is divided into 2 sections. First is the results in Banach spaces. Second is ones in CAT(0) spaces. Each section inlcludes both fixed point theorems and convergence theorems.

## 5.1 Results in uniformly convex Banach Spaces

(1) Let $X$ be a uniformly convex Banach space and $C$ be a nonempty bounded closed convex subset of $X$. Then every commuting family $\mathcal{S}$ of asymptotic pointwise nonexpansive mappings on $C$ has a nonempty closed convex common fixed point set.

(2) Let $X$ be a Banach space, $C$ be a nonempty closed convex subset of $X$ and let $T_1, ..., T_m \in \mathcal{T}_r(C)$. Let $t \in (0,1)$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (3.2.6) is well-defined. Assume that $F = \bigcap_{i=1}^m F(T_i) \neq \emptyset$. Then for each $p \in F$, there are sequences of nonnegative real numbers $\{\gamma_k\}$ and $\{\delta_k\}$ (depending on p) such that $\sum_{k=1}^\infty \gamma_k < \infty$, $\sum_{k=1}^\infty \delta_k < \infty$ and the following statements hold:

(i) $\|y_{ik} - p\| \leq (1+\gamma_k)^i \|x_k - p\|$, for all $i = 1, 2, ..., m-1$;

(ii) $\|x_{k+1} - p\| \leq (1+\delta_k)\|x_k - p\|$;

(iii) $\lim_{k\to\infty} \|x_k - p\|$ exists.

(3) Let $X$ be a uniformly convex Banach space, $C$ be a nonempty closed convex subset of $X$ and $T_1, ..., T_m \in \mathcal{T}_r(C)$. Let $t \in (0,1)$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (3.2.6) is well-defined. Assume that $F = \bigcap_{i=1}^m F(T_i) \neq \emptyset$. Then

(i) $\lim_{k\to\infty} \|x_k - T_i^{n_k} y_{(i-1)k}\| = 0$, for all $i = 1, 2, ..., m$;

(ii) $\lim_{k\to\infty} \|x_k - T_i^{n_k} x_k\| = 0$, for all $i = 1, 2, ..., m$;

(iii) If the set $\mathcal{J} = \{k \in \mathbb{N} : n_{k+1} = 1 + n_k\}$ is quasi-periodic, then $\lim_{k\to\infty} \|x_k - T_i x_k\| = 0$, for all $i = 1, 2, ..., m$.

(4) Let $X$ be a uniformly convex Banach space with the Opial property and $C$ be a nonempty closed convex subset of $X$. Let $T_1, ..., T_m \in \mathcal{T}_r(C)$ be such that $F = \bigcap_{i=1}^m F(T_i) \neq \emptyset$. Let $t \in (0,1)$ and $\{n_k\} \subset \mathbb{N}$ be such that the sequence $\{x_k\}$ in (3.2.6) is well-defined. If the set $\mathcal{J} = \{k : n_{k+1} = 1 + n_k\}$ is quasi-periodic, then the sequence $\{x_k\}$ converges weakly to a common fixed point of the family $\{T_1, T_2, ..., T_m\}$.

(5) Let $X$ be a uniformly convex Banach space and $C$ be a nonempty closed convex subset of $X$. Let $T_1, ..., T_m \in \mathcal{T}_r(C)$ be such that $T_i^l$ is semi-compact for some $i \in \{1, 2, ..., m\}$ and $l \in \mathbb{N}$. Let $t \in (0, 1)$ and $\{n_k\} \subset \mathbb{N}$ be such that the sequence $\{x_k\}$ in (3.2.6) is well-defined. Suppose that $F := \bigcap_{i=1}^m F(T_i) \neq \emptyset$ and the set $\mathcal{J} = \{k : n_{k+1} = 1 + n_k\}$ is quasi-periodic, then the sequence $\{x_k\}$ converges strongly to a common fixed point of the family $\{T_1, T_2, ..., T_m\}$.

(6) Let $X$ be a Banach space, $C$ be a nonempty closed convex subset of $X$ and let $T_1, ..., T_m \in \mathcal{T}_r(C)$ be such that $F := \bigcap_{i=1}^m F(T_i) \neq \emptyset$. Let $t \in (0, 1)$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (3.2.6) is well-defined. Assume that $\sum_{k=1}^{\infty} \sup_{x \in C} b_{n_k}(x) < \infty$. Then there exists a sequence $\{v_k\}$ in $[0, \infty)$ and a nonnegative real number $M$ such that $\sum_{k=1}^{\infty} v_k < \infty$ and the following statements hold for all $p \in F$ :

   (i) $\|x_{k+1} - p\| \leq (1 + v_k)^m \|x_k - p\|$, for all $k \in \mathbb{N}$;

   (ii) $\|x_{k+l} - p\| \leq M\|x_k - p\|$, for all $k, l \in \mathbb{N}$;

(7) Let $X$ be a Banach space, $C$ be a nonempty closed convex subset of $X$ and $T_1, ..., T_m \in \mathcal{T}_r(C)$ be such that $F = \bigcap_{i=1}^m F(T_i) \neq \emptyset$. Let $t \in (0, 1)$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (3.2.6) is well-defined. Assume that $\sum_{k=1}^{\infty} \sup_{x \in C} b_{n_k}(x) < \infty$. Then $\{x_k\}$ converges strongly to a point in $F$ if and only if $\liminf_{k \to \infty} d(x_k, F) = 0$, where $d(x, F) = \inf_{p \in F} d(x, p)$.

(8) Let $X$ be a Banach space, $C$ be a nonempty closed convex subset of $X$ and $T_1, ..., T_m \in \mathcal{T}_r(C)$. Let $t \in (0, 1)$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (3.2.6) is well-defined. Assume that $F = \bigcap_{i=1}^m F(T_i) \neq \emptyset$ and $\sum_{k=1}^{\infty} \sup_{x \in C} b_{n_k}(x) < \infty$. Then the sequence $\{x_k\}$ converges strongly to a point in $p \in F$ if and only if there exists a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ which converges to $p$.

(9) Let $X$ be a uniformly convex Banach space and $C$ be a nonempty closed convex subset of $X$. Let $\{T_1, ..., T_m\} \subset \mathcal{T}_r(C)$ be satisfy Condition $(A'')$. Let $t \in (0, 1)$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (3.2.6) is well-defined. Suppose that $\sum_{k=1}^{\infty} \sup_{x \in C} b_{n_k}(x) < \infty$, $F = \bigcap_{i=1}^m F(T_i) \neq \emptyset$ and the set $\mathcal{J} = \{k \in \mathbb{N} : n_{k+1} = 1 + n_k\}$ is quasi-periodic. Then $\{x_k\}$ converges strongly to a common fixed point of the family $\{T_i : i = 1, 2, ..., m\}$.

## 5.2 Results in CAT(0) Spaces

(1) Let $X$ be a complete CAT(0) space, $C$ be a nonempty bounded closed convex subset of $X$. Then for any commuting family $\mathcal{S}$ of asymptotic pointwise nonexpansive mappings on $C$, the set $\mathcal{F}(\mathcal{S})$ of common fixed points of $\mathcal{S}$ is a nonempty nonexpansive retract of $C$.

(2) Let $X$ be a complete CAT(0) space, $C$ be a nonempty bounded closed convex subset of $X$. Then for any commuting family $\mathcal{S}$ of nonexpansive mappings on $C$, the set $\mathcal{F}(\mathcal{S})$ of common fixed points of $\mathcal{S}$ is a nonempty nonexpansive retract of $C$.

(3) Let $X$ be a complete CAT(0) space, $C$ be a nonempty closed convex subset of $X$ and let $T_1, ..., T_m \in \mathcal{T}_r(C)$. Let $t \in (0,1)$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (4.2.1) is well-defined. Assume that $F = \bigcap_{i=1}^m F(T_i) \neq \emptyset$. Then for each $p \in F$, there are sequences of nonnegative real numbers $\{\gamma_k\}$ and $\{\delta_k\}$ (depending on p) such that $\sum_{k=1}^\infty \gamma_k < \infty$, $\sum_{k=1}^\infty \delta_k < \infty$ and the following statements hold:

  (i) $d(y_{ik}, p) \leq (1+\gamma_k)^i d(x_k, p)$, for all $i = 1, 2, ..., m-1$;

  (ii) $d(x_{k+1}, p) \leq (1+\delta_k) d(x_k, p)$;

  (iii) $\lim_{k\to\infty} d(x_k, p)$ exists.

(4) Let $X$ be a complete CAT(0) space, $C$ be a nonempty closed convex subset of $X$ and $T_1, ..., T_m \in \mathcal{T}_r(C)$. Let $t \in (0,1)$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (4.2.1) is well-defined. Assume that $F = \bigcap_{i=1}^m F(T_i) \neq \emptyset$. Then

  (i) $\lim_{k\to\infty} d(x_k, T_i^{n_k} y_{(i-1)k}) = 0$, for all $i = 1, 2, ..., m$;

  (ii) $\lim_{k\to\infty} d(x_k, T_i^{n_k} x_k) = 0$, for all $i = 1, 2, ..., m$,

  (iii) If the set $\mathcal{J} = \{k \in \mathbb{N} : n_{k+1} = 1 + n_k\}$ is quasi-periodic, then $\lim_{k\to\infty} d(x_k, T_i x_k) = 0$, for all $i = 1, 2, ..., m$.

(5) Let $X$ be a complete CAT(0) space, $C$ be a nonempty closed convex subset of $X$ and $T_1, ..., T_m \in \mathcal{T}_r(C)$. Let $t \in (0,1)$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (4.2.1) is well-defined. Suppose that $F = \bigcap_{i=1}^m F(T_i) \neq \emptyset$ and the set $\mathcal{J} = \{k \in \mathbb{N} : n_{k+1} = 1 + n_k\}$ is quasi-periodic. Then $\{x_k\}$ $\Delta-$converges to a common fixed point of the family $\{T_1, T_2, ..., T_m\}$.

(6) Let $X$ be a complete CAT(0) space, $C$ be a nonempty closed convex subset of $X$ and $T_1, ..., T_m \in \mathcal{T}_r(C)$. Assume that $T_i^l$ is semi-compact for some $i \in \{1, 2, ..., m\}$ and $l \in \mathbb{N}$. Let $t \in (0,1)$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (4.2.1) is well-defined. Suppose that $F = \bigcap_{i=1}^m F(T_i) \neq \emptyset$ and the set $\mathcal{J} = \{k \in \mathbb{N} : n_{k+1} = 1 + n_k\}$ is quasi-periodic. Then $\{x_k\}$ converges strongly to a common fixed point of the family $\{T_1, T_2, ..., T_m\}$.

(7) Let $X$ be a complete CAT(0) space, $C$ be a nonempty closed convex subset of $X$ and $T_1, ..., T_m \in \mathcal{T}_r(C)$ be such that $\sum_{k=1}^\infty \sup_{x\in C} b_{n_k}(x) < \infty$. Let $t \in (0,1)$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (4.2.1) is well-defined. Assume that $F = \bigcap_{i=1}^m F(T_i) \neq \emptyset$. Then

  (i) there exists a sequence $\{v_k\}$ in $[0, \infty)$ such that $\sum_{k=1}^\infty v_k < \infty$ and $d(x_{k+1}, p) \leq (1+v_k)^m d(x_k, p)$, for all $p \in F$ and all $k \in \mathbb{N}$,

  (ii) there exists a constant $M > 0$ such that $d(x_{k+l}, p) \leq M d(x_k, p)$, for all $p \in F$ and $k, l \in \mathbb{N}$.

(8) Let $X$ be a complete CAT(0) space, $C$ be a nonempty closed convex subset of $X$ and $T_1, ..., T_m \in \mathcal{T}_r(C)$ be such that $\sum_{k=1}^\infty \sup_{x\in C} b_{n_k}(x) < \infty$. Let $t \in (0,1)$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (4.2.1) is well-defined. Assume that $F = \bigcap_{i=1}^m F(T_i) \neq \emptyset$. Then $\{x_k\}$ converges strongly to some point in $F$ if and only if $\liminf_{k\to\infty} d(x_k, F) = 0$, where $d(x, F) = \inf_{p\in F} d(x, p)$.

(9) Let $X$ be a complete CAT(0) space, $C$ be a nonempty closed convex subset of $X$ and $T_1, ..., T_m \in \mathcal{T}_r(C)$ be such that $\sum_{k=1}^{\infty} \sup_{x\in C} b_{n_k}(x) < \infty$. Let $t \in (0,1)$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (4.2.1) is well-defined. Assume that $F = \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Then $\{x_k\}$ converges strongly to a point $p \in F$ if and only if there exists a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ which converges to $p$.

(10) Let $X$ be a complete CAT(0) space and $C$ be a nonempty closed convex subset of $X$. Let $\{T_1, ..., T_m\} \subset \mathcal{T}_r(C)$ be satisfy Condition $(A'')$. Assume that $F = \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$ and $\sum_{k=1}^{\infty} \sup_{x\in C} b_{n_k}(x) < \infty$. Let $t \in (0,1)$ and $\{n_k\} \subset \mathbb{N}$ be such that $\{x_k\}$ in (4.2.1) is well-defined. If the set $\mathcal{J} = \{k \in \mathbb{N} : n_{k+1} = 1 + n_k\}$ is quasi-periodic, then $\{x_k\}$ converges strongly to a common fixed point of the family $\{T_1, T_2, ..., T_m\}$.