Chapter 1
Introduction

There are various directions in studying fixed point theory. In this thesis, we will focus on the existence of a fixed point which can be separated into three parts: fixed point theorems for single-valued mappings, fixed point theorems for multivalued mappings and common fixed point theorems for single-valued and multivalued mappings.

1.1 Fixed Point Theorems for Single-Valued Mappings

Let X be a nonempty set and $t : X \to X$ be a mapping. A point x in X is a fixed point of t if

$$tx = x.$$

The set of all fixed points of t will be denoted by $Fix(t)$. Many problems arising in different areas of mathematics, such as optimization, variational analysis and differential equations, can be equivalently formulated as fixed point problem, i.e. one has to find a fixed point of some mapping t. In 1912, Brouwer's fixed point theorem [8] guarantees the existence of a fixed point of continuous mapping on compact convex set in $\mathbb{R}^n$.

Theorem 1.1.1. (Brouwer's fixed point theorem) *Let B be the closed unit ball in $\mathbb{R}^n$ and $t : B \to B$ a continuous mapping. Then t has a fixed point.*

In 1930, Brouwer's fixed point theorem was extended to infinite dimensional spaces by Schauder [54], in the following way:

Theorem 1.1.2. (Schauder's fixed point theorem) *Let C be a compact convex subset of a Banach space X. Suppose $t : C \to C$ is a continuous mapping. Then t has a fixed point.*

1.2 Fixed Point Theorems for Multivalued Mappings

Let X and Y be topological spaces. A multivalued mapping $T : X \to 2^Y$ is a mapping that sends points x in X to subsets Tx of Y. A point x in X is a fixed point of a set-valued map $T : X \to 2^X$, if

$$x \in Tx.$$

The set of all fixed points of T will be denoted by $Fix(T)$. Fixed point theory for multivalued mappings has many useful applications in various fields, for example in game theory or mathematical economics. Therefore, it is natural to extend the fixed point results for single-valued mappings to the setting of multivalued mappings. In 1941, the Kakutani-Fixed Point Theorem [42] was the first fixed point result about multivalued mappings which is a generalization of Brouwer's fixed point theorem. The statement is as follows (see Definition 2.2.1 for the definition of upper semicontinuous):

Theorem 1.2.1. (Kakutani Fixed Point Theorem) *Let E be a nonempty compact convex subset of $\mathbb{R}^n$. Let $T : E \to 2^E$ a upper semicontinuous with nonempty closed and convex valued mapping. Then T has a fixed point.*

In 1950, the multivalued analogue of Schauder's fixed point theorem was given by Bohnenblust and Karlin [7].

Theorem 1.2.2. (Bohnenblust-Karlin Fixed Point Theorem) *Let E be a nonempty compact convex subset of a Banach space X. Let $T : E \to 2^E$ be a upper semicontinuous with nonempty closed and convex valued mapping. Then T has a fixed point.*

Let E be a subset of a metric space (X, d). We shall denote by $CB(X)$ the family of all nonempty bounded and closed subsets of X, $K(X)$ the family of all compact subsets of X, and denote by $KC(X)$ the family of all nonempty compact convex subsets of X. Let $H(\cdot, \cdot)$ be the Hausdorff distance defined on $CB(X)$, i.e.,

$$H(A, B) := \max\{\sup_{a\in A} dist(a, B), \sup_{b\in B} dist(b, A)\}, \ A, B \in CB(X),$$

where $dist(a, B) := \inf\{\|a - b\| : b \in B\}$ is the distance from a point a to a subset B. A multivalued mapping $T : E \to 2^X$ is said to be nonexpansive if

$$H(Tx, Ty) \le d(x, y) \text{ for each } x, y \in X.$$

In 1974, Lim [47] proved the basic result in the nonexpansive theory by using the asymptitic center technique and the notion of regularity.

Theorem 1.2.3. *Let E be a bounded closed and convex subset of a uniformly convex Banach space X. Suppose $T : E \to K(E)$ is nonexpansive mapping. Then T has a fixed point.*

In 1990, Kirk and Massa [45] generalized Theorem 1.2.3 by using asymptotic centers of sequences and nets and obtained the following result (see Section 2.3 for the definition of asymptotic centers):

Theorem 1.2.4. (Kirk and Massa Theorem) *Let E be a nonempty bounded closed and convex subset of a Banach space X and $T : E \to KC(E)$ be a nonexpansive mapping. Suppose that the asymptotic center in E of each bounded sequence of X is nonempty and compact. Then T has a fixed point.*

We call the assumption in Kirk and Massa Theorem *the Kirk-Massa condition*. In 2000, Xu [62] extended Kirk and Massa Theorem to nonexpansive nonself mappings (see Section 2.2 for the definition of inwardness):

Theorem 1.2.5. ([62, Theorem 2.3.1]) *Let E be a nonempty bounded closed and convex subset of a Banach space X satisfying the Kirk-Massa condition. Let $T : E \to KC(X)$ be a nonexpansive and inward mapping. Then T has a fixed point.*

In 2007, Wiśnicki and Wośko [61] introduced an ultrafilter coefficient $DL_{\mathcal{U}}(X)$ for a Banach space X and proved the existence of fixed points by using ultra-asymptotic centers technique (See Section 2.3 for the definition of ultrafilter and ultra-asymptotic center).

Definition 1.2.6. ([61, Definition 5.2]) Let $\mathcal{U}$ be a trivial ultrafilter defined on the set of natural numbers $\mathbb{N}$. The coefficient $DL_{\mathcal{U}}(X)$ of a Banach space X is defined as

$$DL_{\mathcal{U}}(X) = \sup\{\frac{\chi_E(A_{\mathcal{U}}(E, \{x_n\}))}{\chi_E(\{x_n\})}\},$$

where the supremum is taken over all nonempty weakly compact convex subsets E of X and all weakly, not norm-convergent sequences $\{x_n\}$ in E which are regular relative to E.

Theorem 1.2.7. ([61, Theorem 5.3]) *Let E be a nonempty weakly compact convex subset of a Banach space X with $DL_{\mathcal{U}}(X) < 1$. Assume that $T : E \to KC(X)$ is a nonexpansive, inward and $1-\chi$-contractive mapping. Then T has a fixed point.*

In 2009, Dhompongsa and Inthakon [14] introduced the following coefficient by following the concept of $DL_{\mathcal{U}}(X)$:

Definition 1.2.8. ([14, Definition 3.2]) Let $\mathcal{U}$ be a free ultrafilter defined on $\mathbb{N}$. The coefficient $D_{\mathcal{U}}(X)$ of a Banach space X is defined as

$$D_{\mathcal{U}}(X) = \sup\{\frac{\chi_E(\{y_n\})}{\chi_E(\{x_n\})}\},$$

where the supremum is taken over all nonempty weakly compact convex subsets E of X, all sequences $\{x_n\}$ in E which are weakly, not norm-convergent and are regular relative to E and all weakly, not norm-convergent sequences $\{y_n\} \subset A_{\mathcal{U}}(E, \{x_n\})$ which are regular relative to E.

A concept corresponds to the coefficient $D_{\mathcal{U}}(X)$ is the following property:

Definition 1.2.9. ([14, Definition 3.1]) A Banach space X is said to have property (D') if there exists $\lambda \in [0,1)$ such that for any nonempty weakly compact convex subset E of X, any sequence $\{x_n\} \subset E$ which is regular relative to E, and any sequence $\{y_n\} \subset A(E,\{x_n\})$ which is regular relative to E we have

$$r(E,\{y_n\}) \leq \lambda r(E,\{x_n\}).$$

It is shown in [14] that $D_{\mathcal{U}}(X) < 1$ if and only if X satisfies property (D'). Obviously, $DL_{\mathcal{U}}(X) < 1 \Rightarrow (D') \Rightarrow (D)$ (see Definition 3.1.1 for the definition of the property (D)). Several well known spaces have been proved to have property (D') (see for examples [13, 16, 21, 22, 23, 31, 41, 53, 64]). The class of spaces having property (D') include spaces satisfying the Kirk-Massa condition. One of the main results in [14] is the following theorem:

Theorem 1.2.10. ([14, Theorem 1.9]) *Let E be a nonempty weakly compact convex subset of a Banach space X having property (D'). Assume that $T : E \to KC(X)$ is a nonexpansive, inward and $1-\chi$-contractive mapping. Then T has a fixed point.*

Motivated by above research works, we are interested to extend Theorem 1.2.10 to a wider class of mappings.

1.3 Common Fixed Point Theorems for Single-valued and Multivalued Mappings

Let E be a nonempty subset of a Banach space X. A mapping $t : E \to E$ is said to be nonexpansive if $\|tx - ty\| \leq \|x - y\|$, $x, y \in E$. For a pair (t,T) of nonexpansive mappings $t : E \to E$ and $T : E \to 2^X$ defined on a bounded closed and convex subset E of a convex metric space or a Banach space X, we are interested in finding a common fixed point of t and T. Recall that t and T are said to be commuting mappings if $tTx \subset Ttx$ for all $x \in E$. In [17], Dhompongsa, Kaewkhao, and Panyanak obtained a result for the CAT(0) space setting:

Theorem 1.3.1. ([17, Theorem 4.1]) *Let E be a nonempty bounded closed and convex subset of a complete CAT(0) space X, and let $t : E \to E$ and $T : E \to KC(X)$ be nonexpansive mappings. Assume that for some $p \in Fix(t)$,*

$$\alpha p \oplus (1-\alpha)Tx \ \text{ is convex for } x \in E, \alpha \in [0,1].$$

If t and T are commuting, then $Fix(t) \cap Fix(T) \neq \emptyset$.

Shahzad and Markin [57] improved Theorem 1.3.1 by removing the assumption that the nonexpansive multivalued mapping T in that theorem has a convex-valued contractive approximation. They also noted that Theorem 1.3.1 needs the additional assumption that $T(\cdot) \cap E \neq \emptyset$ for that result to be valid.

Theorem 1.3.2. ([57, Theorem 4.2]) *Let E be a nonempty bounded closed and convex subset of a complete CAT(0) space X. Assume $t : E \to E$ and $T : E \to KC(X)$ are nonexpansive mappings and $T(x) \cap E \neq \emptyset$ for each $x \in E$. If the mappings t and T commute, then Fix(t) $\cap$ Fix(T) $\neq \emptyset$.*

Dhompongsa, Kaewcharoen, and Kaewkhao [16] extended Theorem 1.3.1 to uniform convex Banach spaces.

Theorem 1.3.3. ([16, Theorem 4.2]) *Let E be a nonempty bounded closed and convex subset of a uniform convex Banach space X. Assume $t : E \to E$ and $T : E \to KC(E)$ are nonexpansive mappings. If t and T are commuting, then Fix(t) $\cap$ Fix(T) $\neq \emptyset$.*

The result has been improved, generalized, and extended under various assumptions. See for examples, [2, Theorem 3.3], [3, Theorem 3.4], [26, Theorem 4.7], [27, Theorem 3.9], [28, Theorem 5.3], [37, Theorem 5.2], [39, Theorem 3.5], [51, Theorem 4.2], [55, Theorem 3.8], [56, Theorem 3.1].

Motivated by these research works, we are interested to extend Theorem 1.3.3 to a bigger class of Banach spaces while a class of mappings is no longer finite.

This thesis is divided into 5 chapters. Chapter 1 presents a brief history of fixed point theory and research objectives. In Chapter 2, basic well-known facts which are used in later chapters are presented. In Chapter 3, we use the ultra-asymptotic centers technique in establishing the existence of fixed points for nonself multivalued mappings in a Banach space. The results in this Chapter were presented at the 5th Annual Conference on Fixed Point Theory and Applications (CFPTA 2011: 8-9 July 2011, Lampang Rajabhat University) and published in the paper [19]: S. Dhompongsa and N. Nanan, *Fixed point theorems by ways of ultra-asymptotic centers*, Abstract and Applied Analysis Volume 2011, Article ID 826851, 21 pages. In Chapter 4, we prove common fixed point theorems for a commuting family of nonexpansive mappings one of which is multivalued in a Banach space by using nonexpansive retracts as a main tool. The results presented in this Chapter appeared in the paper [50] : Narawadee Nanan and Sompong Dhompongsa, *A common fixed point theorem for a commuting family of nonexpansive mappings one of which is multivalued*, Fixed Point Theory and Applications 2011, 2011:54. The conclusion is given in Chapter 5.