Chapter 3

$\Delta-$ Convergence Theorems for Nonexpansive Semigroups on CAT(0) Spaces

The aim of this chapter is to prove the nonlinear ergodic theorems in the framework of CAT(0) spaces. The first section contains some preliminary results and the main theorem. We also show its applications in the second section. All of results in this chapter have been published in the paper: W. Anakkamatee and S. Dhompongsa, *Rodé's theorem on common fixed points of semigroup of nonexpansive mappings in CAT(0) spaces,* Fixed Point Theory Appl., 2011, 2011:34 [1].

3.1 $\Delta-$ Convergence Theorems

In 2011, Kakavandi and Amini [23] first introduced the notation T_μ for the CAT(0) space setting. Here is a mild generalization of their result (see [23, Lemma 2.1]).

Lemma 3.1. *Let S be a semigroup and C be a closed convex subset of a CAT(0) space X and μ be a mean on $B(S)$. For a bounded function $h : S \to C$, define*

$$\varphi_\mu(y) := \mu_s(d^2(h(s), y))$$

for all $y \in X$. Then φ_μ attains its unique minimum at a point of $\overline{co}\{h(s) : s \in S\}$.

In this case, we put

$$T_\mu(h) := argmin\{y \mapsto \mu_s(d^2(h(s), y))\}, \tag{3.1}$$

and for $h(s)$ of the form $T_s x$ we write $T_\mu(h)$ shortly as $T_\mu x$. We have already mentioned in Chapter 1 that for a Hilbert space X, we can find the unique element $x_0 \in C$ such that

$$\mu_s\langle T_s x, y\rangle = \langle x_0, y\rangle \tag{3.2}$$

for all $y \in X$. The following is our observation for Hilbert spaces.

Remark 3.2. If X is a Hilbert space, then

(i) $T_\mu x = x_0$ where x_0 verifies (3.2), and

(ii) $\|x_0\|^2 = \sup_{y\in X}(2\langle x_0, y\rangle - \|y\|^2)$.

Proof. (i): Let x_0 be such that $\mu_s\langle T_s x, y\rangle = \langle x_0, y\rangle$ for all $y \in X$. We have $\varphi_\mu(x_0) = \varphi_\mu(0)+\|x_0\|^2-2\langle x_0, x_0\rangle = \varphi_\mu(0)-\|x_0\|^2 \leq \varphi_\mu(0)+\|T_\mu x\|^2-2\langle x_0, T_\mu x\rangle = \varphi_\mu(T_\mu x)$. Therefore $x_0 = T_\mu x$.

(ii): For any $x, y \in X$, we know that $\|T_s x-y\|^2 = \|T_s x\|^2-2\langle T_s x, y\rangle+\|y\|^2$. By the linearity of μ and (3.2), we have $\mu_s(\|T_s x - y\|^2) = \mu_s(\|T_s x\|^2) - 2\langle x_0, y\rangle + \|y\|^2$. Thus $\inf_{y\in X} \mu_s(\|T_s x-y\|^2) = \mu_s(\|T_s x\|^2) - \sup_{y\in X}(2\langle x_0, y\rangle - \|y\|^2)$. On the other hand, by (i), $\inf_{y\in X} \mu_s(\|T_s x-y\|^2) = \mu_s(\|T_s x - x_0\|^2) = \mu_s(\|T_s x\|^2) - 2\mu_s\langle T_s x, x_0\rangle + \|x_0\|^2 = \mu_s(\|T_s x\|^2) - \|x_0\|^2$. Hence $\|x_0\|^2 = \sup_{y\in X}(2\langle x_0, y\rangle - \|y\|^2)$. $\square$

Fixed point theorems for nonexpansive semigroups on CAT(0) spaces, stated below, was also proved by Kakavandi and Amini [23].

Lemma 3.3. [23, Lemma 3.1] *If C is a closed convex subset of a CAT(0) space X and $T : C \to C$ is a nonexpansive mapping, then $Fix(T)$ is closed and convex.*

Lemma 3.4. [23, Proposition 3.2] *Let C be a closed convex subset of a CAT(0) space X and S an amenable semigroup. If $\mathcal{S} = \{T_s : s \in S\}$ is a nonexpansive semigroup on C then the following conditions are equivalent:*

(i) $\{T_s x : s \in S\}$ is bounded for some $x \in C$;

(ii) $\{T_s x : s \in S\}$ is bounded for all $x \in C$;

(iii) $Fix(\mathcal{S}) \neq \emptyset$.

Proposition 3.5. [23, Theorem 3.3] *Let C be a closed convex subset of a complete CAT(0) space X, S an amenable semigroup, $\mathcal{S} = \{T_s : s \in S\}$ a nonexpansive semigroup on C with $Fix(\mathcal{S}) \neq \emptyset$. Then $T_\mu x \in Fix(\mathcal{S})$ for any invariant mean μ on $B(S)$.*

We now let S be a commutative semigroup and define a relation $\succcurlyeq$ on S by $s \succcurlyeq t$ if $s = t$ or there exists $u \in S$ such that $s = ut$. When $s \succcurlyeq t$ but $s \neq t$, we simply write $s \succ t$. We can see that S is a directed set with the relation $\succcurlyeq$. Examples of such S are the usual ordered sets $(\mathbb{N}, +, \geq)$ and $([0, \infty), +, \geq)$.

Proposition 3.6. [23, Proposition 4.1] *Let C be a closed convex subset of a complete CAT(0) space X, S a commutative semigroup, $\mathcal{S} = \{T_s : s \in S\}$ a nonexpansive semigroup on C with $Fix(\mathcal{S}) \neq \emptyset$. Then, for each $x \in C$, the net $\{\pi T_s x\}_{s \in S}$ converges to a point Px in $Fix(\mathcal{S})$, where $\pi = \pi_{Fix(\mathcal{S})} : C \to Fix(\mathcal{S})$ is the projection mapping.*

By using the above results, we can obtain a useful property of T_μ.

Proposition 3.7. *Let C be a closed convex subset of a complete CAT(0) space X, S a commutative semigroup, $\mathcal{S} = \{T_s : s \in S\}$ a nonexpansive semigroup on C with $Fix(\mathcal{S}) \neq \emptyset$. Let π be the projection mapping onto $Fix(\mathcal{S})$. Then, for any invariant mean μ on $B(S)$, $T_\mu x = \lim_s \pi T_s x = Px$ for all $x \in C$.*

Proof. Fix $x \in C$ and let $\varepsilon > 0$. By Proposition 3.6, there exists $s_0 \in S$ such that $d(\pi T_s x, Px) < \varepsilon$ for all $s \succcurlyeq s_0$. We know, by Proposition 3.5, that $T_\mu x \in Fix(\mathcal{S})$. Thus

$$\begin{aligned} d(Px, T_s x) &\leq d(Px, \pi T_s x) + d(\pi T_s x, T_s x) \\ &< d(\pi T_s x, T_s x) + \varepsilon \\ &\leq d(T_\mu x, T_s x) + \varepsilon \end{aligned}$$

for all $s \succcurlyeq s_0$. Since $\{T_s x : s \in S\}$ is bounded by Lemma 3.4, there exists $M > 0$ such that $d(T_\mu x, T_s x) < M$ for all $s \in S$. Therefore $d^2(Px, T_s x) \leq d^2(T_\mu x, T_s x) + 2M\varepsilon + \varepsilon^2$ for each $s \succcurlyeq s_0$. Since μ is invariant, we have

$$\begin{aligned} \mu_s(d^2(Px, T_s x)) &= \mu_s(d^2(Px, T_{ss_0} x)) \\ &\leq \mu_s(d^2(T_\mu x, T_{ss_0} x)) + 2M\varepsilon + \varepsilon^2 \\ &= \mu_s(d^2(T_\mu x, T_s x)) + 2M\varepsilon + \varepsilon^2 \end{aligned}$$

for any $\varepsilon > 0$. By the argminimality of $T_\mu x$ (see Lemma 3.1), $T_\mu x = Px$. □

In order to obtain the Rodé's Theorem (Theorem 1.4) in the framework of CAT(0) spaces, we need to restrict the asymptotically invariant net of means $\{\mu_\alpha\}$ to those that satisfy an additional condition: for each $t \in S$,

$$\mu_{\alpha_s}(d^2(T_s x, y)) - \mu_{\alpha_s}(d^2(T_{st} x, y)) \to 0 \text{ uniformly for } y \in C. \tag{3.3}$$

In the Hilbert space setting, condition (3.3) is not required because the weak convergence can obtain from (3.2) directly.

Lemma 3.8. *Let X be a complete CAT(0) space, C a closed convex subset of X, S a commutative semigroup, and $\mathcal{S} = \{T_s : s \in S\}$ a nonexpansive semigroup on C with $Fix(\mathcal{S}) \neq \emptyset$. Suppose $\{\mu_\alpha\}$ is an asymptotically invariant net of means on $B(S)$ satisfying condition (3.3). If $\{T_{\mu_\alpha} x\}$ Δ-converges to x_0, then $x_0 \in Fix(\mathcal{S})$.*

Proof. We first show that, for each $r \in S$,

$$\lim_\alpha d(T_{\mu_\alpha} x, T_r T_{\mu_\alpha} x) = 0. \tag{3.4}$$

If this is not the case, there must be some $\delta > 0$ so that for each α, there exists $\alpha' \succ \alpha$ satisfying $d(T_{\mu_{\alpha'}} x, T_r T_{\mu_{\alpha'}} x) \geq \delta$. Put $\varepsilon = \frac{\delta^2}{2}$. Since the asymptotically invariant net $\{\mu_\alpha\}$ satisfies (3.3), there exists α_0 for which for each $\alpha \succcurlyeq \alpha_0$,

$$\begin{aligned}
\varphi_{\mu_\alpha}(T_r T_{\mu_\alpha} x) &= \mu_{\alpha_s}(d^2(T_s x, T_r T_{\mu_\alpha} x)) \\
&< \mu_{\alpha_s}(d^2(T_r T_s x, T_r T_{\mu_\alpha} x)) + \varepsilon \\
&\leq \mu_{\alpha_s}(d^2(T_s x, T_{\mu_\alpha} x)) + \varepsilon = \varphi_{\mu_\alpha}(T_{\mu_\alpha} x) + \varepsilon.
\end{aligned}$$

Let w be the midpoint of $T_{\mu_{\alpha_0'}} x$ and $T_r T_{\mu_{\alpha_0'}} x$. By (CN) inequality, the following inequalities hold for each $s \in S$:

$$\begin{aligned}
d^2(T_s x, w) &\leq \frac{1}{2} d^2(T_s x, T_{\mu_{\alpha_0'}} x) + \frac{1}{2} d^2(T_s x, T_r T_{\mu_{\alpha_0'}} x) - \frac{1}{4} d^2(T_{\mu_{\alpha_0'}} x, T_r T_{\mu_{\alpha_0'}} x) \\
&\leq \frac{1}{2} d^2(T_s x, T_{\mu_{\alpha_0'}} x) + \frac{1}{2} d^2(T_s x, T_r T_{\mu_{\alpha_0'}} x) - \frac{\delta^2}{4}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\varphi_{\mu_{\alpha_0'}}(w) &\leq \frac{1}{2} \varphi_{\mu_{\alpha_0'}}(T_{\mu_{\alpha_0'}} x) + \frac{1}{2} \varphi_{\mu_{\alpha_0'}}(T_r T_{\mu_{\alpha_0'}} x) - \frac{\delta^2}{4} \\
&< \varphi_{\mu_{\alpha_0'}}(T_{\mu_{\alpha_0'}} x) + \frac{\varepsilon}{2} - \frac{\delta^2}{4} = \varphi_{\mu_{\alpha_0'}}(T_{\mu_{\alpha_0'}} x),
\end{aligned}$$

which is a contradiction and thus (3.4) holds.

We show $x_0 \in Fix(\mathcal{S})$ by supposing on the contrary, i.e., there exists some $r \in S$ such that $T_r x_0 \neq x_0$, i.e., $d(x_0, T_r x_0) := \gamma > 0$. Since $\{T_{\mu_\alpha} x\}$ is a subset of a bounded set $\overline{co}\{T_s x : s \in S\}$, we can get an $M > 0$ so that $d(T_{\mu_\alpha} x, x_0) \leq M$ for all α. Let $0 < \varepsilon < \min\{\frac{\gamma^2}{16M}, 2M\}$. From (3.4), there exists α_0 with the property that $d(T_r T_{\mu_\alpha} x, T_{\mu_\alpha} x) < \varepsilon$ for all $\alpha \succcurlyeq \alpha_0$. Now, for each $\alpha \succcurlyeq \alpha_0$,

$$\begin{aligned} d(T_{\mu_\alpha} x, T_r x_0) &\leq d(T_{\mu_\alpha} x, T_r T_{\mu_\alpha} x) + d(T_r T_{\mu_\alpha} x, T_r x_0) \\ &< d(T_{\mu_\alpha} x, x_0) + \varepsilon. \end{aligned}$$

Thus

$$d^2(T_{\mu_\alpha} x, T_r x_0) < d^2(T_{\mu_\alpha} x, x_0) + 2\varepsilon d(T_{\mu_\alpha} x, x_0) + \varepsilon^2.$$

Using the (CN) inequality, we see that

$$\begin{aligned} d^2\left(T_{\mu_\alpha} x, \frac{x_0 \oplus T_r x_0}{2}\right) &\leq \frac{1}{2} d^2(T_{\mu_\alpha} x, x_0) + \frac{1}{2} d^2(T_{\mu_\alpha} x, T_r x_0) - \frac{1}{4} d^2(x_0, T_r x_0) \\ &\leq \frac{1}{2} d^2(T_{\mu_\alpha} x, x_0) + \frac{1}{2}(d^2(T_{\mu_\alpha} x, x_0) + 2\varepsilon M + \varepsilon^2) - \frac{\gamma^2}{4} \\ &= d^2(T_{\mu_\alpha} x, x_0) + \varepsilon M + \frac{\varepsilon^2}{2} - \frac{\gamma^2}{4} \end{aligned}$$

for all $\alpha \succcurlyeq \alpha_0$. Consequently,

$$\begin{aligned} \limsup_\alpha d^2\left(T_{\mu_\alpha} x, \frac{x_0 \oplus T_r x_0}{2}\right) &\leq \limsup_\alpha d^2(T_{\mu_\alpha} x, x_0) + \varepsilon M + \frac{\varepsilon^2}{2} - \frac{\gamma^2}{4} \\ &< \limsup_\alpha d^2(T_{\mu_\alpha} x, x_0), \end{aligned}$$

contradicting to the fact that $\{x_0\} = A(C, \{T_{\mu_\alpha} x\})$. Therefore $x_0 \in F(\mathcal{S})$. $\square$

Theorem 3.9. *Let X be a complete CAT(0) space with Property (N), C a closed convex subset of X, S a commutative semigroup, and $\mathcal{S} = \{T_s : s \in S\}$ a nonexpansive semigroup on C with $Fix(\mathcal{S}) \neq \emptyset$. Suppose $\{\mu_\alpha\}$ is an asymptotically invariant net of means on $B(S)$ satisfying condition (3.3). Then $\{T_{\mu_\alpha} x\}$ Δ-converges to $Px \in F(\mathcal{S})$ for all $x \in C$, where Px is defined in Proposition 3.6.*

Proof. Let $x \in C$ and $\{\mu_{\alpha'}\}$ be any subnet of $\{\mu_\alpha\}$. By Remark 2.30, there exists a subnet $\{\mu_{\alpha''}\}$ of $\{\mu_{\alpha'}\}$ such that $\{\mu_{\alpha''}\}$ w*-converges to μ for some invariant mean μ on $B(S)$. We know, by Proposition 3.7, that $T_\mu x = Px$. Since the net $\{T_{\mu_{\alpha''}} x\} \subset$

$\overline{co}\{T_s x : s \in S\}$, it is bounded. Then, by Proposition 2.45, there exists a subnet $\{\mu_{\alpha_\beta}\}$ of $\{\mu_{\alpha''}\}$ such that $\{T_{\mu_{\alpha_\beta}} x\}$ Δ-converges to some $x_0 \in C$. Lemma 3.8 guarantees that $x_0 \in Fix(\mathcal{S})$. We claim that $x_0 = T_\mu x \,(= Px)$. So, for every subnet $\{\mu_{\alpha'}\}$ of $\{\mu_\alpha\}$, there exists a subnet $\{\mu_{\alpha_\beta}\}$ of $\{\mu_{\alpha'}\}$ such that $\{T_{\mu_{\alpha_\beta}} x\}$ Δ-converges to Px. By Remark 2.46 (ii), $\{T_{\mu_\alpha} x\}$ Δ-converges to Px.

To show $x_0 = T_\mu x$, we define

$$\overline{T_{\mu_{\alpha_\beta}} x} := \textit{argmin}\{y \mapsto \mu_{\beta_s}(d^2(T_{ss_0}x, y))\}$$

and split the proof into three steps.

Step 1. $\overline{T_{\mu_{\alpha_\beta}} x} \in \overline{co}\{T_s x : s \succcurlyeq s_0\}$.

Suppose $\overline{T_{\mu_{\alpha_\beta}} x} \notin \overline{co}\{T_s x : s \succcurlyeq s_0\}$, by (2.3),

$$d^2(T_{ss_0}x, \overline{T_{\mu_{\alpha_\beta}} x}) \geq d^2(T_{ss_0}x, \pi\overline{T_{\mu_{\alpha_\beta}} x}) + d^2(\overline{T_{\mu_{\alpha_\beta}} x}, \pi\overline{T_{\mu_{\alpha_\beta}} x})$$

for each $s \in S$ where $\pi : C \to \overline{co}\{T_s x : s \succcurlyeq s_0\}$ is the projection mapping. Thus

$$\begin{aligned} \mu_{\alpha_{\beta_s}}(d^2(T_{ss_0}x, \overline{T_{\mu_{\alpha_\beta}} x})) &\geq \mu_{\alpha_{\beta_s}}(d^2(T_{ss_0}x, \pi\overline{T_{\mu_{\alpha_\beta}} x})) + d^2(\overline{T_{\mu_{\alpha_\beta}} x}, \pi\overline{T_{\mu_{\alpha_\beta}} x}) \\ &> \mu_{\alpha_{\beta_s}}(d^2(T_{ss_0}x, \pi\overline{T_{\mu_{\alpha_\beta}} x})). \end{aligned}$$

This impossiblity shows that $\overline{T_{\mu_{\alpha_\beta}} x} \in \overline{co}\{T_s x : s \succcurlyeq s_0\}_{s \succcurlyeq s_0}$.

Step 2. $\lim_\beta d(T_{\mu_{\alpha_\beta}} x, \overline{T_{\mu_{\alpha_\beta}} x}) = 0$.

If this does not hold, there must be some $\eta > 0$ so that for each β, there exists $\beta' \succ \beta$ satisfying $d(T_{\mu_{\alpha_{\beta'}}} x, \overline{T_{\mu_{\alpha_{\beta'}}} x}) \geq \eta$. Put $\varepsilon = \frac{\eta^2}{2}$. Since the asymptotically invariant net $\{\mu_\beta\}$ satisfies (3.3), there exists β_0 such that

$$\left|\mu_{\alpha_{\beta_s}}(d^2(T_s x, \overline{T_{\mu_{\alpha_\beta}} x})) - \mu_{\alpha_{\beta_s}}(d^2(T_{ss_0}x, \overline{T_{\mu_{\alpha_\beta}} x}))\right| < \varepsilon$$

for each $\beta \succcurlyeq \beta_0$.

We suppose first that $\mu_{\alpha_{\beta'_{0_s}}}(d^2(T_{ss_0}x, \overline{T_{\mu_{\alpha_{\beta'_0}}}x})) \leq \mu_{\alpha_{\beta'_{0_s}}}(d^2(T_s x, T_{\mu_{\alpha_{\beta'_0}}}x))$. Set $w = \frac{T_{\mu_{\alpha_{\beta'_0}}}x \oplus \overline{T_{\mu_{\alpha_{\beta'_0}}}x}}{2}$. By (CN) inequality, the following inequalities hold for each $s \in S$:

$$
\begin{aligned}
d^2(T_s x, w) &\leq \frac{1}{2}d^2(T_s x, T_{\mu_{\alpha_{\beta'_0}}}x) + \frac{1}{2}d^2(T_s x, \overline{T_{\mu_{\alpha_{\beta'_0}}}x}) - \frac{1}{4}d^2(T_{\mu_{\alpha_{\beta'_0}}}x, \overline{T_{\mu_{\alpha_{\beta'_0}}}x}) \\
&\leq \frac{1}{2}d^2(T_s x, T_{\mu_{\alpha_{\beta'_0}}}x) + \frac{1}{2}d^2(T_s x, \overline{T_{\mu_{\alpha_{\beta'_0}}}x}) - \frac{\eta^2}{4}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\varphi_{\mu_{\alpha_{\beta'_0}}}(w) &\leq \frac{1}{2}\varphi_{\mu_{\alpha_{\beta'_0}}}(T_{\mu_{\alpha_{\beta'_0}}}x) + \frac{1}{2}\varphi_{\mu_{\alpha_{\beta'_0}}}(\overline{T_{\mu_{\alpha_{\beta'_0}}}x}) - \frac{\eta^2}{4} \\
&< \frac{1}{2}\varphi_{\mu_{\alpha_{\beta'_0}}}(T_{\mu_{\alpha_{\beta'_0}}}x) + \frac{1}{2}\mu_{\alpha_{\beta'_{0_s}}}(d^2(T_{ss_0}x, \overline{T_{\mu_{\alpha_{\beta'_0}}}x})) + \frac{\varepsilon}{2} - \frac{\eta^2}{4} \\
&\leq \varphi_{\mu_{\alpha_{\beta'_0}}}(T_{\mu_{\alpha_{\beta'_0}}}x) + \frac{\varepsilon}{2} - \frac{\eta^2}{4} = \varphi_{\mu_{\alpha_{\beta'_0}}}(T_{\mu_{\alpha_{\beta'_0}}}x),
\end{aligned}
$$

contradicting to the argminimality of $T_{\mu_{\alpha_{\beta'_0}}}x$. For another case, we can show in the same way that $\mu_{\alpha_{\beta'_{0_s}}}(d^2(T_{ss_0}x, w)) < \mu_{\alpha_{\beta'_{0_s}}}(d^2(T_{ss_0}x, \overline{T_{\mu_{\alpha_{\beta'_0}}}x}))$ for some w which also leads to a contradiction.

Step 3. $x_0 = T_\mu x$.

We suppose on the contrary and let $\eta := d(x_0, T_\mu x) > 0$, $I = [T_\mu x, x_0]$ and π_I be the projection mapping onto I. Since $\{T_s x : s \in S\}$ is bounded, there exists $M > 0$ such that $d(T_s x, \pi_I(T_s x)) \leq M$ for all $s \in S$. Set $N_0 > \frac{4(M+\eta)}{5\eta}$ and $\rho = \frac{\eta}{5N_0}$.

Suppose there exists $s_0 \in S$ such that $d(\pi_I(T_s x), x_0) \geq 2\rho$ for all $s \succcurlyeq s_0$. We know, by Step 1, that $\overline{T_{\mu_{\alpha_\beta}}x} \in \overline{co}\{T_s x : s \succcurlyeq s_0\}$. Let $A := \{y \in C : d(\pi_I(y), x_0) > 2\rho\}$. We observe, by using property (N), that A is convex and

$$\overline{co}\{T_s x : s \succcurlyeq s_0\} \subset \overline{A} \subset \{y \in C : d(\pi_I(y), x_0) \geq 2\rho\}.$$

Thus $d(\pi_I(\overline{T_{\mu_{\alpha_\beta}}x}), x_0) \geq 2\rho$. By Step 2 and the nonexpansiveness of π_I,

$$\lim_\beta d(\pi_I(T_{\mu_{\alpha_\beta}}x), \pi_I(\overline{T_{\mu_{\alpha_\beta}}x})) = \lim_\beta d(T_{\mu_{\alpha_\beta}}x, \overline{T_{\mu_{\alpha_\beta}}x}) = 0.$$

Choose β_0 so that $d(\pi_I(T_{\mu_{\alpha_\beta}}x), \pi_I(\overline{T_{\mu_{\alpha_\beta}}x})) < \rho$ for all $\beta \succcurlyeq \beta_0$. Thus $d(\pi_I(T_{\mu_{\alpha_\beta}}x), x_0) > \rho$ for all $\beta \succcurlyeq \beta_0$. But then $x_0 \notin \overline{co}\{T_{\mu_{\alpha_\beta}}x : \beta \succcurlyeq \beta_0\}$ which contradicts to the fact that x_0 is the $\Delta-$ limit of $\{T_{\mu_{\alpha_\beta}}x\}$.

Thus there must be a subnet $\{s'\}$ of S such that $s' \succ s$ for all s and

$$d(\pi_I(T_{s'}x), x_0) < 2\rho = \frac{2\eta}{5N_0} \tag{3.5}$$

for all s'. Hence

$$d(\pi_I(T_{s'}x), T_\mu x) = \eta - d(\pi_I(T_{s'}x), x_0) > \eta - \frac{2\eta}{5N_0}. \tag{3.6}$$

By N_0 satisfies $5N_0\eta^2 > 4\eta M + 4\eta^2$, we have

$$\eta^2 - \frac{4\eta^2}{5N_0} > \frac{4\eta M}{5N_0}. \tag{3.7}$$

From (3.5), (3.6) and (3.7),

$$\begin{aligned} d^2(\pi_I(T_{s'}x), T_\mu x) &> \eta^2 - \frac{4\eta^2}{5N_0} + \left(\frac{2\eta}{5N_0}\right)^2 \\ &> \frac{4\eta M}{5N_0} + \left(\frac{2\eta}{5N_0}\right)^2 \\ &> 2d(x_0, \pi_I(T_{s'}x))d(T_{s'}x, \pi_I(T_{s'}x)) + d^2(x_0, \pi_I(T_{s'}x)). \end{aligned}$$

Using (2.3),

$$\begin{aligned} d^2(T_{s'}x, T_\mu x) &\geq d^2(\pi_I(T_{s'}x), T_\mu x) + d^2(\pi_I(T_{s'}x), T_{s'}x) \\ &> d^2(x_0, \pi_I(T_{s'}x)) + 2d(x_0, \pi_I(T_{s'}x))d(T_{s'}x, \pi_I(T_{s'}x)) \\ &\quad + d^2(\pi_I(T_{s'}x), T_{s'}x) \\ &= (d(x_0, \pi_I(T_{s'}x)) + d(T_{s'}x, \pi_I(T_{s'}x)))^2 \\ &\geq d^2(T_{s'}x, x_0) \end{aligned}$$

for all s'. Since the points x_0 and $T_\mu x$ belong to the set $Fix(\mathcal{S})$, the nets $\{d^2(T_s x, x_0)\}$ and $\{d^2(T_s x, T_\mu x)\}$ are decreasing. So $\lim_s d^2(T_s x, x_0)$ and $\lim_s d^2(T_s x, T_\mu x)$ exist. Hence $\varphi_\mu(T_\mu x) = \lim_s d^2(T_s x, T_\mu x) = \lim_{s'} d^2(T_{s'}x, T_\mu x) = \mu_{s'}(d^2(T_{s'}x, T_\mu x)) \geq \mu_{s'}(d^2(T_{s'}x, x_0)) = \lim_{s'} d^2(T_{s'}x, x_0) = \lim_s d^2(T_s x, x_0) = \varphi_\mu(x_0)$, a contradiction. Thus $x_0 = T_\mu x$. $\square$

It is an interesting open problem to determine whether Theorem 3.9 remains valid when the semigroup is amenable but not commutative.

3.2 Applications

In this section, we consider the applications of Theorem 3.9 on semigroups $\mathbb{N}$ and $[0, \infty)$ under the operation $+$ (the usual addition).

Proposition 3.10. *Let C be a closed convex subset of a complete CAT(0) space X and $T : C \to C$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. Let $S = \mathbb{N}$, $\mathcal{S} = \{T^n : n \in S\}$, $\Lambda = \mathbb{N}$ or $\mathbb{R}^+$ and $\beta_{\lambda k} \geq 0$ be such that $\sum_{k \in S} \beta_{\lambda k} = 1$ for all $\lambda \in \Lambda$. Suppose for all $k \in S$,*

$$\lim_{\lambda \to \infty} \beta_{\lambda k} = 0 \tag{3.8}$$

and for each $m \in S$,

$$\lim_{\lambda \to \infty} \sum_{k=m}^{\infty} |\beta_{\lambda k} - \beta_{\lambda(k-m)}| = 0. \tag{3.9}$$

For any $f = (a_0, a_1, ...) \in B(S)$ let $\mu_\lambda(f) = \sum_{k=0}^{\infty} \beta_{\lambda k} a_k$. Then for each $x \in C$, $\{T_{\mu_\lambda} x\}$ Δ-converges to x_0 for some x_0 in $Fix(T)$.

In particular, if X is a Hilbert space we have $\sum_{k=0}^{\infty} \beta_{\lambda k} T^k x$ converges weakly to x_0 for some x_0 in $Fix(T)$.

Proof. For each $m \in S$,

$$\begin{aligned} |\mu_\lambda(f) - \mu_\lambda(r_m f)| &= \Big| \sum_{k=1}^{\infty} \beta_{\lambda k} a_k - \sum_{k=1}^{\infty} \beta_{\lambda k} a_{k+m} \Big| \\ &\leq \sum_{k=1}^{m-1} |\beta_{\lambda k}||a_k| + \sum_{k=m+1}^{\infty} |\beta_{\lambda k} - \beta_{\lambda(k-m)}||a_k|. \end{aligned}$$

By (3.8) and (3.9) we have $\lim_{\lambda \to \infty} |\mu_\lambda(f) - \mu_\lambda(r_m f)| = 0$ and this shows that the net $\{\mu_\lambda\}$ is asymptotically invariant. Let $x \in C$ and consider a_k of the form $a_k = d^2(T^k x, y)$ where $y \in C$. We see that $\{\mu_\lambda\}$ satisfies (3.3). By Theorem 3.9 we have $\{T_{\mu_\lambda} x\}$ $\Delta-$converges to x_0 for some x_0 in $Fix(T)$.

In Hilbert spaces, by a well-known result in probability theory, we know that

$$\sum_{k=0}^{\infty} \beta_{\lambda k} \Big\| T^k x - \sum_{k=0}^{\infty} \beta_{\lambda k} T^k x \Big\|^2 \le \sum_{k=0}^{\infty} \beta_{\lambda k} \| T^k x - y \|^2$$

for all $y \in C$. So we have $T_{\mu_\lambda} x = \sum_{k=1}^{\infty} \beta_{\lambda k} T^k x$. □

Corollary 3.11 (Bailon Ergodic Theorem)**.** *Let C be a closed convex subset of a Hilbert space H and $T : C \to C$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. Then, for any $x \in C$,*

$$S_n x = \frac{1}{n} \sum_{k=1}^{n} T^k x$$

converges weakly to x_0 for some x_0 in $Fix(T)$.

Proof. Let $\Lambda = \mathbb{N}$ and put, for $\lambda \in \Lambda$ and $k \in \mathbb{N}$,

$$\beta_{\lambda k} = \begin{cases} \frac{1}{\lambda}, & k \le \lambda, \\ 0, & k > \lambda. \end{cases}$$

The result now follows from Proposition 3.10. □

Corollary 3.12. [44, Theorem 3.5.1] *Let C be a closed convex subset of a Hilbert space H and $T : C \to C$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. Then, for any $x \in C$,*

$$S_r x = (1 - r) \sum_{k=1}^{\infty} r^k T^k x$$

converges weakly to x_0 for some x_0 in $Fix(T)$ as $r \uparrow 1$.

Proof. Let $\Lambda = \mathbb{R}^+$ and put, for $\lambda \in \Lambda$ and $k \in \mathbb{N}$,

$$\beta_{\lambda k} = \frac{(\lambda - 1)^k}{\lambda^{k+1}}.$$

Taking $r = \frac{\lambda - 1}{\lambda}$, Proposition 3.10 implies the desired result. □

Now we consider the semigroup $S = [0, \infty)$. Let C be a closed convex subset of a Hilbert space H. A family $\mathcal{S} = \{T(s) : s \in S\}$ is called a *continuous nonexpansive semigroup* on C if $\mathcal{S}$ satisfies the following:

(i) $T(s) : C \to C$ is a nonexpansive mapping for all $s \in S$,

(ii) $T(t+s)x = T(t)T(s)x$ for all $x \in C$ and $t, s \in S$,

(iii) for each $x \in C$, the mapping $s \mapsto T(s)x$ is continuous, and

(iv) $T(0)x = x$ for all $x \in C$.

Proposition 3.13. *Let C be a closed convex subset of a Hilbert space H. Let $S = [0, \infty)$, $\mathcal{S} = \{T(s) : s \in S\}$ a continuous nonexpansive semigroup on C with $Fix(\mathcal{S}) \neq \emptyset$, $\Lambda = \mathbb{R}^+$ and g_λ a density function on S, i.e., $g_\lambda \geq 0$ and $\int_0^\infty g_\lambda(s)ds = 1$ for all $\lambda \in \Lambda$. Suppose g_λ satisfies the following properties: for each $h \in S$,*

$$\lim_{\lambda \to \infty} g_\lambda(s) = 0 \tag{3.10}$$

uniformly on $[0, h]$ and

$$\lim_{\lambda \to \infty} \int_h^\infty |g_\lambda(s) - g_\lambda(s-h)|ds = 0. \tag{3.11}$$

Then, for any $x \in C$,

$$\int_0^\infty g_\lambda(s)T(s)xds$$

converges weakly to some $x_0 \in Fix(\mathcal{S})$ as $\lambda \to \infty$.

Proof. For $f \in B(S)$ we define $\mu_\lambda(f) = \int_0^\infty g_\lambda(s)f(s)ds$ for all $\lambda > 0$. Thus μ_λ is a mean on $B(S)$. For any $h \in S$ we consider

$$\begin{aligned} |\mu_\lambda(f) - \mu_\lambda(r_h f)| &= \Big| \int_0^\infty g_\lambda(s)f(s)ds - \int_0^\infty g_\lambda(s)f(s+h)ds \Big| \\ &\leq \int_0^h |g_\lambda(s)||f(s)|ds + \int_h^\infty |g_\lambda(s) - g_\lambda(s-h)||f(s)|ds. \end{aligned}$$

By (3.10) and (3.11), $\lim_\lambda |\mu_\lambda(f) - \mu_\lambda(r_h f)| = 0$. So $\{\mu_\lambda\}$ is asymptotically invariant. For each $y \in C$, let $f(s) = \|y - T(s)x\|^2$. We see that $\{\mu_\lambda\}$ satisfies (3.3). For each $x \in C$ we know that

$$\int_0^\infty g_\lambda(s) \Big\| \int_0^\infty g_\lambda(s)T(s)xds - T(s)x \Big\|^2 ds \leq \int_0^\infty g_\lambda(s)\|y - T(s)x\|^2 ds$$

for all $y \in C$. Thus

$$T_{\mu_\lambda}x = \int_0^\infty g_\lambda(s)T(s)xds.$$

By using Theorem 3.9, we can conclude the result. □

Corollary 3.14. [44, Theorem 3.5.2] *Let C be a closed convex subset of a Hilbert space H. Suppose $S = [0, \infty)$ and $\mathcal{S} = \{T(s) : s \in S\}$ is a continuous nonexpansive semigroup on C with $F(\mathcal{S}) \neq \emptyset$. Then, for any $x \in C$,*

$$S_\lambda x = \frac{1}{\lambda} \int_0^\lambda T(s) x ds$$

converges weakly to some $x_0 \in Fix(\mathcal{S})$ as $\lambda \to \infty$.

Proof. Let $\Lambda = \mathbb{R}^+$ and put, for $\lambda \in \Lambda$ and $s \in S$, $g_\lambda(s) = \frac{1}{\lambda}\chi_{[0,\lambda]}$. The result now follows from Proposition 3.13. □

Corollary 3.15. [44, Theorem 3.5.3] *Let C be a closed convex subset of a Hilbert space H. Suppose $S = [0, \infty)$ and $\mathcal{S} = \{T(s) : s \in S\}$ is a continuous nonexpansive semigroup on C with $F(\mathcal{S}) \neq \emptyset$. Then, for any $x \in C$,*

$$r \int_0^\infty e^{-rs} T(s) x ds$$

converges weakly to some $x_0 \in Fix(\mathcal{S})$ as $r \downarrow 0$.

Proof. Let $\Lambda = \mathbb{R}^+$ and put, for $\lambda \in \Lambda$ and $s \in S$, $g_\lambda(s) = \frac{1}{\lambda} e^{-\frac{1}{\lambda}s}$. Again we can then apply Proposition 3.13 by taking $r = \frac{1}{\lambda}$. □

By using Lemma 3.1 we can obtain a strong convergence theorem on Hilbert spaces stated as Theorem 3.17 below.

Proposition 3.16. *Let C be a closed convex subset of a Hilbert space H and $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Given $x \in C$ and set $r = r(C, \{T^n x\})$. For each $n \in \mathbb{N}$, define*

$$\Pi_n := \Big\{ p = \{\beta_{nk}\}_{k \geq n} \subset [0,1] : \sum_{k \geq n} \beta_{nk} = 1 \Big\}$$

and

$$V_n := \sup_{p \in \Pi_n} \sum_{k \geq n} \beta_{nk} \|T^k x - \overline{x}_n\|^2$$

where $\overline{x}_n = \sum_{k \geq n} \beta_{nk} T^k x$. Then $V := \lim_{n \to \infty} V_n = r^2$.

Proof. Let $z \in A(C, \{T^n x\})$. We note, by Remark 2.43 (ii), that $z \in F(T)$. Given $\varepsilon > 0$, choose $n_\varepsilon \in \mathbb{N}$ such that $\|T^n x - z\| < r + \varepsilon$ for all $n \geq n_\varepsilon$. Fix $n \geq n_\varepsilon$ and let $p = \{\beta_{nk}\}_{k \geq n} \in \Pi_n$. Thus

$$\sum_{k \geq n} \beta_{nk} \|T^k x - \overline{x}_n\|^2 \leq \sum_{k \geq n} \beta_{nk} \|T^k x - z\|^2 < (r + \varepsilon)^2.$$

So

$$V_n = \sup_{p \in \Pi_n} \sum_{k \geq n} \beta_{nk} \|T^k x - \overline{x}_n\|^2 < (r + \varepsilon)^2.$$

Letting $n \to \infty$, $V = \lim_{n \to \infty} V_n \leq (r + \varepsilon)^2$ for any $\varepsilon > 0$. Hence $V \leq r^2$.

Next, we show $r^2 \leq V$. Indeed, since $z \in \overline{co}\{T^k x : k \geq n\}$ for all $n \in \mathbb{N}$, there exists a sequence $\{\overline{x}_n\}$ with $\overline{x}_n \in co\{T^k x : k \geq n\}$ for each n and $\overline{x}_n \to z$ as $n \to \infty$. Put $\overline{x}_n = \sum_{k \geq n} \beta_{nk} T^k x$. Since $\{T^n x\}$ is bounded, there exists $M > 0$ such that

$$\|T^k x - \overline{x}_n\| + \|T^k x - z\| \leq M.$$

For each $\varepsilon > 0$, choose $n_\varepsilon \in \mathbb{N}$ such that $\|\overline{x}_n - z\| < \varepsilon$, $V_n < V + \varepsilon$ for all $n \geq n_\varepsilon$, and $\|T^k x - z\| > r - \varepsilon$ for all $k \geq n_\varepsilon$. Thus for any $n \geq n_\varepsilon$,

$$\begin{aligned}
\sum_{k \geq n} \beta_{nk} |\|T^k x - \overline{x}_n\|^2 \ &- \ \|T^k x - z\|^2| \\
&= \sum_{k \geq n} \beta_{nk} |\|T^k x - \overline{x}_n\| - \|T^k x - z\|| (\|T^k x - \overline{x}_n\| + \|T^k x - z\|) \\
&= \sum_{k \geq n} \beta_{nk} \|\overline{x}_n - z\| (\|T^k x - \overline{x}_n\| + \|T^k x - z\|) \leq \varepsilon M.
\end{aligned}$$

Hence

$$\begin{aligned}
(r - \varepsilon)^2 &< \sum_{k \geq n} \beta_{nk} \|T^k x - z\|^2 \\
&= \sum_{k \geq n} \beta_{nk} \|T^k x - z\|^2 + \sum_{k \geq n} \beta_{nk} \|T^k x - \overline{x}_n\|^2 - \sum_{k \geq n} \beta_{nk} \|T^k x - \overline{x}_n\|^2 \\
&\leq V_n + \sum_{k \geq n} \beta_{nk} |\|T^k x - \overline{x}_n\|^2 - \|T^k x - z\|^2| < V + \varepsilon + \varepsilon M.
\end{aligned}$$

So $(r - \varepsilon)^2 < V + \varepsilon + \varepsilon M$ for any $\varepsilon > 0$. This implies $r^2 \leq V$. $\square$

Theorem 3.17. *Let C be a closed convex subset of a Hilbert space H and $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Suppose Π_n, V and $\overline{x}_n$ is defined as in Proposition 3.16. If the sequence $\{\overline{x}_n\}$ satisfies*

$$\lim_{n\to\infty}\Big(\sum_{k\geq n}\beta_{nk}\|T^k x-\overline{x}_n\|^2\Big)=V, \tag{3.12}$$

then $\{\overline{x}_n\}$ converges (strongly) to the unique asymptotic center $z \in F(T)$.

Proof. Suppose for some $\varepsilon > 0$, there exists a subsequence $\{\overline{x}_{n_l}\}$ of $\{\overline{x}_n\}$ such that $\|\overline{x}_{n_l} - z\| \geq \varepsilon$ for all $l \in \mathbb{N}$. For each $y \in C$ and $n \in \mathbb{N}$ define

$$\varphi_n(y) := \sum_{k\geq n}\beta_{nk}\|T^k x - y\|^2.$$

Let $0 < \delta < \frac{\varepsilon^2}{8}$ and $z \in A(C, \{T^k x\})$. By (3.12), we choose n_δ such that

$$r^2-\delta = V-\delta < \varphi_{n_l}(\overline{x}_{n_l}) < V+\delta = r^2+\delta$$

for all $n_l \geq n_\delta$ and

$$\|T^k x - z\|^2 < r^2+\delta$$

for all $k \geq n_\delta$. Fix $l \geq n_\delta$ and let $w = \frac{\overline{x}_{n_l}+z}{2}$. By the Parallelogram law, we have for each $k \geq n_l$,

$$\|T^k x - w\|^2 = \frac{1}{2}\|T^k x-\overline{x}_{n_l}\|^2+\frac{1}{2}\|T^k x - z\|^2-\frac{1}{4}\|\overline{x}_{n_l}-z\|^2.$$

Hence

$$\varphi_{n_l}(w) < \frac{1}{2}(r^2+\delta)+\frac{1}{2}(r^2+\delta)-\frac{1}{4}\varepsilon^2 < r^2-\delta < \varphi_{n_l}(\overline{x}_{n_l}).$$

This contradicts to the minimality of $\varphi_{n_l}(\overline{x}_{n_l})$ (using Lemma 3.1). Hence $\{\overline{x}_{n'}\}$ converges to z for all subsequence $\{\overline{x}_{n'}\}$ of $\{\overline{x}_n\}$. We now obtain the desired result. $\square$