Chapter 2
Preliminaries

In this chapter, we present some notions and results to be referred in the next chapters. Section 2.1 concerns with the notion of metric spaces and Banach spaces. Section 2.2 deals with the notion of multivalued mappings. In section 2.3, the terminology of ultrapower of Banach spaces and related notions are given.

2.1 Metric Spaces and Banach Spaces

In this section, we state basic concepts and fundamental results in metric spaces and Banach spaces.

Definition 2.1.1. Let X be a nonempty set and $d : X \times X \to \mathbb{R}$ a function. Then d is called a *metric* on X if the following properties hold for all $x, y, z \in X$:

(i) (Positive Definiteness) $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$;

(ii) (Symmetry) $d(x, y) = d(y, x)$;

(iii) (Triangle Inequality) $d(x, y) \leq d(x, z) + d(z, y)$.

The valued of metric d at (x, y) is called distance between x and y, and the ordered pair (X, d) is called *metric space.*

Example 2.1.2. Some standard examples of metric spaces:

(i) Euclidean space $(\mathbb{R}^n, d)$, for $x = (x_1, ..., x_n), y = (y_1, ..., y_n) \in \mathbb{R}^n$,

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + ... + (x_n - y_n)^2}.$$

(ii) l_∞: the set of all bounded sequences of complex numbers with

$$d(x, y) = \sup_{n\in\mathbb{N}} |x_n - y_n|,$$

where $x = \{x_n\}, y = \{y_n\} \in l_\infty$.

(iii) $C([a, b])$: the set of all continuous real-valued functions on $[a, b]$ with

$$d(x, y) = \max_{t\in[a,b]} |x(t) - y(t)|,$$

where $x, y \in C([a, b])$.

Let $(X,d),(Y,d')$ be metric spaces. A sequence $\{x_n\}$ in X is said to be *convergent* to $x \in X$ if $\lim_{n\to\infty} d(x_n,x) = 0$. In this case, we write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$. For any $\varepsilon > 0$ and $x \in X$, we define the *open ball* with center x and radius ε,

$$B(x,\varepsilon) := \{y \in X : d(x,y) < \varepsilon\}.$$

A mapping $t : X \to Y$ is said to be *continuous at a point* x if for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$d'(tx,ty) < \varepsilon \text{ for all } y \text{ satisfying } d(x,y) < \delta.$$

t is said to be *continuous* if it is continuous at each point of X.

Definition 2.1.3. Let E be a subset of a metric space (X,d).

(i) E is *open* if for $x \in E$, there exists $\varepsilon > 0$ such that $B(x,\varepsilon) \subset E$.

(ii) E is *closed* if $X \setminus E$ is open.

(iii) E is *bounded* if $\delta(E) := \sup_{x,y\in E} d(x,y) < \infty$.

(iv) E is *compact* if every sequence in E has a convergent subsequence.

Proposition 2.1.4. *Let E be a subset of a metric space (X,d).*

(i) E is closed if and only if the limit of every convergent sequence in E is an element of E.

(ii) If E is compact, then E is closed and bounded.

(iii) If X is compact, then E is compact if and only if E is closed.

(iv) If $t : X \to X$ is continuous and E is compact, then $t(E)$ is also compact.

Definition 2.1.5. *A linear space or vector space* X over the field $\mathbb{F}$ (the real valued $\mathbb{R}$ or the complex number $\mathbb{C}$) is a set X together with an internal binary operation "+" called *addition* and a scalar multiplication carrying (α,x) in $\mathbb{F}\times X$ to αx in X satisfying the following for all $x,y,z \in X$ and $\alpha,\beta \in \mathbb{F}$:

(i) $x + y = y + x$;

(ii) $x + (y + z) = (x + y) + z$;

(iii) there exists an element $0 \in X$ called *the zero vector* of X such that $x + 0 = x$ for all $x \in X$;

(iv) given $x \in X$, there exists an element $-x \in X$ called *the additive inverse or the negative* of x such that $x + (-x) = 0$;

(v) $\alpha(x + y) = \alpha x + \alpha y$;

(vi) $(\alpha + \beta)x = \alpha x + \beta x$;

(vii) $(\alpha\beta)x = \alpha(\beta x)$;

(viii) $1x = x$.

Example 2.1.6. The following are vector spaces.

(i) The set of real-valued functions of a real variable on $[0, 1]$. Addition and multiplication by a scalar are defined as follows:

$$\begin{aligned} f + g &: (f + g)(t) = f(t) + g(t), \\ \alpha f &: (\alpha f)(t) = \alpha f(t). \end{aligned}$$

(ii) The set of n-tuples of real numbers:

$$\mathbb{R}^n = \{(a_1, a_2, .., a_n) : a_i \in \mathbb{R}, \text{ for all } i = 1, 2, .., n\},$$

where addition and scalar multiplication are defined by

$$\begin{aligned} (a_1, a_2, .., a_n) + (b_1, b_2, .., b_n) &= (a_1 + b_1, a_2 + b_2, .., a_n + b_n), \\ \alpha(a_1, a_2, .., a_n) &= (\alpha a_1, \alpha a_2, .., \alpha a_n). \end{aligned}$$

Definition 2.1.7. Let E be a nonempty subset of a vector space X over field $\mathbb{F}$.

(i) E is said to be *linear subspace* of X if for all $x, y \in E$ and $\alpha, \beta \in \mathbb{F}$, we have $\alpha x + \beta y \in E$.

(ii) E is *convex* if $\lambda x + (1 - \lambda)y \in E$ for $x, y \in E$ and $\lambda \in [0, 1]$.

(iii) The *convex hull* of E, $co(E)$=$\cap\{K \subset X : K \supset E$ and K is convex $\}$. It is easy to see that $co(E) = \{\sum_{i=1}^{n} \alpha_i x_i \,|\, x_i \in E, \alpha_i \geq 0$ and $\sum_{i=1}^{n} \alpha_i = 1\}$.

(iv) The *closed convex hull* of E,

$$\overline{co}(E) = \cap\{K \subset X : K \supset E \text{ and } K \text{ is closed and convex }\}.$$

Definition 2.1.8. A function $\|\cdot\|$ from a (real) linear space X into $\mathbb{R}$ is called a *norm* if it satisfies the following properties for all $x, y \in X$ and $\alpha \in \mathbb{R}$:

(i) $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$;

(ii) $\|\alpha x\| = |\alpha|\|x\|$;

(iii) $\|x + y\| \leq \|x\| + \|y\|$.

From this norm we can define a metric, induced by the norm $\|\cdot\|$, by

$$d(x, y) = \|x - y\| \text{ for } x, y \in X.$$

A linear space X equipped with the norm $\|\cdot\|$ is called a *normed linear space.*

A sequence $\{x_n\}$ in a normed space $(X, \|\cdot\|)$ is said to be *convergent* to $x \in X$ if $\lim_{n\to\infty} \|x_n - x\| = 0$. In this case, we write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$. A sequence $\{x_n\}$ in X is *Cauchy* if $\lim_{n,m\to\infty} \|x_n - x_m\| = 0$.

Definition 2.1.9. A normed linear space $(X, \|\cdot\|)$ (or simply X) which is *complete*, i.e. every Cauchy sequence in X is convergent, is called a *Banach space.*

Example 2.1.10.

(i) Euclidean space $\mathbb{R}^n$ and unitary space $\mathbb{C}^n$ are Banach spaces with norm

$$\|x\| = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}.$$

(ii) Space l_∞ is a Banach space with norm $\|x\| = \sup_i |x_i|$.

(iii) $L_2[a, b]$: the vector space of all continuous real-valued functions on $[a, b]$ forms a normed space with norm

$$\|x\| = \left(\int_a^b x(t)^2 dt\right)^{1/2}.$$

This space is not complete.

A function $f : X \to \mathbb{R}$ is said to be *linear* if $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ for all $x, y \in X$ and $\alpha, \beta \in \mathbb{R}$. In addition, if there exists a real number $M > 0$ such that $|f(x)| \leq M\|x\|$ for all $x \in X$, we say that f is a *bounded linear functional.*

Definition 2.1.11. Let X be a normed space. The *dual space* of X, denote by X^*, is the space of all bounded linear functionals on X with the operator norm defined by:

$$\|f\| = \sup\{|f(x)| : x \in B_X\} = \sup\{|f(x)| : x \in S_X\}$$

where $B_X = \{x \in X : \|x\| \leq 1\}$ is the unit ball of X and $S_X = \{x \in X : \|x\| = 1\}$ is the unit sphere of X.

It is not difficult to see that X^* equipped with the norm is a Banach space. The topology induced by a norm is too strong in the sense that it has many open sets. Indeed, in order that each bounded sequence in X has a norm convergent subsequence, it is necessary and sufficient that X be finite dimensional. This fact leads us to consider other weaker topologies on normed spaces which are related to the linear structure of the spaces to search for subsequential extraction principles. So it is worthwhile to define the weaker topology for a Banach space X.

Definition 2.1.12. Let X be a normed space. The topology for X induced by the topologizing family X^* is the weak topology of X or the topology $\sigma(X, X^*)$.

That is, the weak topology of a normed space is the weakest topology for the space such that every member of the dual space is continuous.

We say that a sequence (x_n) in X converges weakly to x, denoted by

$$w - \lim_n x_n = x \text{ or } x_n \rightharpoonup x,$$

if and only if

$$\lim_n f(x_n) = f(x)$$

for all $f \in X^*$. A topological property that holds with respect to the weak topology is said to be a *weak* property or to hold *weakly*. For examples, a subset K of X is *weakly closed* if it is closed in the weak topology; that is, if it contains the weak limit of each of its weakly convergent sequences. *Weakly open* sets are now taken as those sets whose complements are weakly closed. The resulting topology on X is called the weak topology on X. Sets which are compact in this topology are said to be *weakly compact*.

We now collect for subsequent use some well-known properties of the weak topology. Despite the fact that proofs of these results can be found in any standard functional analysis text we included selected details.

Proposition 2.1.13. (Eberlein-Smulian Theorem) *For any weakly closed subset E of a Banach space the following are equivalent.*

(i) E is weakly compact.

(ii) Each sequence $\{x_n\}$ in E has a weak converges subsequence to a point of E.

Proposition 2.1.14.

(i) A convex subset of a Banach space is weakly closed if and only if it is norm closed.

(ii) A closed convex subset of a weakly compact set is itself weakly compact.

(iii) A weakly compact subset of a Banach space is bounded.

2.2 Multivalued Mappings

In this section, we collect the definition of classes of multivalued mappings that we use in this thesis.

Definition 2.2.1. Let E be a nonempty subset of a Banach space X. A multivalued mapping $T : E \to CB(X)$ is said to

(i) be contraction if there exists $k \in [0, 1)$ such that $H(Tx, Ty) \leq k\|x - y\|$ for $x, y \in E$;

(ii) satisfy (C_λ) for some $\lambda \in (0, 1)$ if for each $x, y \in E$,

$$\lambda dist(x, Tx) \leq \|x - y\| \text{ implies } H(Tx, Ty) \leq \|x - y\|;$$

(iii) be continuous at $x \in E$ if for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, x) > 0$ such that $\|x - y\| < \delta$ for $y \in E$ implies that $H(Tx, Ty) < \varepsilon$;

(iv) be uniformly continuous on E if every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $\|x - y\| < \delta$ for $x, y \in E$ implies that $H(Tx, Ty) < \varepsilon$;

(v) be upper semicontinuous on E if for every sequence $\{x_n\}$ in E such that $x_n \to x \in E$, for every sequence $y_n \in Tx_n$ with $y_n \to y$ we have $y \in Tx$.

It is easy to show that nonexpansiveness implies condition (C_λ) for every $\lambda \in (0, 1)$ but the converse does not hold.

Example 2.2.2. Define a mapping T on $[0, 5]$ by

$$T(x) = \begin{cases} [0, \frac{x}{5}], x \neq 5; \\ \{1\}, x = 5. \end{cases}$$

Then T satisfies condition $(C_{\frac{1}{2}})$ but it is not nonexpansive.
First, we will show that T satisfies condition $(C_{\frac{1}{2}})$. If $x = 5$ and $y \in [0, 5)$ be such that $\frac{1}{2}dist(x, Tx) \leq \|x - y\|$, then $\frac{1}{2}dist(5, \{1\}) = 2 \leq \|x - y\|$. Thus

$$H(Tx, Ty) = H(\{1\}, [0, \frac{y}{5}]) = 1 \leq 2 \leq \|x - y\|.$$

If $x, y \in [0, 5)$, then

$$H(Tx, Ty) = H([0, \frac{x}{5}], [0, \frac{y}{5}]) = \frac{1}{5}\|x - y\| \leq \|x - y\|.$$

Therefore T satisfies condition $(C_{\frac{1}{2}})$. Take $x = \frac{9}{2}$ and $y = 5$. We have

$$H(Tx, Ty) = H([0, \frac{9}{2}], \{1\}) = 1 > \frac{1}{2} = \|x - y\|.$$

Hence T fails to be nonexpansive.

Next example shows that a mapping T satisfying condition (C_λ) for some $\lambda \in (0,1)$ can be discontinuous:

Example 2.2.3. Let $\lambda \in (0,1)$ and $a = \frac{2(\lambda+1)}{\lambda(\lambda+2)}$. Define a mapping $T : [0, \frac{2}{\lambda}] \to KC([0, \frac{2}{\lambda}])$ by

$$Tx = \begin{cases} \{\frac{x}{2}\} & \text{if } x \neq \frac{2}{\lambda}, \\ [\frac{1}{\lambda}, a] & \text{if } x = \frac{2}{\lambda}. \end{cases}$$

Clearly, $\frac{1}{\lambda} < a < \frac{2}{\lambda}$ and T is nonexpansive on $[0, \frac{2}{\lambda})$. Thus, we only verify that, for $x \in [0, \frac{2}{\lambda})$,

$$\lambda dist(x, Tx) \leq \|x - \frac{2}{\lambda}\| \Rightarrow H\left(Tx, T\frac{2}{\lambda}\right) \leq \|x - \frac{2}{\lambda}\| \tag{2.2.1}$$

and

$$\lambda dist\left(\frac{2}{\lambda}, T\frac{2}{\lambda}\right) \leq \|\frac{2}{\lambda} - x\| \Rightarrow H\left(T\frac{2}{\lambda}, Tx\right) \leq \|\frac{2}{\lambda} - x\|. \tag{2.2.2}$$

If $\lambda dist(x, Tx) \leq \|x - \frac{2}{\lambda}\|$, then $x \leq \frac{4}{\lambda(\lambda+2)}$ and

$$H\left(Tx, T\frac{2}{\lambda}\right) = a - \frac{x}{2} \leq \frac{2}{\lambda} - x = \|x - \frac{2}{\lambda}\|.$$

Hence (2.2.1) holds. If $\lambda dist(\frac{2}{\lambda}, T\frac{2}{\lambda}) \leq \|\frac{2}{\lambda} - x\|$, then $x \leq \frac{4}{\lambda(\lambda+2)}$ and

$$H\left(T\frac{2}{\lambda}, Tx\right) = a - \frac{x}{2} \leq \frac{2}{\lambda} - x = \|\frac{2}{\lambda} - x\|.$$

Thus (2.2.2) holds. Therefore, T satisfies condition (C_λ). Clearly, T is upper semicontinuous but not continuous (and hence T is not nonexpansive).

Let (X, d) be a metric space and $B \subset X$. The *Hausdorff (or ball) measure of noncompactness* is defined as

$$\chi(B) = \inf\{r > 0, B \subset \cup_{i=1}^{n} B(x_i, r) \text{ with } x_i \in X\}.$$

The *Kuratowski measure of noncompactness* is defined as

$$\beta(B) = \inf\{r > 0, B \subset \cup_{i=1}^{n} D_i \text{ and } \delta(D_i) < r\}.$$

A multivalued mapping $T : E \to 2^X$ is called $\phi-$*condensing* (resp. $1 - \phi$-*contractive*) where ϕ is a measure of noncompactness, if for each bounded subset B of E with $\phi(B) > 0$, there holds the inequality

$$\phi(T(B)) < \phi(B) \qquad \text{(resp. } \phi(T(B)) \leq \phi(B)),$$

where $T(B) = \cup_{x \in B} Tx$. Recall that the *inward* set of E at $x \in E$ is defined by

$$I_E(x) = \{x + \alpha(y - x) : \alpha \geq 1, y \in E\}.$$

A multivalued mapping $T : E \to 2^X$ is said to be *inward* (resp. *weakly inward*) on E if

$$Tx \subset I_E(x) \qquad (\text{resp. } Tx \subset \overline{I_E(x)}) \text{ for all } x \in E.$$

The following theorem is very useful in order to prove the results on fixed points for multivalued mappings.

Theorem 2.2.4. ([12, Theorem 11.5]) *Let E be a nonempty bounded closed and convex subset of a Banach space X and $T : E \to KC(X)$ an upper semicontinuous and χ-condensing mapping. If $T(x) \cap \overline{I_E(x)} \neq \emptyset$ for all $x \in E$, then T has a fixed point.*

2.3 Ultrapower of Banach Spaces

Ultrapowers of a Banach space are proved to be useful in many branches of mathematics. Many results can be seen more easily when treated in this setting. Throughout the section I will denote an index set, usually the natural numbers $\mathbb{N}$ for the most situations in metric fixed point theory.

Definition 2.3.1. Let $\mathcal{F}$ be a *filter* on I, that is $\mathcal{F} \subset 2^I$, satisfying:

(i) If $A \in \mathcal{F}$ and $A \subset B \subset I$, then $B \in \mathcal{F}$.

(ii) If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

Example 2.3.2.

(i) The power set of I, 2^I, defines a filter.

(ii) The Fréchet filter $\{A \subset I : I \setminus A \text{ is finite }\}$.

(iii) For some fixed $i_0 \in I, \mathcal{F}_{i_0} := \{A \subset I : i_0 \in A\}$. Filters of the form $\mathcal{F}_{i_0}$ for some $i_0 \in I$ are called *trivial* (or *non-free*) filters, otherwise, they are called *nontrivial* (or *free*).

Definition 2.3.3. A filter $\mathcal{U}$ on I is called an *ultrafilter* if it is maximal with respect to the ordering of filters on I given by set-inclusion. That is, if $\mathcal{U} \subset \mathcal{F}$ and $\mathcal{F}$ is filter on I, then $\mathcal{F} = \mathcal{U}$.

Lemma 2.3.4. *A filter $\mathcal{U} \subset 2^I$ is an ultrafilter on I if and only if for every $A \subset I$ precisely one of A or $I \setminus A$ is in $\mathcal{U}$.*

It will henceforth be a standing assumption that all the filter and ultrafilters with we deal are nontrivial.

Definition 2.3.5. Let $\{x_n\}$ be a sequence in a Hausdorff topological space X and $\mathcal{U}$ a ultrafilter on I. The sequence $\{x_n\}$ is said to *converge* to x with respect to $\mathcal{U}$, denoted by

$$\lim_{n\to\mathcal{U}} x_n = x,$$

if for each neighborhood U of x, $\{n \in I : x_n \in U\} \in \mathcal{U}$. Limits along $\mathcal{U}$ are unique and if $\mathcal{U}$ is on $\mathbb{N}$ and $\{x_n\}$ is a bounded sequence in $\mathbb{R}$, then

$$\liminf_{n\to\infty} x_n \leq \lim_{n\to\mathcal{U}} x_n \leq \limsup_{n\to\infty} x_n.$$

Moreover, if E is a closed subset of X and $\{x_n\} \subset E$, then $\lim_{n\to\mathcal{U}} x_n$ belongs to E whenever it exists.

Remark 2.3.6. Suppose $\{x_n\}$ converges to x in the topology of the space X. Then $\{x_n\}$ converges to x with respect to any ultrafilter $\mathcal{U}$. Let X be a metric space. If $\mathcal{U}$ is an ultrafilter and $\lim_{n\to\mathcal{U}} x_n = x$, with $\{x_n\} \subset X$, then there exists a subsequence of $\{x_n\}$ which converges to x.

Now we are in position to define the ultrapower of a Banach space X. Let $\mathcal{U}$ be a nontrivial ultrafilter on $\mathbb{N}$. We can form the substitution space

$$l_\infty(X) := \{\{x_n\} : x_n \in X \text{ for all } n \in \mathbb{N} \text{ and } \|\{x_n\}\| = \sup_n \|x_n\| < \infty\}.$$

Then,

$$\mathcal{N} = \{\{x_n\} \in l_\infty(X) : \lim_{n\to\mathcal{U}} \|x_n\| = 0\}.$$

is a closed linear subspace of $l_\infty(X)$.

Definition 2.3.7. ([4, 34, 43, 58]) The *Banach space ultrapower* of X over $\mathcal{U}$ is defined to be the Banach quotient

$$(X)_\mathcal{U} := l_\infty(X)/\mathcal{N},$$

with the quotient norm given by $\|\{x_n\}_\mathcal{U}\| = \lim_{n\to\mathcal{U}} \|x_n\|$, where $\{x_n\}_\mathcal{U}$ is the equivalence class of $\{x_n\}$.

One can proof that $\tilde{X} = (X)_{\mathcal{U}}$ is a Banach space. It is also clear that X is isometric to a subspace of $\tilde{X}$ by the canonical embedding $x \mapsto \{x, x, ...\}_{\mathcal{U}}$. If $E \subset X$, we shall use the symbols $\dot{E}$ and $\dot{x}$ to denote the image of E and x in $\tilde{X}$ under this isometry respectively and denote

$$\tilde{E} = \{\{x_n\}_{\mathcal{U}} \in \tilde{X} : x_n \in E \text{ for all } n \in \mathbb{N}\}.$$

Thus $\dot{x} = \{x, x, ...\}_{\mathcal{U}}$ and $\dot{E} = \{\dot{x} \in \tilde{X} : x \in E\}$.

If $T : E \to CB(X)$ is a multivalued mapping, we define a corresponding multivalued mapping $\tilde{T} : \tilde{E} \to CB(\tilde{X})$ by

$$\tilde{T}(\{x_n\}_{\mathcal{U}}) := \{\{u_n\}_{\mathcal{U}} \in \tilde{X} : u_n \in Tx_n \text{ for all } n \in \mathbb{N}\},$$

where $\{x_n\}_{\mathcal{U}} \in \tilde{E}$. Moreover, the set $\tilde{T}(\{x_n\}_{\mathcal{U}})$ is bounded and closed (see [4, 43]). The Hausdorff metric on $CB(\tilde{X})$ will be denoted by $\tilde{H}$.

Proposition 2.3.8. ([61, Proposition 3.1]) *For every $\{x_n\}_{\mathcal{U}}$ and $\{y_n\}_{\mathcal{U}}$ in $\tilde{E}$,*

$$\tilde{H}(\tilde{T}\{x_n\}_{\mathcal{U}}, \tilde{T}\{y_n\}_{\mathcal{U}}) = \lim_{n \to \mathcal{U}} H(Tx_n, Ty_n).$$

Proposition 2.3.9. ([18, Page 37], [61, Proposition 3.2]) *Let E be a nonempty subset of a Banach space X and $T : E \to CB(X)$.*

(i) If T is convex-valued, then $\tilde{T}$ is convex-valued;

(ii) If T is compact-valued, then $\tilde{T}$ is compact-valued and $\tilde{T}\dot{x} = \dot{(Tx)}$ for every $x \in E$;

(iii) If T is nonexpansive, then $\tilde{T}$ is nonexpansive.

The following method and results deal with the concept of asymptotic centers. Let E be a nonempty closed convex subset of a Banach space X and $\{x_n\}$ a bounded sequence in X. For $x \in X$, define the *asymptotic radius* of $\{x_n\}$ at x as the number

$$r(x, \{x_n\}) = \limsup_{n \to \infty} \|x_n - x\|.$$

Let

$$r(E, \{x_n\}) = \inf\{r(x, \{x_n\}) : x \in E\}$$

and

$$A(E, \{x_n\}) = \{x \in E : r(x, \{x_n\}) = r(E, \{x_n\})\}.$$

The number $r(E,\{x_n\})$ and the set $A(E,\{x_n\})$ are, respectively, called the *asymptotic radius* and *asymptotic center* of $\{x_n\}$ relative to E.

It was noted in [34] that if E is nonempty and weakly compact, then $A(E,\{x_n\})$ is nonempty and weakly compact, and if E is convex, then $A(E,\{x_n\})$ is convex.

Definition 2.3.10. A bounded sequence $\{x_n\}$ in X is called *regular relative* to a bounded subset E of a Banach space X if $r(E,\{x_n\}) = r(E,\{x_{n'}\})$ for each subsequence $\{x_{n'}\}$ of $\{x_n\}$; further, $\{x_n\}$ is called *asymptotically uniform relative* to E if $A(E,\{x_n\}) = A(E,\{x_{n'}\})$ for each subsequence $\{x_{n'}\}$ of $\{x_n\}$.

Lemma 2.3.11. ([34, Lemma 15.2],[47, Theorem 1]) *Let E be a subset of a Banach space X, $\{x_n\}$ a bounded sequence in X. Then $\{x_n\}$ has a subsequence which is regular relative to E.*

Let $\mathcal{U}$ be a nontrival ultrafilter on $\mathbb{N}$. Wiśnicki and Wośko [61] defined the *ultra-asymptotic radius* of $\{x_n\}$ relative to E by

$$r_{\mathcal{U}}(E,\{x_n\}) = \inf\{\lim_{n\to\mathcal{U}} \|x_n - x\| : x \in E\},$$

and the *ultra-asymptotic center* of $\{x_n\}$ relative to E by

$$A_{\mathcal{U}}(E,\{x_n\}) = \{x \in E : \lim_{n\to\mathcal{U}} \|x_n - x\| = r_{\mathcal{U}}(E,\{x_n\})\}.$$

Notice that the above notions have a natural interpretation in the ultrapower $\tilde{X}$:

$$r_{\mathcal{U}}(E,\{x_n\}) = \inf_{x\in E} \|\{x_n\}_{\mathcal{U}} - \dot{x}\|$$

is the relative Chebyshev radius of $\{x_n\}_{\mathcal{U}}$, and,

$$\dot{(A_{\mathcal{U}}(E,\{x_n\}))} = \dot{E} \cap B_{\tilde{X}}(\{x_n\}_{\mathcal{U}}, r)$$

is the relative Chebyshev center of $\{x_n\}_{\mathcal{U}}$ relative to $\dot{E}$ in the ultrapower $\tilde{X}$. (Here $B_{\tilde{X}}(\{x_n\}_{\mathcal{U}}, r)$ denotes the ball in $\tilde{X}$ centered at $\{x_n\}_{\mathcal{U}}$ and of radius $r = r_{\mathcal{U}}(E,\{x_n\})$.) It is not difficult to see that $A_{\mathcal{U}}(E,\{x_n\})$ is a nonempty weakly compact convex set if E is. It should be noted that, in general, $A(E,\{x_n\})$ and $A_{\mathcal{U}}(E,\{x_n\})$ may be different.

Example 2.3.12. ([61]) Let $\mathcal{U}$ be a nontrivial ultrafilter defined on $\mathbb{N}$ such that $\{2,4,6,...\} \in \mathcal{U}$ and let

$$x_n = \begin{cases} e_n, & n = 1,3,5,...; \\ \{-3,0,0,...\}, & n = 2,4,6,... \end{cases}$$

be a sequence of elements in $X = l_2$. Then $r(X,\{x_n\}) = \frac{5}{3}$, $r_{\mathcal{U}}(X,\{x_n\}) = 0$, $\chi_E(\{x_n\}) = 1$, $A(X,\{x_n\}) = \{(-\frac{4}{3},0,0,...)\}$ and $A_{\mathcal{U}}(X,\{x_n\}) = \{(-3,0,0,...)\}$.

The notion of the asymptotic radius is closely related to the notion of the relative Hausdorff measure of noncompactness of a bounded set A defined by Domínguez and Lorenzo [23] as $\chi_E(A) = \inf\{r > 0, B \subset \cup_{i=1}^n B(x_i, r) \text{ with } x_i \in E\}$.

Proposition 2.3.13. ([61, Proposition 4.5]) *If $\{x_n\}$ is a bounded sequence which is regular relative to E, then*

$$r(E, \{x_n\}) = r_{\mathcal{U}}(E, \{x_n\}) = \chi_E(\{x_n\}).$$

From Proposition 2.3.13, we have for $w \in A(E, \{x_n\})$,

$$\lim_{n \to \mathcal{U}} \|x_n - w\| \leq \limsup_{n \to \infty} \|x_n - w\| = r(E, \{x_n\}) = r_{\mathcal{U}}(E, \{x_n\}).$$

Therefore, $A(E, \{x_n\}) \subset A_{\mathcal{U}}(E, \{x_n\})$.

The following result plays an important role in our proofs.

Lemma 2.3.14. ([14, Lemma 3.3]) *Let E be a nonempty closed and convex subset of a Banach space X and $\{x_n\}$ a bounded sequence in X which is regular relative to E. For each $\{y_n\} \subset A_{\mathcal{U}}(E, \{x_n\})$, there exists a subsequence $\{x_{n'}\}$ of $\{x_n\}$ such that $\{y_n\} \subset A(E, \{x_{n'}\})$.*