Chapter 3
Fixed Point Theorems via Technique of Ultra-asymptotic Centers

The main aim of this chapter is to present fixed point theorems for nonself multivalued mappings in Banach spaces having Property (D') and Kirk-Massa condition by using ultra-asymptotic centers technique.

3.1 Property (D')

Let E be a nonempty subset of a Banach space X and t a mapping on E. Recall that a subset K of E is said to be t-invariant if $t(K) \subset K$. A sequence $\{x_n\}$ in E is called an approximate fixed point sequence (afps for short) for t if

$$\lim_{n\to\infty} \|x_n - tx_n\| = 0.$$

Analogously for a multivalued mapping $T : E \to CB(X)$, a subset K of E is said to be T-invariant if $T(x) \cap E \neq \emptyset$ for all $x \in K$. A sequence $\{x_n\}$ in E is called an approximate fixed point sequence (afps for short) for T if

$$\lim_{n\to\infty} dist(x_n, Tx_n) = 0.$$

In 2006, Dhompongsa et al. [13] introduced a property for a Banach space X which is weaker that property (D'):

Definition 3.1.1. ([13, Definition 3.1]) A Banach space X is said to have *property* (D) if there exists $\lambda \in [0, 1)$ such that for any nonempty weakly compact convex subset E of X, any sequence $\{x_n\} \subset E$ which is regular and asymptotically uniform relative to E, and any sequence $\{y_n\} \subset A(E, \{x_n\})$ which is regular and asymptotically uniform relative to E we have

$$r(E, \{y_n\}) \leq \lambda r(E, \{x_n\}).$$

Theorem 3.1.2. ([13, Theorem 3.6]) *Let E be a nonempty weakly compact convex subset of a Banach space X which has property (D). Assume that $T : E \to KC(E)$ is a nonexpansive mapping. Then T has a fixed point.*

Butsan, Dhompongsa and Takahashi [11] introduced a new condition for mappings and obtained a fixed point theorem in a Banach space having property (D):

Definition 3.1.3. ([11, Definition 3.1]) Let $t : E \to E$ be a mapping on a subset E of a Banach space X. Then t is said to satisfy *condition* $(*)$ if

(1) for each t-invariant subset K of E, t has an afps in K, and

(2) for each pair of t-invariant subsets K and W of E, $A(W, \{x_n\})$ is t-invariant for each afps $\{x_n\}$ in K.

Theorem 3.1.4. ([11, Theorem 3.5]) *Let E be a weakly compact convex subset of a Banach space X having property (D). Let $t : E \to E$ satisfy conditon $(*)$. If t is continuous, then t has a fixed point.*

In fact, we can replace "continuity" by a weaker condition, namely "$I-t$ is strongly demiclosed at 0": for every sequence $\{x_n\}$ in E strongly converges to $z \in E$ and such that $x_n - tx_n \to 0$ we have $z = tz$ (cf. [30]).

In this section, motivated by Theorem 1.2.10 and above research works, we will extend Theorem 1.2.10 and 3.1.4 for multivalued nonself mappings for spaces having property (D'). As consequences,

(i) Theorem 1.2.10 is generalized to a larger class of mappings;

(ii) Theroem 3.1.4 is generalized to nonself multivalued mappings for spaces having property (D').

First we need the following lemma:

Lemma 3.1.5. *Let E be a nonempty subset of a Banach space X and $T : E \to CB(X)$. Then*

(1) If T is uniformly continuous, then $\tilde{T}$ is uniformly continuous;

(2) If T is continuous at $z \in E$, then $\tilde{T}$ is continuous at $\dot{z}$.

Proof. (1) Let $\varepsilon > 0$. Since T is uniformly continuous, there exists $\delta > 0$ such that $H(Tx, Ty) < \varepsilon$ for each $x, y \in E$ with $\|x-y\| < \delta$. Suppose $\{x_n\}_{\mathcal{U}}, \{y_n\}_{\mathcal{U}} \in \tilde{E}$ and $\|\{x_n\}_{\mathcal{U}} - \{y_n\}_{\mathcal{U}}\| < \delta$. Let $A = \{n : \|x_n - y_n\| < \delta\}$ and $B = \{n : H(Tx_n, Ty_n) < \varepsilon\}$. Since $A \in \mathcal{U}$ and $A \subset B$, we have $B \in \mathcal{U}$. Thus, by Proposition 2.3.8 $\tilde{H}(\tilde{T}\{x_n\}_{\mathcal{U}}, \tilde{T}\{y_n\}_{\mathcal{U}}) \le \varepsilon$.

(2) Let $\varepsilon > 0$. Since T is continuous at z, there exists $\delta > 0$ such that $H(Tx, Tz) < \varepsilon$ for each $x \in E$ with $\|x - z\| < \delta$. If $\{x_n\}_{\mathcal{U}} \in \tilde{E}$ such that $\|\{x_n\}_{\mathcal{U}} - \dot{z}\| < \delta$, then letting $A = \{n : \|x_n - z\| < \delta\}$ and $B = \{n : H(Tx_n, Tz) < \varepsilon\}$, we see that $A \in \mathcal{U}$ and $B \in \mathcal{U}$. Thus by Proposition 2.3.8, $\tilde{H}(\tilde{T}\{x_n\}_{\mathcal{U}}, \tilde{T}\dot{z}) \le \varepsilon$. □

We now define a multivalued version of condition $(*)$ in Definition 3.1.3.

Definition 3.1.6. Let E be a nonempty subset of a Banach space X. A mapping $T : E \to CB(X)$ is said to satisfy *condition* $(*)$ if

(1) T has an afps in E, and

(2) T has an afps in $A(E, \{x_{n'}\})$ for some subsequence $\{x_{n'}\}$ of any given afps $\{x_n\}$ for T in E.

The following is our main theorem:

Theorem 3.1.7. *Let E be a weakly compact convex subset of a Banach space X having property (D'). Assume that $T : E \to KC(X)$ is a multivalued mapping satisfying condition $(*)$. If T is continuous, then T has a fixed point.*

Proof. The proof follows by adapting the proof of [14, Theorem 1.9] . By (1) of Definition 3.1.6, let $\{x_n^0\}$ be an afps for T in E . We can assume by Proposition 2.3.11 that $\{x_n^0\}$ is weakly convergent and regular relative to E. Condition (2) of Definition 3.1.6 gives us a subsequence $\{x_{n_0}^0\}$ of $\{x_n^0\}$ so that the center $A(E, \{x_{n_0}^0\})$ contains an afps for T. Denote $A^0 = A(E, \{x_{n_0}^0\})$ and let $\{x_n^1\}$ be an afps in A^0. Assume that $\{x_n^1\}$ is weakly convergent and regular relative to E. As before, T has an afps in $A(E, \{x_{n_1}^1\})$ for some subsequence $\{x_{n_1}^1\}$ of $\{x_n^1\}$. Since X has property (D'), put $\lambda = D_{\mathcal{U}}(X) < 1$. Then, by Proposition 2.3.13 and Definition 1.2.8,

$$r(E, \{x_{n_1}^1\}) = \chi_E(\{x_{n_1}^1\}) \le \lambda \chi_E(\{x_{n_0}^0\}) = \lambda r(E, \{x_{n_0}^0\}).$$

Continue the procedure to obtain for each $m \ge 0$, a weakly convergent and regular sequence $\{x_{n_m}^m\}$ relative to E in $A^{m-1} := A(E, \{x_{n_{m-1}}^{m-1}\})$ such that

$$\lim_{n\to\infty} dist(x_{n_m}^m, Tx_{n_m}^m) = 0,$$

and for all $m \ge 1$,

$$r(E, \{x_{n_m}^m\}) \le \lambda r(E, \{x_{n_{m-1}}^{m-1}\}).$$

Consequently,

$$r(E, \{x_{n_m}^m\}) \le \lambda r(E, \{x_{n_{m-1}}^{m-1}\}) \le \cdots \le \lambda^m r(E, \{x_{n_0}^0\}).$$

We show that $\{\{x_{n_m}^m\}_{\mathcal{U}}\}_{m\ge1}$ is a Cauchy sequence in $\tilde{X}$. Indeed, for each $m \ge 1$, take an element $\dot{y}_m \in \dot{A}^{m-1}$. Then

$$\|\dot{x}_{n_m}^m - \dot{y}_m\| \le \|\dot{x}_{n_m}^m - \{x_{n_{m-1}}^{m-1}\}_{\mathcal{U}}\| + \|\{x_{n_{m-1}}^{m-1}\}_{\mathcal{U}} - \dot{y}_m\| \le 2r(E, \{x_{n_{m-1}}^{m-1}\}),$$

for all $m \geq 1$ and hence

$$\|\{x^m_{n_m}\}_{\mathcal{U}} - \{x^{m-1}_{n_{m-1}}\}_{\mathcal{U}}\| \leq \|\{x^m_{n_m}\}_{\mathcal{U}} - \dot{y}_m\| + \|\dot{y}_m - \{x^{m-1}_{n_{m-1}}\}_{\mathcal{U}}\| \leq 3r(E, \{x^{m-1}_{n_{m-1}}\}).$$

Thus,

$$\|\{x^m_{n_m}\}_{\mathcal{U}} - \{x^{m-1}_{n_{m-1}}\}_{\mathcal{U}}\| \leq 3\lambda^{m-1} r(E, \{x^0_{n_0}\}),$$

implying that $\{x^m_{n_m}\}_{\mathcal{U}}$ is a Cauchy sequence, and hence converges to some $\{z_n\}_{\mathcal{U}}$ in $\tilde{E}$ as $m \to \infty$. Next, we will show that $\{z_n\}_{\mathcal{U}} \in \dot{E}$. For each $m \geq 0$,

$$\begin{aligned} dist(\{z_n\}_{\mathcal{U}}, \dot{E}) &\leq \|\{z_n\}_{\mathcal{U}} - \{x^m_{n_m}\}_{\mathcal{U}}\| + dist(\{x^m_{n_m}\}_{\mathcal{U}}, \dot{E}) \\ &\leq \|\{z_n\}_{\mathcal{U}} - \{x^m_{n_m}\}_{\mathcal{U}}\| + \|\{x^m_{n_m}\}_{\mathcal{U}} - \dot{x}^{m+1}_{1_{m+1}}\| \\ &= \|\{z_n\}_{\mathcal{U}} - \{x^m_{n_m}\}_{\mathcal{U}}\| + r(E, \{x^m_{n_m}\}) \\ &\leq \|\{z_n\}_{\mathcal{U}} - \{x^m_{n_m}\}_{\mathcal{U}}\| + \lambda^m r(E, \{x^0_{n_0}\}). \end{aligned}$$

Taking $m \to \infty$, we see that

$$dist(\{z_n\}_{\mathcal{U}}, \dot{E}) = 0.$$

Thus, it follows that there exists $z \in E$ such that $\{z_n\}_{\mathcal{U}} = \dot{z}$. By Lemma 3.1.5, $\tilde{T}$ is continuous at $\dot{z}$, and thus $\tilde{H}(\tilde{T}\{x^m_{n_m}\}_{\mathcal{U}}, \tilde{T}\dot{z}) \to 0$ as $m \to \infty$. For every $m \geq 0$,

$$dist(\dot{z}, \tilde{T}\dot{z}) \leq \|\dot{z} - \{x^m_{n_m}\}_{\mathcal{U}}\}\| + dist(\{x^m_{n_m}\}_{\mathcal{U}}, \tilde{T}\{x^m_{n_m}\}_{\mathcal{U}}) + \tilde{H}(\tilde{T}\{x^m_{n_m}\}_{\mathcal{U}}, \tilde{T}\dot{z}).$$

Taking $m \to \infty$, we then obtain $\dot{z} \in \tilde{T}\dot{z}$. By Proposition 2.3.9, $\tilde{T}\dot{z} = (\dot{Tz})$ and therefore $z \in Tz$. □

Remark 3.1.8. The proof presented here based on a standard proof appeared in a series of papers [14, 23, 24, 61]. However, we cannot follow their proof directly to be able to obtain a result for larger classes of spaces and mappings. We choose an ultralimit approach by using an ultra-asymptotic center as our main tool. As mentioned earlier, this powerful tool was introduced in [61] by Wiśnicki and Wośko. Thus, our proof may not be totally new, but it significantly improves, generalizes, or extends many known results:

(i) Theorem 3.1.7 (as well as Theorem 3.2.3) unifies many known theorems in one. Examples of mappings in both theorems are given throughout the rest of the paper.

(ii) Theorem 3.1.7 improves condition $(*)$ in Definition 3.1.3 in which the mappings under consideration only are single-valued and are self-mappings. Consequently, Theorem 3.1.4 is improved significantly. Obviously, Theorem 1.2.10 is a special case of Theorem 3.1.7.

(iii) In Remark 3.2.2(ii) below, we show the following implication:

$$(**) + (A) \Rightarrow (*).$$

Thus results in [2, Corollary 3.5, Corollary 3.6], [6, Theorem 1, Theorem 2], [15, Theorem 3.3], [30, Theorem 5], [36, Theorem 2.4], [45, Theorem 2], [49, Theorem 4.2, Corollary 4.3, Theorem 4.4], [59, Theorem 2.6], and Theorem 1.2.5 are either improved, generalized, or extended. See Remark 3.2.4, Corollary 3.2.6, and Corollary 3.2.7. See also Remark 3.2.13(i) and (ii).

We now give some examples of mappings satisfying condition $(*)$. We will see that the ultra-center $A_{\mathcal{U}}(E, \{x_n\})$ plays a significant role in verifying condition (2) of condition $(*)$ for a given mapping.

Nonexpansive Mappings

We will show by following the proof of Theorem 5.3 in [61] that if $T : E \to KC(X)$ is nonexpansive and $1-\chi$-contractive such that $Tx \subset I_E(x)$ for every $x \in E$. Then T satisfies condition $(*)$. The main tools are Theorem 2.2.4 and Lemma 2.3.14.

Proposition 3.1.9. *Let E be a nonempty weakly compact convex subset of a Banach space X. Assume that $T : E \to KC(X)$ is nonexpansive and $1 - \chi$-contractive such that $Tx \subset I_E(x)$ for every $x \in E$. Then T satisfies condition $(*)$.*

Proof. First, we will show that T has an afps in E. Let $y_0 \in E$ and consider, for each $n \geq 1$, the contraction $T_n : E \to KC(X)$ defined by

$$T_n(x) = \frac{1}{n}y_0 + (1 - \frac{1}{n})Tx, \qquad x \in E.$$

It is not difficult to see that $T_n(x) \subset I_E(x)$ for every $x \in E$. Since T is $1 - \chi$-contractive, T_n is $(1 - \frac{1}{n}) - \chi$-contractive and by Theorem 2.2.4, there exists a fixed point x_n of T_n. Clearly, $\{x_n\}$ is an afps for T in E.

Next, let us see that T has an afps in $A(E, \{x_{n'}\})$ for some subsequence $\{x_{n'}\}$ of an afps $\{x_n\}$ for T in E. Let $\{x_n\}$ be an afps in E. By Proposition 2.3.11, we can assume that $\{x_n\}$ is regular relative to E. Let $A_{\mathcal{U}} := A_{\mathcal{U}}(E, \{x_n\})$. We show that

$$Tx \cap I_{A_{\mathcal{U}}}(x) \neq \emptyset \text{ for every } x \in A_{\mathcal{U}}. \qquad (3.1.1)$$

Let $x \in A_{\mathcal{U}}$. Observe first that $\{x_n\}_{\mathcal{U}} \in \tilde{T}\{x_n\}_{\mathcal{U}}$. By Proposition 2.3.9, $\tilde{T}\dot{x} = (\dot{Tx})$ is compact and hence there exists $u \in Tx$ such that

$$\|\{x_n\}_{\mathcal{U}} - \dot{u}\| = \tilde{H}(\tilde{T}\{x_n\}_{\mathcal{U}}, \tilde{T}\dot{x}) \leq \|\{x_n\}_{\mathcal{U}} - \dot{x}\| = r_{\mathcal{U}}(E, \{x_n\}). \qquad (3.1.2)$$

Since $u \in Tx \subset I_E(x)$, there exists $\alpha \geq 1$ and $y \in E$ such that $u = x + \alpha(y - x)$. If $\alpha = 1$ then $u = y \in E$ and it follows from (3.1.2) that $u \in A_{\mathcal{U}}$. If $\alpha > 1$ then $y = \frac{1}{\alpha}u + (1 - \frac{1}{\alpha})x$ and therefore we have

$$\|\{x_n\}_{\mathcal{U}} - \dot{y}\| \leq \frac{1}{\alpha}\|\{x_n\}_{\mathcal{U}} - \dot{u}\| + (1 - \frac{1}{\alpha})\|\{x_n\}_{\mathcal{U}} - \dot{x}\| \leq r_{\mathcal{U}}(E, \{x_n\}).$$

Hence $y \in A_{\mathcal{U}}$ and consequently $u \in I_{A_{\mathcal{U}}}(x)$. Thus (3.1.1) is justified.
Fix $y_0 \in A_{\mathcal{U}}$ and consider for each $n \geq 1$, the contraction $T_n : A_{\mathcal{U}} \to KC(X)$ defined by

$$T_n(x) = \frac{1}{n}y_0 + (1 - \frac{1}{n})Tx, \qquad x \in A_{\mathcal{U}}.$$

As before, T_n is $(1 - \frac{1}{n}) - \chi$-contractive and by Theorem 2.2.4, there exists a fixed point $z_n \in A_{\mathcal{U}}$ of T_n. Again, as above, $\{z_n\}$ is an afps for T in $A_{\mathcal{U}}$. By Lemma 2.3.14, there exists a subsequence $\{x_{n'}\}$ of $\{x_n\}$ such that $\{z_n\} \subset A(E, \{x_{n'}\})$. □

Diametrically Contractive Mappings

In [38], Istratescu introduced a new class of mappings:

Definition 3.1.10. ([38]) A mapping t defined on a complete metric space (X, d) is said to be diametrically contractive if $\delta(tK) < \delta(K)$ for all closed subsets K with $0 < \delta(K) < \infty$. (Here $\delta(K) := \sup\{d(x, y) : x, y \in K\}$ denotes the diameter of $K \subset X$.)

Xu [63] proved the fixed point theorem for a diametrically contractive mapping in the framework of Banach spaces.

Theorem 3.1.11. ([63, Theorem 2.3]) *Let E be a weakly compact subset of a Banach space X and let $t : E \to E$ be a diametrically contractive mapping. Then t has a fixed point.*

Dhompongsa and Yingtaweesittikul [20] defined a multivalued version of mappings in Theorem 3.1.11 which is weaker than the condition required in Definition 3.1.10.

Theorem 3.1.12. ([20, Theorem 2.2]) *Let E be a weakly compact subset of a Banach space X and let $T : E \to KC(X)$ be a multivalued mapping such that $\delta(TK \cap K) < \delta(K)$ for all closed sets K with $\delta(K) > 0$ and E is invariant under T. Then T has a unique fixed point.*

Moreover, in [20] example of a mapping that satisfies condition in Theorem 3.1.12 but does not satisfy condition in Theorem 3.1.11 is given.

Example 3.1.13. Let $Tx = [0, x - \log(x+1)]$ for $x \in [0, 100]$. If A is a bounded closed subset of $[0, 100]$, then for some $a, b > 0$ we have $A \subset [a, b]$, and $\delta(A) = b - a$. Clearly, $TA \subset [0, b - \log(b+1)]$. Therefore $TA \cap A \subset [a, b - \log(b+1)]$. This implies that $\delta(TA \cap A) < \delta(A)$. 0 is the unique fixed point of T. We observe that T does not satisfy the condition in Theorem 3.1.11 because $\delta(T[0.6, 1]) = 0.7 > 0.4 = \delta([0.6, 1])$.

The following result extends Theorem 3.1.12 partially.

Proposition 3.1.14. *Let E be a nonempty weakly compact convex subset of a Banach space X and let $T : E \to KC(X)$ be a multivalued mapping such that $\delta(TK) \leq \delta(K)$ for all closed sets K with $\delta(K) > 0$ and E is invariant under T. Then T satisfies condition $(*)$.*

Proof. First, we will see that T has an afps in E. Let $y_0 \in E$ and consider, for each $n \geq 1$, the contraction $T_n : E \to KC(X)$ defined by

$$T_n(x) = \frac{1}{n} y_0 + (1 - \frac{1}{n}) Tx, \qquad x \in E.$$

For $x \in E$, let $a \in Tx \cap E$. Thus $\frac{1}{n} y_0 + (1 - \frac{1}{n}) a \in T_n x \cap E$ and therefore $T_n x \cap E \neq \emptyset$ for every $x \in E$. We show that $\delta(T_n K) < \delta(K)$ for all closed sets K with $\delta(K) > 0$. Let K be a closed subset of E with $\delta(K) > 0$. For $x, y \in T_n K$, there exist $x', y' \in TK$ such that

$$\begin{aligned} x &= \frac{1}{n} y_0 + (1 - \frac{1}{n}) x', \\ y &= \frac{1}{n} y_0 + (1 - \frac{1}{n}) y'. \end{aligned}$$

and this entails $\|x - y\| = (1 - \frac{1}{n}) \|x' - y'\| \leq (1 - \frac{1}{n}) \delta(TK)$. Hence $\delta(T_n K) \leq (1 - \frac{1}{n}) \delta(TK) < \delta(K)$. By Theorem 3.1.12, there exists a fixed point x_n of T_n, and thus the sequence $\{x_n\}$ forms an afps for T in E.

Next, let us see that T has an afps in $A(E, \{x_{n'}\})$ for some subsequence $\{x_{n'}\}$ of an afps $\{x_n\}$ for T in E. Let $\{x_n\}$ be an afps in E. We can assume that $\{x_n\}$ is regular relative to E. Let $A_{\mathcal{U}} = A_{\mathcal{U}}(E, \{x_n\})$. First, we show that

$$A_{\mathcal{U}} \cap Tx \neq \emptyset \text{ for every } x \in A_{\mathcal{U}}. \tag{3.1.3}$$

Let $x \in A_{\mathcal{U}}$ and for each $n \geq 1$, we see that

$$H(Tx_n, Tx) \leq \delta(T\{x_n, x\}) \leq \delta(\{x_n, x\}) = \|x_n - x\|.$$

Take $y_n \in Tx_n$ so that

$$\|x_n - y_n\| = dist(x_n, Tx_n),$$

and select $z_n \in Tx$ for each n such that

$$\|z_n - y_n\| = dist(y_n, Tx).$$

Let $\lim_{n\to\mathcal{U}} z_n = z \in Tx$. Note that

$$\|x_n - z\| \leq \|x_n - y_n\| + \|y_n - z_n\| + \|z_n - z\|.$$

We obtain

$$\begin{aligned}\lim_{n\to\mathcal{U}} \|x_n - z\| &\leq \lim_{n\to\mathcal{U}} \|y_n - z_n\| = \lim_{n\to\mathcal{U}} dist(y_n, Tx) \leq \lim_{n\to\mathcal{U}} H(Tx_n, Tx) \\ &\leq \lim_{n\to\mathcal{U}} \|x_n - x\| = r_{\mathcal{U}}(E, \{x_n\})\end{aligned}$$

proving that $z \in A_{\mathcal{U}}$. Thus (3.1.3) is satisfied. Fix $y_0 \in A_{\mathcal{U}}$ and consider, for each $n \geq 1$, the contraction $T_n : A_{\mathcal{U}} \to KC(X)$ defined by

$$T_n(x) = \frac{1}{n} y_0 + (1 - \frac{1}{n}) Tx, \qquad x \in A_{\mathcal{U}}.$$

For $x \in A_{\mathcal{U}}$, let $a \in A_{\mathcal{U}} \cap Tx$. Thus $\frac{1}{n} y_0 + (1 - \frac{1}{n}) a \in A_{\mathcal{U}} \cap T_n x$. Therefore $A_{\mathcal{U}} \cap T_n x \neq \emptyset$ for every $x \in A_{\mathcal{U}}$. Let K be a closed subset of E with $\delta(K) > 0$. As before, $\delta(T_n K) \leq (1 - \frac{1}{n}) \delta(TK) < \delta(K)$. By Theorem 3.1.12 (or we can apply Theorem 2.2.4), there exists a fixed point z_n of T_n. Again, as above, $\{z_n\}$ is an afps for T in $A_{\mathcal{U}}$. Finally, by Lemma 2.3.14, there exists a subsequence $\{x_{n'}\}$ of $\{x_n\}$ such that $\{z_n\} \subset A(E, \{x_{n'}\})$. $\square$

3.2 Kirk-Massa Condition

We aim to extend Theorem 1.2.4 to a wider class of mappings. Thus, the domains of mappings are more general than the ones in Section 3.1. Obviously, every space that satisfies the Kirk-Massa condition always has property (D'). Thus, particularly, the fixed point result in Section 3.1 holds for uniform convex Banach spaces, uniformly convex in every direction (UCED) and spaces satisfying the Opial condition.

Definition 3.2.1. Let $\mathcal{U}$ be a free ultrafilter defined on $\mathbb{N}$. Let E be a bounded closed and convex subset of a Banach space X. A mapping $T : E \to CB(X)$ is said to satisfy condition $(**)$ if it fulfills the following conditions.

(1) T has an afps in E, and

(2) if $\{x_n\}$ is an afps for T in E and $x \in E$, then $\lim_{n\to\mathcal{U}} H(Tx_n, Tx) \leq \lim_{n\to\mathcal{U}} \|x_n - x\|$.

Remark 3.2.2.

(i) Let E be a bounded closed and convex subset of a Banach space X, and let a mapping $T : E \to KC(X)$ satisfy condition $(**)$ and E is T-invariant. If in addition, T satisfies:

(A) every T-invariant, closed, and convex subset possesses an afps,

then T satisfies condition $(*)$.

Proof. By (1) of condition $(**)$, let $\{x_n\}$ be an afps for T in E. From Proposition 2.3.11 by passing through a subsequence, we may assume that $\{x_n\}$ is regular relative to E. Let $A_{\mathcal{U}} = A_{\mathcal{U}}(E, \{x_n\})$ and $x \in A_{\mathcal{U}}$. The compactness of Tx_n implies that for each n we can take $y_n \in Tx_n$ so that

$$\|x_n - y_n\| = dist(x_n, Tx_n).$$

Since Tx is compact, select $z_n \in Tx$ for each n such that

$$\|z_n - y_n\| = dist(y_n, Tx).$$

Let $\lim_{n\to\mathcal{U}} z_n = z \in Tx$. Note that

$$\|x_n - z\| \leq \|x_n - y_n\| + \|y_n - z_n\| + \|z_n - z\|.$$

We obtain

$$\begin{aligned} \lim_{n\to\mathcal{U}} \|x_n - z\| &\leq \lim_{n\to\mathcal{U}} \|y_n - z_n\| = \lim_{n\to\mathcal{U}} dist(y_n, Tx) \leq \lim_{n\to\mathcal{U}} H(Tx_n, Tx) \\ &\leq \lim_{n\to\mathcal{U}} \|x_n - x\| = r_{\mathcal{U}}(E, \{x_n\}) \end{aligned}$$

proving that $z \in A_{\mathcal{U}}$ and hence $A_{\mathcal{U}} \cap Tx \neq \emptyset$ for all $x \in A_{\mathcal{U}}$ i.e. $A_{\mathcal{U}}$ is T-invariant. By assumption, there exists an afps in $A_{\mathcal{U}}$. By Lemma 2.3.14, there exists a subsequence $\{x_{n'}\}$ of $\{x_n\}$ such that $\{z_n\} \subset A(E, \{x_{n'}\})$. Thus, T satisfies condition $(*)$. □

We wonder if we can drop condition (A) in proving the implication: $(**) \Rightarrow (*)$.

(ii) A mapping that satisfies condition $(*)$ need not satisfy condition $(**)$. Consider a mapping $T : [0, \frac{1}{2}] \to 2^{[0,\frac{1}{2}]}$ defined by $T(x) = [\sqrt{x}, \sqrt[3]{x}]$. Since 0 is a fixed point of T, the sequence $\{x_n\}$ given by $x_n \equiv 0$ for all n forms an afps for T. Thus, T fulfills condition (1) of Definition 3.1.6. If $\{x_n\}$ is an afps for T, then $\{x_n\}$ converges to 0 and $A(E, \{x_n\}) = \{0\}$. This implies that $A(E, \{x_n\})$ has an afps for T, and T satisfies condition $(*)$. On the other hand, for the afps $\{x_n\}$ given by $x_n \equiv 0$, if $x \in (0, \frac{1}{2}]$, then

$$\lim_{n\to\mathcal{U}} H(Tx_n, Tx) = \sqrt[3]{x} > x = \lim_{n\to\mathcal{U}} \|x_n - x\|.$$

Thus, T fails to satisfy condition $(**)$.

As mentioned earlier, it is unclear if a mapping satisfies condition $(**)$ also satisfies condition $(*)$.

The main idea of the proof of the following theorem is originated from Kirk and Massa [45].

Theorem 3.2.3. *Let E be a nonempty bounded closed and convex subset of a Banach space X satisfying the Kirk-Massa condition. Let $T : E \to KC(X)$ be a multivalued mapping satisfying condition $(**)$. If T is an upper semicontinuous mapping and invariant under E, then T has a fixed point.*

Proof. Let $\{x_n\}$ be an afps for T in E. From Proposition 2.3.11 by passing through a subsequence, we may assume that $\{x_n\}$ is regular relative to E. Let $A_{\mathcal{U}} = A_{\mathcal{U}}(E, \{x_n\})$. The compactness of Tx_n implies that for each n we can take $y_n \in Tx_n$ such that

$$\|x_n - y_n\| = dist(x_n, Tx_n).$$

If $x \in A_{\mathcal{U}}$, since Tx is compact, select $z_n \in Tx$ for each n such that

$$\|z_n - y_n\| = dist(y_n, Tx).$$

Let $\lim_{n\to\mathcal{U}} z_n = z \in Tx$. Note that

$$\|x_n - z\| \le \|x_n - y_n\| + \|y_n - z_n\| + \|z_n - z\|.$$

Thus

$$\begin{aligned}\lim_{n\to\mathcal{U}} \|x_n - z\| &\le \lim_{n\to\mathcal{U}} \|y_n - z_n\| = \lim_{n\to\mathcal{U}} dist(y_n, Tx) \le \lim_{n\to\mathcal{U}} H(Tx_n, Tx) \\ &\le \lim_{n\to\mathcal{U}} \|x_n - x\| = r_{\mathcal{U}}(E, \{x_n\}) \qquad (3.2.1)\end{aligned}$$

proving that $z \in A_{\mathcal{U}}$ and hence $A_{\mathcal{U}} \cap Tx \neq \emptyset$ for all $x \in A_{\mathcal{U}}$. By assumption, $A(E, \{x_n\})$ is nonempty and compact which implies that $A_{\mathcal{U}}$ is also nonempty and compact. Now define a mapping $F : A_{\mathcal{U}} \to KC(A_{\mathcal{U}})$ by $Fx := A_{\mathcal{U}} \cap Tx$ for all $x \in A_{\mathcal{U}}$. Thus F is upper semicontinuous. Indeed, let $\{u_n\} \subset A_{\mathcal{U}}$ be such that $\lim_{n\to\infty} u_n = u$ and $v_n \in Fu_n$ be such that $\lim_{n\to\infty} v_n = v$. Since T is upper semicontinuous and $A_{\mathcal{U}}$ is compact, we have $v \in Tu$ and $v \in A_{\mathcal{U}}$, that is $v \in Fu$. By Theorem 1.2.2, F and hence T, has a fixed point in $A_{\mathcal{U}}$. □

Remark 3.2.4.

(i) Recently, García-Falset, Lorens-Fuster and Morena-Gálvez [29] proved that condition (2) of Definition 3.2.1 is equivalent to

$$\limsup_{n\to\infty} H(Tx_n, Tx) \leq \limsup_{n\to\infty} \|x_n - x\|.$$

Therefore, we obtain the following result:

Corollary 3.2.5. *Let E be a nonempty bounded closed and convex subset of a Banach space X satisfying the Kirk-Massa condition. Let $T : E \to KC(X)$ be a multivalued mapping satisfying these conditions.*

(1) T has an afps in E, and

(2) if $\{x_n\}$ is an afps for T in E and $x \in E$, then

$$\limsup_{n\to\infty} H(Tx_n, Tx) \leq \limsup_{n\to\infty} \|x_n - x\|.$$

If T is an upper semicontinuous mapping, then T has a fixed point.

However, we can not follow the prove of Theorem 3.2.3 line-by-line to obtain this corollary. Therefore, the ultra-asymptitic center is a powerful tool in our proof.

(ii) In [49, Definition 3.1], the following concept of mappings is defined: A mapping $t : E \to E$ satisfy condition (L) on E provided that it fulfills the following two conditions.

(1) If a set $D \subset E$ is nonempty, closed, convex and t-invariant, then there exists an afps for t in D.

(2) For any afps $\{x_n\}$ of t in E and each $x \in E$,

$$\limsup_{n\to\infty} \|x_n - tx\| \leq \limsup_{n\to\infty} \|x_n - x\|.$$

Therefore, Remark 3.2.2 (i) shows that the class of mappings satisfying condition $(*)$ contains and extends mappings satisfying condition (L) as a single-valued version.

(iii) If, in addition, mappings in Theorem 3.2.3 also satisfy conditon (A), then the condition on "upper semicontinuity" can be dropped. This is because an afps in a compact set can be chosen so that its asymptotic center is only a singleton, and a fixed point can be easily derived. Consequently, Theorem 3.1.7 can be extended to a bigger class of domains, namely the bounded, closed, and convex ones. And the following results are immediate:

Corollary 3.2.6. ([49, Theorem 4.2]) *Let E be a nonempty compact convex subset of a Banach space X and $t : E \to E$ a mapping satisfying condition (L). Then, t has a fixed point.*

Corollary 3.2.7. ([49, Corollary 4.3]) *Let E be a nonempty compact convex subset of a Banach space X and $t : E \to E$ a mapping satisfying condition (L). Suppose that the asymptotic center in E of each sequence in E is nonempty and compact. Then, t has a fixed point.*

We give some examples of mappings satisfying condition $(**)$. The first example is of course the mapping described in Theorem 1.2.5.

Condition (C_λ)

Follows from [40, Lemma 2.8], we obtain the following result:

Proposition 3.2.8. *Let E be a nonempty bounded closed and convex subset of a Banach space X. If $T : E \to CB(E)$ satisfies condition (C_λ) for some $\lambda \in (0,1)$, then T satisfies condition (A), that is, T satisfies condition (1) of condition $(**)$.*

Generalized Nonexpansive Mappings

Let E be a nonempty subset of a Banach space X. Following [6], a mapping $t : E \to X$ is a generalized nonexpansive mapping if for some nonnegative constants $\alpha_1, ..., \alpha_5$ with $\sum_{i=1}^{5} \alpha_i = 1$,

$$\|tx - ty\| \leq \alpha_1\|x - y\| + \alpha_2\|x - tx\| + \alpha_3\|y - ty\| + \alpha_4\|x - ty\| + \alpha_5\|y - tx\|,$$

for each $x, y \in E$.

We will use the following equivalent condition:
For some nonnegative constants α, β, γ with $\alpha + 2\beta + 2\gamma \leq 1$, for all $x, y \in E$.

$$\|tx - ty\| \leq \alpha\|x - y\| + \beta(\|x - tx\| + \|y - ty\|) + \gamma(\|x - ty\| + \|y - tx\|).$$

We introduce a multivalued version of these mappings. Let $T : E \to CB(X)$ be a multivalued mapping. T is called a generalized nonexpansive mapping if there exist nonnegative constants α, β, γ with $\alpha + 2\beta + 2\gamma \leq 1$ such that for each $x, y \in E$, there holds

$$H(Tx, Ty) \leq \alpha\|x - y\| + \beta(dist(x, Tx) + dist(y, Ty)) + \gamma(dist(x, Ty) + dist(y, Tx)).$$

Obviously, nonexpansive mapping implies generalized nonexpansive mapping. The following example shows that the converse is not true:

Example 3.2.9. Define a mapping T on $[0,5]$ by

$$T(x)=\begin{cases}[0,\frac{x}{5}], x\neq 5;\\ \{1\}, x=5.\end{cases}$$

In Example 2.2.2. we know that T is not nonexpansive. It is easy to see that T is generalized nonexpansive where $\beta\in[\frac{1}{4},\frac{1}{2}]$ and $\alpha,\gamma\in[0,1]$ such that $\alpha+2\beta+2\gamma\leq 1$. Indeed, for $x,y\neq 5$ or $x,y=5$, we know that T is a contraction mapping which implies generalized nonexpansive. For $x\neq 5$ and $y=5$,

$$\begin{aligned} H(Tx,Ty) &= H([0,\frac{x}{5}],\{1\})=1=\frac{1}{4}\cdot 4\leq\beta\cdot 4=\beta|5-1|=\beta dist(y,Ty)\\ &\leq \alpha\|x-y\|+\beta(dist(x,Tx)+dist(y,Ty))+\gamma(dist(x,Ty)+dist(y,Tx)). \end{aligned}$$

Proposition 3.2.10. *Let E be a nonempty subset of a Banach space X. Suppose $T:E\to CB(X)$ is a generalized nonexpansive mapping, then T satisfies (2) of condition $(**)$.*

Proof. Let $\{x_n\}$ be an afps for T in E and $x\in E$. By assumption, we obtain

$$\begin{aligned} H(Tx_n,Tx) &\leq \alpha\|x_n-x\|+\beta(dist(x_n,Tx_n)+dist(x,Tx))\\ &\quad+\gamma(dist(x_n,Tx)+dist(x,Tx_n)). \end{aligned} \tag{3.2.2}$$

Since $dist(x,Tx)\leq\|x-x_n\|+dist(x_n,Tx_n)+H(Tx_n,Tx)$,
$dist(x_n,Tx)\leq dist(x_n,Tx_n)+H(Tx_n,Tx)$, $dist(x,Tx_n)\leq\|x-x_n\|+dist(x_n,Tx_n)$,

$$dist(x,Tx)\leq\lim_{n\to\mathcal{U}}\|x-x_n\|+\lim_{n\to\mathcal{U}}H(Tx_n,Tx), \tag{3.2.3}$$

$$\lim_{n\to\mathcal{U}}dist(x_n,Tx)\leq\lim_{n\to\mathcal{U}}H(Tx_n,Tx), \tag{3.2.4}$$

and

$$\lim_{n\to\mathcal{U}}dist(x,Tx_n)\leq\lim_{n\to\mathcal{U}}\|x-x_n\|. \tag{3.2.5}$$

By (3.2.2),

$$\begin{aligned} \lim_{n\to\mathcal{U}}H(Tx_n,Tx) &\leq \alpha\lim_{n\to\mathcal{U}}\|x_n-x\|\\ &\quad+\beta\lim_{n\to\mathcal{U}}\|x-x_n\|+\beta\lim_{n\to\mathcal{U}}H(Tx_n,Tx)\\ &\quad+\gamma\lim_{n\to\mathcal{U}}H(Tx_n,Tx)+\gamma\lim_{n\to\mathcal{U}}\|x-x_n\|. \end{aligned}$$

Thus

$$(1-\beta-\gamma)\lim_{n\to\mathcal{U}}H(Tx_n,Tx) \leq (\alpha+\beta+\gamma)\lim_{n\to\mathcal{U}}\|x-x_n\|,$$

and therefore

$$\lim_{n\to\mathcal{U}}H(Tx_n,Tx)\leq\lim_{n\to\mathcal{U}}\|x_n-x\|.$$

□

Takahashi Generalized Nonexpansive Mappings

Definition 3.2.11. Let E be a nonempty subset of a Banach space X. A mapping $t : E \to X$ is said to be a Takahashi generalized nonexpansive mapping if for some $\alpha, \beta \in [0,1]$ with $\alpha + 2\beta \le 1$, there holds

$$\|tx - ty\|^2 \le \alpha\|x-y\|^2 + \beta(\|y-tx\|^2 + \|x-ty\|^2) \text{ for } x, y \in E.$$

The following are examples of Takahashi generalized nonexpansive mappings:

- nonexpansive mappings t : $\|tx - ty\| \le \|x - y\|$;
- nonspreading mappings t [46] : $2\|tx-ty\|^2 \le \|y-tx\|^2 + \|x-ty\|^2$;
- hybrid mappings t [60] : $3\|tx-ty\|^2 \le \|y-tx\|^2 + \|x-ty\|^2$;
- mappings t [60] : $2\|tx-ty\|^2 \le \|x-y\|^2 + \|y-tx\|^2$;
- mappings t : $3\|tx-ty\|^2 \le 2\|y-tx\|^2 + \|x-ty\|^2$.

We define a multivalued version of Takahashi generalized nonexpansive mappings and prove that these mappings satisfy (2) of condition $(**)$.

Proposition 3.2.12. *Let E be a nonempty subset of a Banach space X. For nonnegative constants α, β with $\alpha + 2\beta \le 1$. If $T : E \to KC(X)$ is a multivalued mapping such that*

$$H^2(Tx,Ty) \le \alpha\|x-y\|^2 + \beta(dist^2(x,Ty) + dist^2(y,Tx)),$$

then T satisfies (2) of condition $(**)$.

Proof. Let $\{x_n\}$ be an afps for T in E and $x \in E$. By (3.2.4) and (3.2.5),

$$\begin{aligned}\lim_{n\to\mathcal{U}} H^2(Tx_n,Tx) &\le \alpha \lim_{n\to\mathcal{U}} \|x_n - x\|^2 \\ &\quad + \beta \lim_{n\to\mathcal{U}} dist^2(x,Tx_n) + \beta \lim_{n\to\mathcal{U}} dist^2(x_n,Tx) \\ &\le \alpha \lim_{n\to\mathcal{U}} \|x_n - x\|^2 \\ &\quad + \beta \lim_{n\to\mathcal{U}} \|x_n - x\|^2 + \beta \lim_{n\to\mathcal{U}} H^2(Tx_n,Tx).\end{aligned}$$

Thus

$$(1-\beta) \lim_{n\to\mathcal{U}} H^2(Tx_n,Tx) \le (\alpha+\beta) \lim_{n\to\mathcal{U}} \|x - x_n\|^2.$$

Therefore

$$\lim_{n\to\mathcal{U}} H(Tx_n,Tx) \le \lim_{n\to\mathcal{U}} \|x_n - x\|.$$

□

Remark 3.2.13.

(i) If T is a mapping in Example 3.2.9, then T also satisfies condition in Theorem 3.2.12.

(ii) If $t : E \to E$ is a generalized nonexpansive mapping with any of the following conditions holds, then t satisfies condition $(**)$:

(1) $\alpha + 2\beta + 2\gamma < 1$ (see [52, Theorem 4]);

(2) $\alpha + 2\beta + 2\gamma = 1$ and $\beta > 0, \gamma > 0, \alpha \geq 0$ (see [6, Theorem 1]);

(3) $\alpha + 2\beta + 2\gamma = 1$ and $\beta > 0, \gamma = 0, \alpha > 0$ (see [35, Theorem 1.1]);

(4) $\alpha + 2\beta + 2\gamma = 1$ and $\beta = 0, \gamma > 0, \alpha \geq 0$ (see [5, Lemma 2.1]).

(iii) Regarding the proof of Theorem 3.2.3, the fixed point result also holds for weak*-nonexpansive mappings $T : E \to KC(E)$([1]): for each $x, y \in E$ and $u_x \in Tx$ such that $\frac{1}{2}\|x - u_x\| \leq \|x - y\|$, there exists $u_y \in Ty$ such that $\|u_x - u_y\| \leq \|x - y\|$. Thereby [1, Theorem 1.7] is extended to another circumstance.