Chapter 2
Basic Concepts and Preliminaries

For understanding the thesis, we collect some notations, terminologies and elementary results. Although details are included in some cases, many well-known results are merely stated without proofs.

2.1 Metric Spaces

Definition 2.1. A *metric* (or *distance function*) d on X is a real-valued function defined on $X \times X$ satisfying the following conditions for any $x, y, z \in X$:

(M1) $d(x, y) \geq 0$;

(M2) $d(x, y) = 0$ if and only if $x = y$;

(M3) $d(x, y) = d(y, x)$ (symmetry);

(M4) $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality).

A pair (X, d) is called a *metric space* if d is a metric on X. Whenever it can be done without causing confusion, we denote the metric space (X, d) by the symbol X.

Example 2.2. Examples of metric spaces.

(i) *Euclidean space* $\mathbb{R}^n$. Let $\mathbb{R}^n$ be the set of all ordered n-tuples of real numbers. Then $\mathbb{R}^n$ is a metric space with the metric defined by

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$$

for all $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ in $\mathbb{R}^n$.

(ii) *Sequence space* ℓ^2. Let ℓ^2 be the set of all sequences $x = \{x_n\} \subset \mathbb{R}$ such that $\sum_{n=1}^{\infty} |x_n|^2 < \infty$. Then ℓ^2 is a metric space with the metric defined by

$$d(x, y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^2 \right)^{\frac{1}{2}}$$

for all $x = \{x_n\}$ and $y = \{y_n\}$ in ℓ^2.

(iii) *Sequence space* ℓ^∞. Let ℓ^∞ be the set of all bounded sequences $x = \{x_n\} \subset \mathbb{R}$. Then ℓ^∞ is a metric space with the metric defined by

$$d(x, y) = \sup_{n \in \mathbb{N}} |x_n - y_n|$$

for all $x = \{x_n\}$ and $y = \{y_n\}$ in ℓ^∞.

(iv) *Function space* $B(S)$. Let S be a nonempty set and $B(S)$ be the set of all bounded real-valued function on S. Then $B(S)$ is a metric space with the metric defined by

$$d(f, g) = \sup_{t \in S} |f(t) - g(t)|$$

for all $f, g \in B(S)$.

(v) *Space of bounded closed subsets* $FB(X)$. Let X be a metric space and $FB(X)$ a family of nonempty bounded closed subsets of X. For each $A, B \in FB(X)$, define the *Hausdorff distance* (or *Hausdorff metric*) on $FB(X)$ by

$$H(A, B) := \max \left\{ \sup_{a \in A} dist(a, B), \sup_{b \in B} dist(b, A) \right\}, \tag{2.1}$$

where $dist(a, B) := \inf_{b \in B} d(a, b)$ is the distance from the point a to the subset B. Then $(FB(X), H)$ is a metric space.

Definition 2.3. Let X be a metric space. The set

$$B(x_0, r) := \{x \in X : d(x_0, x) < r\},$$

where $r > 0$ and $x_0 \in X$, is called the *open ball* of radius r and centre x_0. The set

$$\overline{B}(x_0, r) := \{x \in X : d(x_0, x) \leq r\},$$

where $r > 0$ and $x_0 \in X$, is called the *closed ball* of radius r and centre x_0.

A subset G of X is called an *open set*, if there exists $r > 0$ such that $B(x, r) \subset G$ for any $x \in X$. A subset F of X is called a *closed set* if its complement $X \setminus F$ is open.

Definition 2.4. For any nonempty subset C of a metric space X, we define the *diameter* $\delta(C)$ of C by

$$\delta(C) = \sup_{x,y \in C} d(x, y).$$

In case $\delta(C)$ is finite, we call C a *bounded set*.

Definition 2.5. Let D be a nonempty set and $\succcurlyeq$ a relation on D. Then the ordered pair $(D, \succcurlyeq)$ is said to be *directed* if

(i) $\succcurlyeq$ is reflexive: $\alpha \succcurlyeq \alpha$ for all $\alpha \in D$;

(ii) $\succcurlyeq$ is transitive: whenever $\alpha \succcurlyeq \beta$ and $\beta \succcurlyeq \gamma \Rightarrow \alpha \succcurlyeq \gamma$ for all $\alpha, \beta, \gamma \in D$;

(iii) for any two elements α and β, there exists γ such that $\gamma \succcurlyeq \alpha$ and $\gamma \succcurlyeq \beta$.

In this case, we call D a *directed set* with the relation $\succcurlyeq$.

A *net* (or a *generalized sequence*) in X is a mapping from a directed set D into X. We use the notation $\{x_\alpha : \alpha \in D\}$ to stand for a net (sometimes it is simply written as $\{x_\alpha\}$). When we consider $\mathbb{N}$ as the directed set with the usual order $\geq$, we call $\{x_n\}$ a *sequence* in X.

Definition 2.6. Let $\{x_\alpha : \alpha \in D\}$ be a net in a set X and let D' be another directed set. A net $\{x_{\alpha_\beta} : \beta \in D'\}$ in X is said to be a *subnet* of $\{x_\alpha : \alpha \in D\}$ if it satisfies the following conditions:

(i) $\{x_{\alpha_\beta} : \beta \in D'\} \subset \{x_\alpha : \alpha \in D\}$;

(ii) for any $\alpha \in D$, there exists $\beta_0 \in D'$ such that $\alpha_\beta \succcurlyeq \alpha$ for all $\beta \succcurlyeq \beta_0$.

Definition 2.7. Let X be a metric space. A net $\{x_\alpha\}$ in X is said to be *convergent* if it converges to some $x_0 \in X$, i.e., for any $\varepsilon > 0$ there exists $\alpha_0 \in D$ such that $d(x_\alpha, x_0) < \varepsilon$ for all $\alpha \succcurlyeq \alpha_0$. In this case we write $\lim_\alpha x_\alpha = x_0$ and call x_0 the limit of $\{x_\alpha\}$.

When a sequence $\{x_n\}$ converges to a point $x_0 \in X$, we usually use the notation $\lim_{n\to\infty} x_n = x_0$ or $x_n \to x_0$ as $n \to \infty$.

Definition 2.8. A sequence $\{x_n\}$ in a metric space X is said to be *Cauchy* if for any $\varepsilon > 0$ there exists a positive integer n_0 such that $d(x_m, x_n) < \varepsilon$ for all $m, n \geq n_0$.

Definition 2.9. A metric space X is said to be *complete* if every Cauchy sequence is convergent.

Definition 2.10. A subset C of a metric space X is said to be *compact* if every sequence in C has a convergent subsequence in C.

The following is a general property of compact sets in metric spaces but the converse is not true.

Theorem 2.11. (cf. [33]) *A compact subset of a metric space is closed and bounded.*

2.2 Banach Spaces and Hilbert Spaces

A *vector space* (or *linear space*) over field $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$) is the set X of objects along with binary operations $+$ (called the addition) and $\cdot$ (called the scalar multiplication) satisfying the following conditions for any $x, y \in X$ and $\alpha, \beta \in \mathbb{F}$:

(V1) $x = y = y + x$;

(V2) $x + (y + x) = (x + y) + z$;

(V3) there exists a unique $0 \in X$ (called the zero element) such that $x + 0 = x$;

(V4) there exists a unique $-x \in X$ (called the negative of x) such that $x + (-x) = 0$;

(V5) $\alpha(x + y) = \alpha x + \alpha y$;

(V6) $(\alpha + \beta)x = \alpha x + \beta x$;

(V7) $(\alpha\beta)x = \alpha(\beta x)$;

(V8) $1 \cdot x = x$.

A nonempty subset Y of a vector space X is called a *subspace* of X if Y is also a vector space with respect to the binary operations defined on X. As a simple proof, we can show that Y is a subspace if and only if $\alpha x + \beta y \in y$ for all $x, y \in y$ and $\alpha, \beta \in \mathbb{F}$.

Definition 2.12. Let C be a subset of a vector space X. Then C is said to be *convex* if $(1-\alpha)x + \alpha y \in C$ for all $x, y \in C$ and all scalar $\alpha \in [0, 1]$.

Definition 2.13. Let C be a subset (not necessarily convex) of a vector space X. Then the *convex hull* of C in X, denoted by $co(C)$ is the intersection of all convex subsets of X containing C.

Definition 2.14. A *norm* $\|\cdot\|$ on a vector space X is a real-valued function defined on X which satisfies the followings for any $x, y \in X$ and $\alpha \in \mathbb{F}$:

(N1) $\|x\| \geq 0$;

(N2) $\|x\| = 0$ if and only if $x = 0$;

(N3) $\|\alpha x\| = |\alpha|\|x\|$;

(N4) $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality).

A *normed space* X is a vector space with a norm. It is easy to see that every normed space is a metric space with the metric $d_{\|\cdot\|}$ (called the metric induced by the norm) defined by

$$d_{\|\cdot\|}(x, y) = \|x - y\|$$

for all $x, y \in X$. When $(X, d_{\|\cdot\|})$ is a complete metric space, we call X a *Banach space*.

Example 2.15. Examples of Banach spaces.

(i) *Euclidean space* $\mathbb{R}^n$. The space $\mathbb{R}^n$ is a Banach space with the norm defined by

$$\|x\| = \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}} = \sqrt{|x_1|^2 + \cdots + |x_n|^2}$$

for all $x = (x_1, ..., x_n) \in \mathbb{R}^n$.

(ii) *Sequence space* ℓ^2. The space ℓ^2 is a Banach space with the norm defined by

$$\|x\| = \left(\sum_{n=1}^{\infty} |x_n|^2\right)^{\frac{1}{2}}$$

for all $x = \{x_n\} \in \ell^2$.

(iii) *Sequence space* ℓ^∞. The space ℓ^∞ is a Banach space with the norm defined by

$$\|x\| = \sup_{n \in \mathbb{N}} |x_n|$$

for all $x = \{x_n\} \in \ell^\infty$.

(iv) *Function space* $B(S)$. The space $B(S)$ is a Banach space with the norm defined by

$$\|f\| = \sup_{t \in S} |f(t)|$$

for all $f \in B(S)$.

(v) *Dual space* X^*. Let X^* be the set of all continuous linear functional on a normed space X. Then X^* is a Banach space space under the norm

$$\|f\| = \sup_{\|x\|=1} |f(x)|$$

for all $f \in X^*$.

Definition 2.16. Let x_0 be a vector in a Banach space X and $f \in X^*$. For each $\varepsilon > 0$, define

$$U(x_0 : f, \varepsilon) = \{x \in X : |f(x - x_0)| < \varepsilon\}.$$

The *weak topology* on X is the topology generated by the class of all sets which are expressible in the form $U(x_0 : f, \varepsilon)$.

Definition 2.17. Let f_0 be a vector in X^* and $x \in X$. For each $\varepsilon > 0$, define

$$U(f_0 : x, \varepsilon) = \{f \in X^* : |f_0(x) - f(x)| < \varepsilon\}.$$

The *weak* topology* on X is the topology generated by the class of all sets which are expressible in the form $U(f_0 : x, \varepsilon)$.

Theorem 2.18 (Alaoglu's Theorem)**.** *Let X be a normed space. Then the closed unit ball*

$$\{f \in X^* : \|f\| \leq 1\}$$

is compact in the weak* topology.

In this thesis, we consider about some Banach spaces with the special conditions. Here is their definitions and important properties.

Definition 2.19. A Banach space X is said to be *strictly convex* if $\|x\| = \|y\| = 1$ and $x \neq y$ implies

$$\left\|\frac{x+y}{2}\right\| < 1.$$

Definition 2.20. A Banach space X is said to be *uniformly convex* if for any $\varepsilon \in (0, 2]$ there exists $\delta > 0$ such that

$$\left\|\frac{x+y}{2}\right\| \leq 1 - \delta,$$

for all $x, y \in X$ satisfying $\|x\| \leq 1,\ \|y\| \leq 1$ and $\|x - y\| \geq \varepsilon$.

Theorem 2.21. (cf. [44]) *Every uniformly convex Banach space is strictly convex.*

Definition 2.22. Let X be a vector space over field $\mathbb{R}$. An *inner product* on X is a real-valued function $\langle \cdot, \cdot \rangle$ defined on $X \times X$ with the following properties:

(I1) $\langle x, x \rangle \geq 0$ for all $x \in X$;

(I2) $\langle x, x \rangle = 0$ if and only if $x = 0$;

(I3) $\langle x, y \rangle = \langle y, x \rangle$;

(I4) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$ for all $x, y, z \in X$ and $\alpha, \beta \in \mathbb{R}$.

An *inner product space* is a vector space with an inner product. It is easy to see that every inner product space is a normed space with the norm (called the norm induced by the inner product)

$$\|x\| = \sqrt{\langle x, x \rangle}$$

for all $x \in X$. We call X a *Hilbert space* when it is complete under the metric induced by the norm.

Example 2.23. Example of Hilbert spaces.

(i) *Euclidean space* $\mathbb{R}^n$. The space $\mathbb{R}^n$ is a Hilbert space with the inner product defined by

$$\langle x, y\rangle = x_1y_1 + \cdots + x_ny_n$$

for all $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ in $\mathbb{R}^n$.

(ii) *Sequence space* ℓ^2. The space ℓ^2 is a Hilbert space with the inner product defined by

$$\langle x, y\rangle = \sum_{n=1}^{\infty} x_ny_n$$

for all $x = \{x_n\}$ and $y = \{y_n\}$ in ℓ^2.

The following properties are important for inner product spaces (see [33] for more details).

Proposition 2.24 (The Cauchy-Schwarz inequality)**.** *Let X be an inner product space. Then*

$$|\langle x, y\rangle| \leq \|x\|\|y\|,$$

for all $x, y \in X$.

Proposition 2.25 (The parallelogram law)**.** *Let X be an inner product space. Then*

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2,$$

for all $x, y \in X$.

Theorem 2.26 (Reisz's representation theorem)**.** *Let X be a Hilbert space and $f \in X^*$. Then there exists a unique element $y_0 \in X$ such that $f(x) = \langle x, y_0\rangle$ for each $x \in X$. In this case, $\|f\| = \|y_0\|$.*

2.3 Fixed Point Theorems for Nonexpansive Semigroups

In this section, we collect some definitions and the well-known fixed point theorems for nonexpansive semigroups in Hilbert spaces.

Definition 2.27. Let S be a semigroup. A continuous linear functional $\mu \in B(S)^*$ is called a *mean* on $B(S)$ if $\|\mu\| = \mu(1) = 1$.

For any $f \in B(S)$ we use the following notation: $\mu(f) = \mu_s(f(s))$.

When we consider the semigroup $S = \mathbb{N}$ under the usual addition, we can see that $B(S) = B(\mathbb{N}) = \ell^\infty$. A continuous linear functional $\mu \in \ell^\infty$ is said to be a *Banach limit* if it is a mean on ℓ^∞ and $\mu_n(x_n) = \mu_n(x_{n+1})$ for all $x = \{x_1, x_2, ...\} \in \ell^\infty$.

Definition 2.28. Let S be a semigroup. A mean μ on $B(S)$ is said to be *left invariant* [resp. *right invariant*] if $\mu_s(f(ts)) = \mu_s(f(s))$ [resp. $\mu_s(f(st)) = \mu_s(f(s))$] for all $f \in B(S)$ and for all $t \in S$.

We will say that μ is an *invariant mean* if it is both left and right invariant. If $B(S)$ has an invariant mean, we call S an *amenable semigroup*.

It is well-known that every commutative semigroup is amenable [13]. For each $s \in S$ and $f \in B(S)$, we define elements $l_s f$ and $r_s f$ in $B(S)$ by $(l_s f)(t) = f(st)$ and $(r_s f)(t) = f(ts)$ for any $t \in S$, respectively.

Definition 2.29. A net $\{\mu_\alpha\}$ of means on $B(S)$ is said to be *asymptotically invariant* if

$$\lim_\alpha(\mu_\alpha(l_s f) - \mu_\alpha(f)) = 0 = \lim_\alpha(\mu_\alpha(r_s f) - \mu_\alpha(f)).$$

Remark 2.30. For each asymptotically net of means $\{\mu_\alpha\}$, there exists a subnet $\{\mu_{\alpha'}\}$ of $\{\mu_\alpha\}$ such that $\{\mu_{\alpha'}\}$ w*-converges to some invariant mean in $B(S)^*$

Proof. Since $\{\mu_\alpha\}$ is a closed subset of $\{\gamma \in B(S)^* : \|\gamma\| \leq 1\}$, by the Alaoglu's Theorem, it is compact in the weak* topology. So there exists a subnet $\{\mu_{\alpha'}\}$ of $\{\mu_\alpha\}$ such that $\{\mu_{\alpha'}\}$ w*-converges to some μ in $B(S)^*$. We can obtain, from the proof of Theorem 3.4.4 in [44], that μ is invariant. □

Proposition 2.31. (cf. [44]) *Let μ be a right invariant mean on $B(S)$. Then,*

$$\sup_s \inf_t f(ts) \leq \mu(f(s)) \leq \inf_s \sup_t f(ts)$$

for each $f \in B(S)$. Similarly, let μ be a left invariant mean on $B(S)$. Then,

$$\sup_s \inf_t f(st) \leq \mu(f(s)) \leq \inf_s \sup_t f(st)$$

for each $f \in B(S)$.

Remark 2.32. If $\lim_s f(s) = a$ for some $a \in \mathbb{R}$ and $\{s'\}$ is a subnet of $\{s\}$ satisfying $s' \succ s$ for each s, then

$$\mu_{s'}(f(s')) = a.$$

Proof. This is an easy consequence of Proposition 2.31 since $\mu_{s'}(f(s')) = \mu_s(f(s')) = \lim_s f(s') = a$. $\square$

Theorem 2.33. (cf. [44]) *Let S be an amenable semigroup and C be a closed convex subset of a Hilbert space X. Let $\mathcal{S} = \{T_s : s \in S\}$ be a nonexpansive semigroup on C. Then the followings are equivalent:*

(i) $\{T_s x : s \in S\}$ *is bounded for some* $x \in C$*;*

(ii) $\{T_s x : s \in S\}$ *is bounded for all* $x \in C$*;*

(iii) $Fix(\mathcal{S}) \neq \emptyset$.

Theorem 2.34. (cf. [44]) *Let S be an amenable semigroup and C be a closed convex subset of a Hilbert space X. Let $\mathcal{S} = \{T_s : s \in S\}$ be a nonexpansive semigroup on C with $Fix(\mathcal{S}) \neq \emptyset$. Then for each invariant mean μ on B(S), T_μ satisfies the followings:*

(i) $T_\mu : C \to C$ *is nonexpansive;*

(ii) $T_\mu T_s = T_\mu = T_s T_\mu$ *for all* $s \in S$*;*

(iii) $T_\mu x \in \overline{co}\{T_s x : t \in S\}$ *for all* $x \in C$.

2.4 CAT(0) Spaces

For studying the definition and basic properties of CAT(0) spaces, we begin with introducing some geometric concepts in metric spaces.

Definition 2.35. Let X be a metric space. A *geodesic* joining $x \in X$ to $y \in X$ is a mapping c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$.

The image γ of c is called a *geodesic (or metric) segment* joining x and y. When it is unique this geodesic is denoted $[x, y]$. Write $c(\alpha\, 0 + (1 - \alpha)l) = \alpha x \oplus (1 - \alpha)y$ and for $\alpha = \frac{1}{2}$ we write $\frac{1}{2}x \oplus \frac{1}{2}y$ as $\frac{x \oplus y}{2}$, the midpoint of x and y.

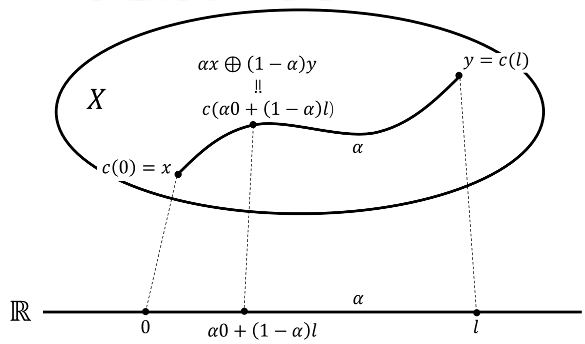

Figure 2.1: Geodesic segment

The space X is said to be a *(uniquely) geodesic space* if every two points of X are joined by a (unique) geodesic. We can define a convex set in a geodesic space as follow:

Definition 2.36. A subset C of a geodesic space X is said to be *convex* if C includes every geodesic segment joining between x and y for all $x, y \in C$.

A *geodesic triangle* $\triangle(x, y, x)$ in a geodesic space X consists of three points x, y, z in X (the vertices of $\triangle$) and three geodesic segments joining between each pair of its vertices. A *comparison triangle* for a geodesic triangle $\triangle(x, y, x)$ is a triangle $\overline{\triangle}(x, y, z) := \triangle(\overline{x}, \overline{y}, \overline{v})$ in the Euclidean plane $\mathbb{R}^2$ such that $d(x, y) = d(\overline{x}, \overline{y})$, $d(x, z) = d(\overline{x}, \overline{z})$ and $d(y, z) = d(\overline{y}, \overline{z})$.

Definition 2.37. A geodesic space X is said to be a *CAT(0) space* if every geodesic triangle in X is at least as thin as its comparison triangle in the Euclidean plane, i.e.,

$$d(a, b) \leq d_{\mathbb{R}^2}(\overline{a}, \overline{b})$$

for any $a, b \in \triangle(x, y, z)$ and $\overline{a}, \overline{b} \in \overline{\triangle}(x, y, z)$ (see Figure 2.2).

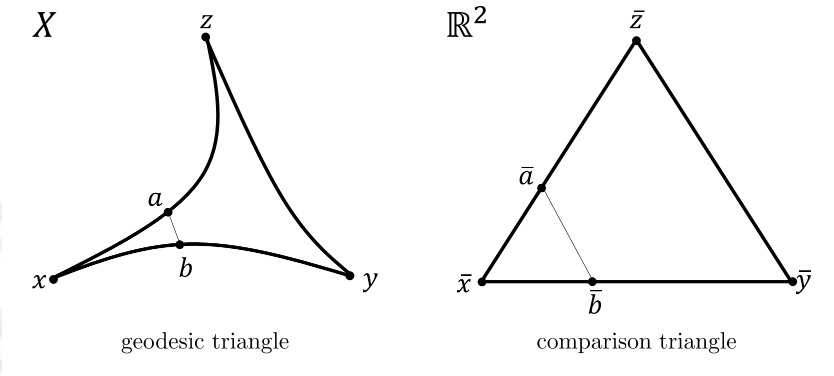

Figure 2.2: Geodesic triangle in a CAT(0) space and its comparison triangle

Example 2.38. Example of CAT(0) spaces.

(i) Hilbert spaces.

(ii) $\mathbb{R}-$trees: A uniquely geodesic space X is an $\mathbb{R}-$tree if

$$[x, y] \cap [y, z] = \{y\} \Longrightarrow [x, z] = [x, y] \cup [y, z].$$

(iii) Classical hyperbolic space H^n: A vector space $\mathbb{R}^{n+1}$ is a hyperbolic space of n-dimension if

$$\sum_{i=1}^{n} u_i^2 - u_{n+1}^2 = -1 \text{ and } u_{n+1} > 0$$

for all $(u_1, ..., u_{n+1}) \in \mathbb{R}^{n+1}$.

(iv) Let $X = \{0\} \times [0, 1] \times \mathbb{R}$ and $Y = [-1, 1] \times \{0\} \times \{0\} \cup \{0\} \times [0, 1] \times \{0\}$. Then $Z = X \cup Y$ is a CAT(0) space[1] (see Figure 2.3).

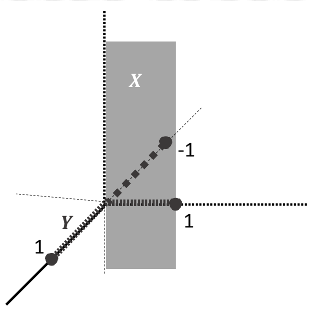

Figure 2.3: An example of CAT(0) spaces

[1]This example, constructed by Dr. Santi Tasena, is the one of CAT(0) space which does not be a Hilbert space or an $\mathbb{R}-$tree.

Let X be a geodesic space. The *Alexandrov angle* between two geodesics $c_1 : [0, t_1] \to X$ and $c_2 : [0, t_2] \to X$ with $c_1(0) = c_2(0)$ is the number $\xi \in [0, \pi]$ such that

$$\xi = \limsup_{t_1, t_2 \to 0} \measuredangle(c_1(t_1), c_1(0), c_2(t_2)),$$

where $\measuredangle(c_1(t_1), c_1(0), c_2(t_2))$ is the interior angle at $\overline{c_1(0)}$ in $\overline{\triangle}(c_1(t_1), c_1(0), c_2(t_2))$.

The following is the characterization of CAT(0) spaces (see [9] for more details):

Theorem 2.39. *The followings are equivalent for a uniquely geodesic space X*

(i) X is a CAT(0) space.

*(ii) X satisfies the **(CN) inequality**: If $x, y \in X$ and $\alpha \in (0, 1)$, then*

$$d^2\left(z, \alpha x \oplus (1 - \alpha)y\right) \leq \alpha d^2(z, x) + (1 - \alpha)d^2(z, y) - \alpha(1 - \alpha)d^2(x, y),$$

for all $z \in X$.

*(iii) X satisfies the **law of cosine**: If $a = d(x, z), b = d(y, z), c = d(x, y)$ and ξ is the Alexandrov angle at z between $[x, z]$ and $[y, z]$, then*

$$c^2 \geq a^2 + b^2 - 2ab\cos\xi.$$

Lemma 2.40. [9, Proposition 2.2] *Let X be a CAT(0) space. Then for each $p, q, x, y \in X$ and $\alpha \in [0, 1]$,*

$$d(\alpha p \oplus (1 - \alpha)q, \alpha x \oplus (1 - \alpha)y) \leq \alpha d(p, x) + (1 - \alpha)d(q, y).$$

In particular, a CAT(0) space is of hyperbolic type, i.e., it satisfies the below inequality:

$$d(p, \alpha x \oplus (1 - \alpha)y) \leq \alpha d(p, x) + (1 - \alpha)d(p, y). \tag{2.2}$$

For any nonempty subset C of X, let $\pi = \pi_C$ be the projection mapping from X to C. It is known by [9, p.176 - 177] that if C is closed and convex, the mapping π is well-defined, nonexpansive, and satisfies

$$d^2(x, y) \geq d^2(x, \pi x) + d^2(\pi x, y) \textit{ for all } x \in X \textit{ and } y \in C \tag{2.3}$$

(see Figure 2.4).

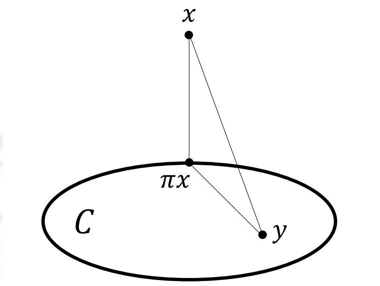

Figure 2.4: The projection mapping

Definition 2.41. [18, Definition 5.13] A complete CAT(0) space X has the *property of the nice projection onto geodesics* (*property (N)* for short) if, given any geodesic segment $I \subset X$, it is the case that $\pi_I(m) \in [\pi_I(x), \pi_I(y)]$ for any x, y in X and $m \in [x, y]$ (see Figure 2.5).

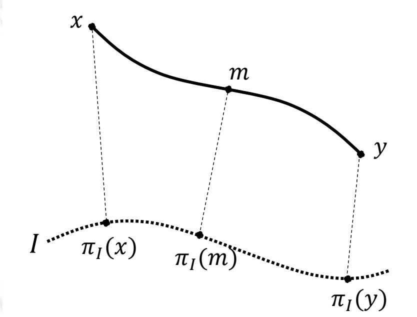

Figure 2.5: Property (N)

As noted in [18], we do not know of any example of a CAT(0) space which does not enjoy the property (N).

Definition 2.42. For any bounded net $\{x_\alpha\}$ in a closed convex subset C of a CAT(0) space X, put

$$r(x, \{x_\alpha\}) = \limsup_{\alpha} d(x, x_\alpha)$$

for each $x \in C$. The *asymptotic radius* of $\{x_\alpha\}$ on C is given by

$$r(C, \{x_\alpha\}) = \inf_{x \in C} r(x, \{x_\alpha\}),$$

and the *asymptotic center* of $\{x_\alpha\}$ in C is the set

$$A(C, \{x_\alpha\}) = \{x \in C : r(x, \{x_\alpha\}) = r(C, \{x_\alpha\})\}.$$

It is known that in a complete CAT(0) space, $A(C, \{x_\alpha\})$ consists of exactly one point and $A(X, \{x_\alpha\}) = A(C, \{x_\alpha\})$ [17].

Remark 2.43.

(i) Let D, D' be directed sets. If $\{x_{\alpha_\beta} : \beta \in D'\}$ is a subnet of a bounded net $\{x_\alpha : \alpha \in D\}$, then $r(C, \{x_{\alpha_\beta}\}) \leq r(C, \{x_\alpha\})$.

(ii) Let C be a closed convex subset of a CAT(0) space X, $T : C \to C$ a nonexpansive mapping and $x \in C$. If $\{T^n x\}$ is bounded and $z \in A(C, \{T^n x\})$, then $z \in F(T)$.

Proof. (i) Let $\alpha_0 \in D$. By the definition of subnets, there exists $\beta_0 \in D'$ such that $\nu(\beta) \succcurlyeq \alpha_0$ for all $\beta \succcurlyeq \beta_0$. For each $x \in C$, we have $\sup_{\alpha \succcurlyeq \alpha_0} d(x, x_\alpha) \geq \sup_{\beta \succcurlyeq \beta_0} d(x, x_{\alpha_\beta})$. Thus

$$\sup_{\alpha \succcurlyeq \alpha_0} d(x, x_\alpha) \geq \inf_{\beta_1} \sup_{\beta \succcurlyeq \beta_1} d(x, x_{\alpha_\beta}),$$

for all α_0. Hence

$$r(x, \{x_\alpha\}) = \inf_{\alpha_0} \sup_{\alpha \succcurlyeq \alpha_0} d(x, x_\alpha) \geq r(x, \{x_{\alpha_\beta}\})$$

Since the above inequality holds for all $x \in C$, we see that

$$r(C, \{x_\alpha\}) = \inf_{x \in C} r(x, \{x_\alpha\}) \geq \inf_{x \in C} r(x, x_{\nu(\beta)}) = r(C, \{x_{\nu(\beta)}\}).$$

(ii) By the nonexpansiveness of T, $\limsup_n d(T^n x, Tz) = \limsup_n d(TT^n x, Tz)$ $\leq \limsup_n d(T^n x, z)$. As every asymptotic center is unique, we have $z = Tz$. $\square$

2.5 Weak Convergence in CAT(0) Spaces

In 1976, Lim [30] introduced a concept of convergence in a general metric space, called strong $\Delta-$convergence. Many years later, Kirk and Panyanak [25] introduced a concept of convergence in a CAT(0) space, called $\Delta-$convergence.

Definition 2.44. A net $\{x_\alpha\}$ in a CAT(0) space X is said to $\Delta-$*converge* to $x \in X$ if x is the unique asymptotic center of $\{u_\beta\}$ for every subnet $\{u_\beta\}$ of $\{x_\alpha\}$. In this case, we write $\Delta - \lim_\alpha x_\alpha = x$ and call x the $\Delta-$limit of $\{x_\alpha\}$.

Proposition 2.45. [25, Proposition 3.4] *Every bounded net in X has a $\Delta-$convergent subnet.*

Remark 2.46.

(i) Let D be a directed set, $\{x_\alpha : \alpha \in D\}$ a net in X and $x \in X$. If $\limsup_\alpha d(x, x_\alpha) > \rho$ for some $\rho > 0$, Then there exists a subnet $\{x_{\beta_\alpha}\}$ of $\{x_\alpha\}$ such that $d(x, x_{\beta_\alpha}) \geq \rho$ for all α.

(ii) Let $\{x_\alpha\}$ be a net in X. Then $\{x_\alpha\}$ $\Delta-$converges to $x \in X$ if and only if every subnet $\{x_{\alpha'}\}$ of $\{x_\alpha\}$ has a subnet $\{x_{\alpha''}\}$ which $\Delta-$converges to x.

Proof. (i): For each $\alpha \in D$ we have $\sup_{\alpha' \succcurlyeq \alpha} d(x, x_{\alpha'}) > \rho$. Thus there exists $\beta_\alpha \succ \alpha$ such that $d(x, x_{\beta_\alpha}) \geq \rho$, and this holds for all α. Set a set $D' = \{\beta_\alpha : \alpha \in D\}$. Clearly, D' is a directed set. Let $\alpha_0 \in D$, thus $\beta_\alpha \succcurlyeq \alpha_0$ for all $\beta_\alpha \succcurlyeq \beta_{\alpha_0}$. This shows that $\{x_{\beta_\alpha}\}$ is a subnet of $\{x_\alpha\}$ satisfying $d(x, x_{\beta_\alpha}) \geq \rho$ for all α.

(ii): It is easy to see that if $\{x_\alpha\}$ $\Delta-$converges to x, then every subnet of $\{x_\alpha\}$ also $\Delta-$converges to x. On the other hand, suppose $\{x_\alpha\}$ does not $\Delta-$converge to x. Thus there exists a subnet $\{x_\beta\}$ of $\{x_\alpha\}$ such that $x \notin A(C, \{x_\beta\})$, and so $\limsup_\beta d(x, x_\beta) > \rho > r(C, \{x_\beta\})$ for some $\rho > 0$. By (i), there exists a subnet $\{x_{\gamma_\beta}\}$ of $\{x_\beta\}$ satisfying $d(x, x_{\gamma_\beta}) \geq \rho$ for all β. By assumption, there exists a subnet $\{x_{(\gamma_\beta)_\eta}\}$ of $\{x_{\gamma_\beta}\}$ $\Delta-$converging to x. Using Remark 2.43, $\rho \leq \limsup_\gamma d(x, x_{(\gamma_\beta)_\eta}) = r(C, \{x_{(\gamma_\beta)_\eta}\}) \leq r(C, \{x_{\gamma_\beta}\}) \leq r(C, \{x_\beta\})$, a contradiction. □

In 2008, Berg and Nikolaev [8] introduced a concept of quasilinearization as follow. For any metric space X, we call an order pair $(a, b) \in X \times X$ a vector and denote it by $\overrightarrow{ab}$. The *quasilinearization* is defined as a map $\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}d^2(a, d) + \frac{1}{2}d^2(b, c) - \frac{1}{2}d^2(a, c) - \frac{1}{2}d^2(b, d)$$

for all $a, b, c, d \in X$. By using the above concept, Kakavandi and Amini [22] introduced another concept of weak convergence in CAT(0) spaces, called w-convergence.

Definition 2.47. A sequence $\{x_n\}$ in a CAT(0) space X is said to *w-converge* to $x \in X$ if $\lim_{n\to\infty} \langle \overrightarrow{x_n x}, \overrightarrow{ab} \rangle = 0$ for all $a, b \in X$.

Proposition 2.48. [22, Proposition 2.5] *For sequences in a complete CAT(0) space X, w-convergence implies $\Delta-$convergence (to the same limit).*

A simple example shows that the converse of this proposition does not hold:

Example 2.49. Consider an $\mathbb{R}-$tree in $\mathbb{R}^\infty$ defined as follow: Let $\{e_n\}$ be the standard basis, $x_0 = e_1 = (1, 0, 0, 0, ...)$, and for each n, let $x_n = x_0 + e_{n+1}$. An $\mathbb{R}-$tree is formed by the segments $[x_1, x_n]$ for $n = 0, 1, 2, ...$ (see Figure 2.6).

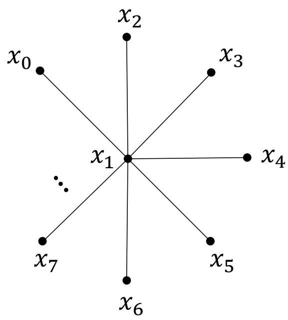

Figure 2.6: The $\mathbb{R}-$tree in Example 2.49

It is easy to see that $\{x_n\}$ Δ-converges to x_1. But $\{x_n\}$ does not w-converge to x_1 since $\langle \overrightarrow{x_n x_1}, \overrightarrow{x_0 x_1} \rangle = -1$ for all $n \geq 2$. Thus a bounded sequence does not necessary contain an w-convergent subsequence.

2.6 Convex Combination in CAT(0) Spaces

In this section we give a method for finding the convex combination in CAT(0) spaces introduced by Dhompongsa et al. [16]. Let $\{v_1, v_2, ..., v_n\} \subset X$ and $\lambda_1, \lambda_2, ..., \lambda_n \in (0, 1)$ with $\sum_{i=1}^{n} \lambda_i = 1$. Following [12] we write, by induction,

$$\bigoplus_{i=1}^{n} \lambda_i v_i := (1 - \lambda_n)\Big(\frac{\lambda_1}{1 - \lambda_n} v_1 \oplus \frac{\lambda_2}{1 - \lambda_n} v_2 \oplus \cdots \oplus \frac{\lambda_{n-1}}{1 - \lambda_n} v_{n-1}\Big) \oplus \lambda_n v_n. \tag{2.4}$$

Note for an example that $\frac{1}{3}v_1 \oplus \frac{1}{3}v_2 \oplus \frac{1}{3}v_3$ and $\frac{1}{3}v_1 \oplus \frac{1}{3}v_3 \oplus \frac{1}{3}v_2$ are not necessary coincide (see Figure 2.7).

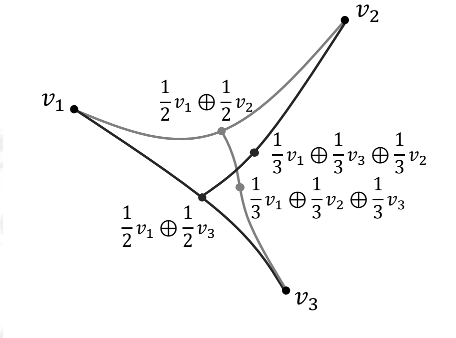

Figure 2.7: Convex combination of three points in a CAT(0) space

It is straightforward, using (2.2), to show that

$$d\left(\bigoplus_{i=1}^{n} \lambda_i v_i, x\right) \le \sum_{n=1}^{n} \lambda_i d(v_i, x) \tag{2.5}$$

for each $x \in X$.

Let $\{\lambda_n\}$ be a given sequence in $(0,1)$ such that $\sum_{n=1}^{\infty} \lambda_n = 1$, let $\{v_n\}$ be a bounded sequence in X and let v_0 be an arbitrary point in X. Let $\lambda_n' = \sum_{i=n+1}^{\infty} \lambda_i$ and assume that $\sum_{i=n}^{\infty} \lambda_i' \to 0$ as $n \to \infty$. We present the description of $\bigoplus_{n=1}^{\infty} \lambda_n v_n$, defined in [16], as follow. First, we set

$$s_n := \lambda_1 v_1 \oplus \lambda_2 v_2 \oplus \cdots \oplus \lambda_n v_n \oplus \lambda_n' v_0.$$

Thus, by (2.4),

$$s_n = \left(\sum_{i=1}^{n} \lambda_i\right) w_n \oplus \lambda_n' v_0, \tag{2.6}$$

where $w_1 = v_1$ and for each $n \ge 2$,

$$w_n = \frac{\lambda_1}{\sum_{i=1}^{n} \lambda_i} v_1 \oplus \frac{\lambda_2}{\sum_{i=1}^{n} \lambda_i} v_2 \oplus \cdots \oplus \frac{\lambda_n}{\sum_{i=1}^{n} \lambda_i} v_n.$$

We can see that $\{s_n\}$ is a Cauchy sequence. Thus $s_n \to x$ as $n \to \infty$ for some $x \in X$. Write

$$x = \bigoplus_{n=1}^{\infty} \lambda_n v_n.$$

By (2.6), $d(s_n, w_n) \le \lambda_n' d(w_n, v_0)$, it is seen that $\lim_{n\to\infty} s_n = \lim_{n\to\infty} w_n$. Thus the limit x is independent of the choice of v_0. It had been noted in [16] that the sequence $\left\{\lambda_n = \frac{1}{2^n}\right\}$ satisfies the condition $\sum_{i=n}^{\infty} \lambda_i' \to 0$ as $n \to \infty$ but this does not generally follow from the condition $\sum_{n=1}^{\infty} \lambda_n = 1$. For example, consider the sequence $\left\{\lambda_n = \frac{1}{M}\frac{1}{n^{3/2}}\right\}$ where $M = \sum_{i=1}^{\infty} \frac{1}{n^{3/2}}$. Observe that $M\sum_{i=n}^{\infty} \lambda_i' \geq \infty$. Thus $\sum_{i=n}^{\infty} \lambda_i' \geq \infty \nrightarrow 0$ as $n \to \infty$.

The following is an extension of Bruck's result (Theorem 1.7).

Lemma 2.50. [16, Lemma 3.8] *Let C be a nonempty closed convex subset of a complete CAT(0) space X, let $\{t_n : n \in \mathbb{N}\}$ be a family of single-valued nonexpansive mappings on C. Suppose $\bigcap_{n=1}^{\infty} Fix(t_n)$ is nonempty. Define $t : C \to C$ by*

$$tx = \bigoplus_{n=1}^{\infty} \lambda_n t_n x$$

for all $x \in C$ where $\{\lambda_n\} \subset (0,1)$ with $\sum_{n=1}^{\infty} \lambda_n = 1$ and $\sum_{i=n}^{\infty} \lambda_i' \to 0$ as $n \to \infty$. Then t is nonexpansive and $Fix(t) = \bigcap_{n=1}^{\infty} Fix(t_n)$.