Chapter 4
Common Fixed Point Theorems via Technique of Nonexpansive Retracts

The main propose of this chapter is to obtain common fixed point theorems for a commuting family of nonexpansive mappings one of which is multivalued mappings by using nonexpansive retracts as a main tool.

4.1 Motivations and Basic Concepts

Recall that a bounded closed and convex subset E of a Banach space X has the *fixed point property for nonexpansive mappings* (FPP) (respectively, *for multivalued nonexpansive mappings* (MFPP)) if every nonexpansive mapping of E into E has a fixed point (respectively, every nonexpansive mapping of E into 2^E with compact convex values has a fixed point).

The following concepts were introduced by Bruck [9, 10]. For a bounded closed and convex subset E of a Banach space X, a mapping $t : E \to X$ is said to satisfy the *conditional fixed point property* (CFP) if either t has no fixed points, or t has a fixed point in each nonempty bounded closed convex set that leaves t invariant. A set E is said to have the *hereditary fixed point property for nonexpansive mappings* (HFPP) if every nonempty bounded closed convex subset of E has the *fixed point property for nonexpansive mappings; E is said to have the conditional fixed point property for nonexpansive mappings* (CFPP) if every nonexpansive $t : E \to E$ satisfies (CFP).

A direct consequence of Theorem 3.1.2 is that every weakly compact convex subset of a space having property (D) has both (MFPP) for multivalued nonexpansive mappings and (CFPP). The class of spaces having property (D) contains several well-known ones including k-uniformly rotund, nearly uniformly convex, uniformly convex in every direction spaces, and spaces satisfying Opial condition (see [3,19-23] and references therein).

For a subset F of E, a mapping $r : E \to F$ is a *retraction* if r is continuous and

$$r(x) = x, \text{ for every } x \in F.$$

A subset F is a nonexpansive retract of E if there exists a retraction of E onto F which is a nonexpansive mapping.

Example 4.1.1. Let $F = \{(x, 0) : x \in \mathbb{R}\} \subset \mathbb{R}^2$. Define a mapping $r : \mathbb{R}^2 \to F$ by $r((x, y)) = (x, 0)$ for $(x, y) \in \mathbb{R}^2$. Then F is nonexpansive retract of $\mathbb{R}^2$.

The following result was proved by Bruck:

Theorem 4.1.2. ([10, Theorem 1]) *Let E be a nonempty closed convex subset of a Banach space X. Suppose E is weakly compact or bounded and separable. Suppose E has both (FPP) and (CFPP). Then for any commuting family S of nonexpansive self-mappings of E, the set $F(S)$ of common fixed points of S is a nonempty nonexpansive retract of E.*

The object of this chapter is to extend Theorems 1.3.3 and 4.1.2 for a commuting family S of nonexpansive mappings one of which is multivalued. As consequences,

(i) Theorem 1.3.3 is extended to a bigger class of Banach spaces while a class of mappings is no longer finite;

(ii) Theorem 4.1.2 is extended so that one of its members in S can be multivalued.

The following result is a main tool of this chapter:

Theorem 4.1.3. ([9, Theorem 1]) *Let E be a nonempty closed convex subset of a Banach space X. Suppose E is locally weakly compact and F is a nonempty subset of E. Let $N(F) = \{f | f : E \to E$ is nonexpansive and $fx = x$ for all $x \in F\}$. Suppose that for each z in E, there exists h in $N(F)$ such that $h(z) \in F$. Then, F is a nonexpansive retract of E.*

Let (M, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ is a map c from a closed interval $[0, r] \subset \mathbb{R}$ to X such that $c(0) = x, c(r) = y$ and $d(c(t), c(s)) = |t - s|$ for all $s, t \in [0, r]$. The mapping c is an isometry and $d(x, y) = r$. The image of c is called a *geodesic segment* joining x and y which when unique is denoted by $\text{seg}[x, y]$. A metric space (M, d) is said to be of *hyperbolic type* if it is a metric space that contains a family L of geodesic segments such that (a) each two points x, y in M are endpoints of exactly one member $\text{seg}[x, y]$ of L, and (b) if $p, x, y \in M$ and $m \in \text{seg}[x, y]$ satisfies $d(x, m) = \alpha d(x, y)$ for $\alpha \in [0, 1]$, then $d(p, m) \leq (1 - \alpha)d(p, x) + \alpha d(p, y)$. M is said to be *metrically convex* if for any two points $x, y \in M$ with $x \neq y$ there exists $z \in M, x \neq z \neq y$, such that $d(x, y) = d(x, z) + d(z, y)$. Obviously, every metric space of hyperbolic type is always metrically convex. The converse is true provided that the space is complete: If (M, d) is a complete metric space and metrically convex, then (M, d) is of hyperbolic type (cf. [34, Page 24]). Clearly, every nonexpansive retract is of hyperbolic type.

Proposition 4.1.4. ([33, Proposition 2]) *Suppose* (M, d) *is of hyperbolic type, let* $\{\alpha_n\} \subset [0, 1)$, *if* $\{x_n\}$ *and* $\{y_n\}$ *are sequences in* M *which satisfy for all* i, n,

(i) $x_{n+1} \in seg[x_n, y_n]$ *with* $d(x_n, x_{n+1}) = \alpha_n d(x_n, y_n)$,

(ii) $d(y_{n+1}, y_n) \leq d(x_{n+1}, x_n)$,

(iii) $d(y_{i+n}, x_i) \leq d < \infty$,

(iv) $\alpha_n \leq b < 1$, *and*

(v) $\sum_{s=0}^{\infty} \alpha_s = +\infty$.

Then $\lim_{n\to\infty} d(y_n, x_n) = 0$.

4.2 Main Results

We begin with a useful result in order to prove our main theorem. If in Theorem 4.1.3, we put $F = Fix(t)$ where $t : E \to E$ is nonexpansive, then it was noted in [10, Remark 1] that a retraction $c \in N(F)$ can be chosen so that $cW \subset W$ for all t-invariant closed and convex subsets W of E. With the same proof, we can show that the same result is valid for $F = F(S)$. In this case, we define $N(F(S)) = \{f | f : E \to E$ is nonexpansive, $Fix(f) \supset F(S)$, $f(W) \subset W$ whenever W is a closed convex S-invariant subset of $E\}$. Here, by an "S-invariant" subset, we mean a subset that is left invariant under every member of S.

Lemma 4.2.1. *Let* E *be a nonempty weakly compact convex subset of a Banach space* X *and let* S *be any commuting family of nonexpansive self-mappings of* E. *Suppose that* E *has (FPP) and (CFPP). Then,* $F(S)$ *is a nonempty nonexpansive retract of* E, *and a retraction* c *can be chosen so that every* S*-invariant closed and convex subset of* E *is also* c*-invariant.*

Proof. Note by Theorem 4.1.2 that $F(S)$ is nonempty. According to Theorem 4.1.3, it suffices to show that for each z in E, there exists h in $N(F(S))$ such that $h(z) \in F(S)$.

Let $z \in E$ and $K = \{f(z) | f \in N(F(S))\} \subset E$. Since K is weakly compact convex and invariant under every member in S, we obtain by Theorem 4.1.2 that

$F(S) \cap K \neq \emptyset$, i.e., there exists h in $N(F(S))$ such that $h(z) \in F(S)$. Theorem 4.1.3 then implies that $F(S)$ is a nonexpansive retract of E, where a retraction is chosen from $N(F(S))$. □

We are ready to prove our first main result.

Theorem 4.2.2. *Let E be a weakly compact convex subset of a Banach space X. Suppose E has (MFPP) and (CFPP). Let S be any commuting family of nonexpansive self-mappings of E. If $T : E \to KC(E)$ is a multivalued nonexpansive mapping that commutes with every member of S, then $F(S) \cap Fix(T) \neq \emptyset$.*

Proof. Let c be a nonexpansive retraction of E onto $F(S)$ obtained in Lemma 4.2.1. Set $Ux := Tcx$ for $x \in E$. Clearly,

$$H(Ux, Uy) = H(Tcx, Tcy) \leq \|cx - cy\| \leq \|x - y\| \text{ for } x, y \in E.$$

Thus, U is nonexpansive, and since E has (MFPP), there exists $p \in Up = Tcp$. Since Tcp is S-invariant, by the property of c, Tcp is also c-invariant, i.e., $cp \in Tcp$. Therefore, $F(S) \cap Fix(T) \neq \emptyset$. □

For a subset A and $\varepsilon > 0$, the ε-neighborhood of A is defined as $B_\varepsilon(A) := \{y \in X : \|x - y\| < \varepsilon, \exists x \in A\}$. Note that if A is convex, then $B_\varepsilon(A)$ is also convex. The following proposition is needed for a proof of Theorem 4.2.4.

Proposition 4.2.3. *Let A be a compact convex subset of a Banach space X and let a nonempty subset F of A be a nonexpansive retract of A. Suppose a mapping $U : A \to KC(A)$ is upper semicontinuous and satisfies:*

(i) $c(Ux) \subset Ux$ for all $x \in F$ where c is a nonexpansive retraction of A onto F,

(ii) F is U-invariant.

Then, U has a fixed point in F.

Proof. Let $\varepsilon > 0$. Since F is compact, there exists a finite ε-dense subset $\{z_1, z_2, \ldots, z_n\}$ of F, i.e., $F \subset \bigcup_{i=1}^{n} B(z_i, \frac{\varepsilon}{2})$. Put $K := \overline{co}(z_1, z_2, \ldots, z_n)$ and define $Vx = \overline{B}_\varepsilon(Ucx) \cap K$ for $x \in K$. Clearly, $V : K \to KC(K)$. For $x \in K$, $cx \in F$ thus by (ii) there exists $y \in Ucx \cap F$. Then, choose z_i for some i such that $\|z_i - y\| \leq \frac{\varepsilon}{2}$. Therefore, $z_i \in \overline{B}_\varepsilon(Ucx) \cap K$, i.e., Vx is nonempty for $x \in K$. We now show that V is upper semicontinuous. Let $\{x_n\}$ be a sequence in K converging to some $x \in K$ and $y_n \in Vx_n$ with $y_n \to y$. Choose $a_n \in Ucx_n$ such that

$\|y_n - a_n\| \leq \varepsilon$. As A is compact, we may assume that $a_n \to a$ for some $a \in A$. By upper semicontinuity of U, $a \in Ucx$. Consider

$$\|y - a\| \leq \|y - y_n\| + \|y_n - a_n\| + \|a_n - a\|.$$

By letting $n \to \infty$, we obtain $\|y - a\| \leq \varepsilon$, i.e., $y \in Vx$ and the proof that V is upper semicontinuous is complete. By Theorem 1.2.2, there exists $p_\varepsilon \in Vp_\varepsilon$, that is, $\|p_\varepsilon - b_\varepsilon\| \leq \varepsilon$ for some $b_\varepsilon \in Ucp_\varepsilon$.

By the assumption on U, we see that $cb_\varepsilon \in Ucp_\varepsilon$ and $\|cp_\varepsilon - cb_\varepsilon\| \leq \|p_\varepsilon - b_\varepsilon\| \leq \varepsilon$. Taking $\varepsilon = \frac{1}{n}$ and write q_n for $cp_{\frac{1}{n}}$ and b_n for $cb_{\frac{1}{n}}$, we obtain a sequence $\{q_n\} \subset F$ and $b_n \in Uq_n \cap F$ with $\|q_n - b_n\| \to 0$. By the compactness of F, we assume that $q_n \to q$ and $b_n \to b$. It is seen that $q = b \in Uq$. □

The following is our second main result:

Theorem 4.2.4. *Let E be a weakly compact convex subset of a Banach space X satisfying the Kirk-Massa condition. Let S be any commuting family of nonexpansive self-mappings of E. Suppose $T : E \to KC(E)$ is a multivalued mapping satisfying condition (C_λ) for some $\lambda \in (0, 1)$ that commutes with every member of S. If T is upper semicontinuous, then $F(S) \cap Fix(T) \neq \emptyset$.*

Proof. As observed earlier, E has both (FPP) and (CFPP), thus we start with a nonexpansive retraction c of E onto $F(S)$ obtained by Lemma 4.2.1. For each $x \in F(S)$, Tx is invariant under every member of S and Tx is convex, thus Tx is c-invariant. Clearly, c is a nonexpansive retraction of Tx onto $Tx \cap F(S)$ that is nonempty by Theorem 4.1.2.

Next, we show that there exists an afps for T in $F(S)$. Let $x_0 \in F(S)$. Since $Tx_0 \cap F(S) \neq \emptyset$, we can choose $y_0 \in Tx_0 \cap F(S)$. Since $F(S)$ is of hyperbolic type, there exists $x_1 \in F(S)$ such that

$$\|x_0 - x_1\| = \lambda\|x_0 - y_0\| \text{ and } \|x_1 - y_0\| = (1 - \lambda)\|x_0 - y_0\|.$$

Choose $y'_1 \in Tx_1$ for which $\|y_o - y'_1\| = dist(y_0, Tx_1)$. Set $y_1 = cy'_1$. Clearly, $\|y_0 - y_1\| = \|cy_0 - cy'_1\| \leq \|y_0 - y'_1\|$. Therefore, we can choose $y_1 \in Tx_1 \cap F(S)$ so that $\|y_0 - y_1\| = dist(y_0, Tx_1)$. In this way, we will find a sequence $\{x_n\} \subset F(S)$ satisfying

$$\|x_n - x_{n+1}\| = \lambda\|x_n - y_n\| \text{ and } \|x_{n+1} - y_n\| = (1 - \lambda)\|x_n - y_n\|,$$

where $y_n \in Tx_n \cap F(S)$ and $\|y_n - y_{n+1}\| = dist(y_n, Tx_{n+1})$.
Since $\lambda dist(x_n, Tx_n) \leq \lambda\|x_n - y_n\| = \|x_n - x_{n+1}\|$,

$$\|y_n - y_{n+1}\| \leq H(Tx_n, Tx_{n+1}) \leq \|x_n - x_{n+1}\|.$$

From Proposition 4.1.4, $\lim_{n\to\infty} \|y_n - x_n\| = 0$ and $\{x_n\}$ is an afps for T in $F(S)$. Assume that $\{x_n\}$ is regular relative to E. By the Kirk-Massa condition, $A := A(E, \{x_n\})$ is assumed to be nonempty compact and convex. Define $Ux = Tx \cap A$ for $x \in A$. We are going to show that Ux is nonempty for each $x \in A$. First, let $r := r(E, \{x_n\})$. If $r = 0$ and if $x \in A$, then $x_n \to x$ and $y_n \to x$. Using upper semicontinuity of T, we see that $x \in Tx$, i.e., $F(S) \cap Fix(T) \neq \emptyset$.
Therefore, we assume for the rest of the proof that $r > 0$. Let $x \in A$. If for some subsequence $\{x_{n_k}\}$ of $\{x_n\}$, $\lambda dist(x_{n_k}, Tx_{n_k}) \geq \|x_{n_k} - x\|$ for each k, we have

$$0 = \limsup_{n\to\infty} \lambda dist(x_{n_k}, Tx_{n_k}) \geq \limsup_{n\to\infty} \|x_{n_k} - x\| \geq r$$

since $\{x_n\}$ is regular relative to E and this is a contradiction. Therefore,

$$\lambda dist(x_n, Tx_n) \leq \|x_n - x\| \text{ for sufficiently large } n. \tag{4.2.1}$$

Now, we show that Ux is nonempty. Choose $y_n \in Tx_n$ so that $\|x_n - y_n\| = dist(x_n, Tx_n)$ and choose $z_n \in Tx$ such that $\|y_n - z_n\| = dist(y_n, Tx)$. As Tx is compact, we may assume that $\{z_n\}$ converges to $z \in Tx$. Using (4.2.1) and the fact that T satisfies condition (C_λ), we have

$$\begin{aligned} \|x_n - z\| &\leq \|x_n - y_n\| + \|y_n - z_n\| + \|z_n - z\| \\ &= \|x_n - y_n\| + dist(y_n, Tx) + \|z_n - z\| \\ &\leq \|x_n - y_n\| + H(Tx_n, Tx) + \|z_n - z\| \\ &\leq \|x_n - y_n\| + \|x_n - x\| + \|z_n - z\| \text{ for sufficiently large } n. \end{aligned}$$

Taking supremum limit in the above inequalities to obtain

$$\limsup_{n\to\infty} \|x_n - z\| \leq \limsup_{n\to\infty} \|x_n - x\| = r.$$

This implies that $z \in Ux$ proving that Ux is nonempty as claimed.
Now, we show that U is upper semicontinuous. Let $\{z_k\}$ be a sequence in A converging to some $z \in A$ and $y_k \in Uz_k$ with $y_k \to y$. Consider the following estimates:

$$\begin{aligned} \limsup_{n\to\infty} \|x_n - y\| &\leq \limsup_{n\to\infty} \|x_n - y_k\| + \limsup_{n\to\infty} \|y_k - y\| \\ &= r(E, \{x_n\}) + \limsup_{n\to\infty} \|y_k - y\| \text{ for each } k. \end{aligned}$$

Letting $k \to \infty$, it follows that

$$\limsup_{n\to\infty} \|x_n - y\| \leq r(E, \{x_n\}).$$

Hence $y \in A$. From upper semicontinuity of T, $y \in Tz$. Therefore, $y \in Uz$ and thus U is upper semicontinuous. Put $F := F(S) \cap A$. Since A is c-invariant, it is clear that F is a nonexpansive retract of A by the retraction c. Now, if $x \in F$, then Ux is S-invariant which implies Ux is c-invariant. Therefore, condition (i) in Proposition 4.2.3 is justified. To verify condition (ii), we let $x \in F$. Take $y \in Ux$. It is obvious that $cy \in Ux \cap F(S)$, so U satisfies condition (ii) of Proposition 4.2.3. Upon applying Proposition 4.2.3 we obtain a fixed point in F of U and thus of T and we are done. □

Now, we are going to prove the last main theorem.

Theorem 4.2.5. *Let E be a weakly compact convex subset of a Banach space X. Suppose E has (MFPP) and (CFPP). Let S be any commuting family of nonexpansive self-mappings of E. If $T : E \to KC(E)$ is a multivalued nonexpansive mapping that commutes with every member of S. Suppose in addition that T satisfies:*

(i) there exists a nonexpansive mapping $s : E \to E$ such that $sx \in Tx$ for each $x \in E$,

(ii) Fix(T)$= \{x \in E : Tx = \{x\}\} \neq \emptyset$.

Then, $F(S) \cap Fix(T)$ is a nonempty nonexpansive retract of E.

Proof. By (i) and (ii), it is seen that $Fix(T) = Fix(s)$. Note by Theorem 4.2.2 that $F(S) \cap Fix(s)$ is nonempty. Let c be a retraction from E onto $F(S)$ obtained by Lemma 4.2.1. Here, c belongs to the set $N(F(S)) = \{f | f : E \to E$ is nonexpansive, $Fix(f) \supset F(S)$, $f(W) \subset W$ whenever W is a closed convex S-invariant subset of $E\}$. Put $F = F(S) \cap Fix(s)$ and let $N(F) = \{f | f : E \to E$ is nonexpansive, $Fix(f) \supset F\}$. Let $z \in E$ and consider the weakly compact and convex set $K := \{f(z) | f \in N(F)\}$. It is left to show that $h(z) \in F$ for some $h \in N(F)$. Since K is S-invariant, K is therefore c-invariant. It is evident that K is s-invariant. Thus $sc : K \to K$. Therefore, sc has a fixed point, say x, in K, i.e., $sc(x) = x$. By (i), $sc(x) \in Tcx$. Since Tcx is c-invariant, we have $cx \in Tcx$. That is $cx \in Fix(T) = Fix(s)$. Hence $scx = x = cx$, i.e., $cx \in F(S) \cap Fix(s)$, and the proof is complete. □

Remark 4.2.6.

(i) As corollaries, the results in Theorems 4.2.2 and 4.2.5 are valid for spaces X having property (D).

(ii) Theorem 4.2.5 can be viewed as a generalization of Theorem 4.1.2 for weakly compact convex domains.

When S consists of only the identity mapping of E, we immediately have the following corollary:

Corollary 4.2.7. *Let E be a weakly compact convex subset of a Banach space X. Suppose E has (MFPP). If $T : E \to KC(E)$ is a multivalued nonexpansive mapping satisfying:*

(i) there exists a nonexpansive mapping $s : E \to E$ such that $sx \in Tx$ for each $x \in E$,

(ii) Fix(T)$= \{x \in E : Tx = \{x\}\} \neq \emptyset$.

Then $Fix(T)$ is a nonempty nonexpansive retract of E.

Of course, when T is single valued, condition (i) is satisfied. Even a very simple example shows that condition (ii) in Corollary 4.2.7 may not be dropped.

Example 4.2.8. Let X be the Hilbert space $\mathbb{R}^2$ with the usual norm, and let $f : [0,1] \to [0,1]$ be a continuous function that is strictly concave, $f(0) = \frac{1}{2}$ and $f(1) = 1$. Moreover let $f'(x) \leq 1$ for $x \in [0,1]$. Let $T : [0,1]^2 \to KC([0,1]^2)$ be defined by $T(x,y) = [0,x] \times [f(x),1]$. We show that T is nonexpansive, but Fix(T)$\neq \{x : Tx = \{x\}\}$ and Fix(T) is not metrically convex. If $(x_1,y_1),(x_2,y_2) \in [0,1]^2$, then

$$H(T(x_1,y_1),T(x_2,y_2)) = |x_1 - x_2| \leq \|(x_1,y_1) - (x_2,y_2)\|.$$

Hence T is nonexpansive. However, $a = (0,\frac{1}{2})$ is a fixed point but $Ta \neq \{a\}$. Finally, Fix(T) is not metrically convex since, putting $b = (1,1)$, we see that $b \in Tb$, but $\frac{a+b}{2} = (\frac{1}{2},\frac{3}{4}) \notin T\frac{a+b}{2}$ since f is strictly concave.

The following example show a mapping that satisfies condition in Theorem 4.2.7.

Example 4.2.9. Let a mapping $T : [0,1] \to 2^{[0,1]}$ defined by $T(x) = [\frac{x}{4},\frac{x}{2}]$ for $x \in [0,1]$.

$$H(Tx,Ty) = H([\frac{x}{4},\frac{x}{2}],[\frac{y}{4},\frac{y}{2}]) = \frac{1}{2}\|x-y\| \leq \|x-y\|, \text{ for } x,y \in [0,1].$$

Thus, T is nonexpansive and $Fix(T) = \{0\}$. Moreover, there exists a nonexpansive mapping $s : [0,1] \to [0,1]$ such that $sx = \frac{x}{2} \in Tx$ for $x \in [0,1]$. Therefore, T satisfies condition in Theorem 4.2.7.

In [9, Lemma 6] it was stated that: Let E be a nonempty weakly compact convex subset of a Banach space X. Suppose E has (HFPP). Suppose F is a nonempty nonexpansive retract of E and $t : E \to E$ is a nonexpansive mapping which leaves F invariant. Then $Fix(t) \cap F$ is a nonempty nonexpansive retract of E.

Here, we have a multivalued version (with a similar proof) of this result.

Corollary 4.2.10. *Let E and T be as in Corollary 4.2.7. Suppose F is a nonexpansive retract of E by a retraction c. If Tx is c-invariant for each $x \in F$, then $Fix(T) \cap F$ is a nonempty nonexpansive retract of E.*