

Chapter 4

Approximation of a Common Fixed Point of Nonexpansive Mappings on Convex Metric Spaces

We introduce the iteration method, called the modified Ishikawa iteration method, for approximating a common fixed point of a family of single-valued nonexpansive mappings and a multivalued nonexpansive mapping. We present the results shown for strictly convex Banach spaces in Section 4.1 and for CAT(0) spaces in section 4.2. We publish the results of this chapter in the paper: W. Anakkamatee and S. Dhompongsa, *An approximation of a common fixed point of nonexpansive mappings on convex metric spaces*, Fixed Point Theory Appl., 2012, 2012:112 [2].

Along Chapter 4 and Chapter 5, we use the notation t to stand for a single-valued mapping and T for a multivalued mapping.

4.1 Strictly Convex Banach Spaces

The following result is a generalization of a result of [35, Lemma 1.3].

Lemma 4.1. *Let C be a compact subset of a strictly convex Banach space X , let $\{\alpha_n\}$ be a sequence of real numbers such that $\alpha_n \in [a, b] \subset (0, 1)$ for all $n \in \mathbb{N}$, and let $\{u_n\}$, $\{v_n\}$ be sequences of C satisfying, for some $c \geq 0$,*

$$(i) \limsup_{n \rightarrow \infty} \|u_n\| \leq c,$$

$$(ii) \limsup_{n \rightarrow \infty} \|v_n\| \leq c, \text{ and}$$

$$(iii) \lim_{n \rightarrow \infty} \|\alpha_n u_n + (1 - \alpha_n)v_n\| = c.$$

Then $\lim_{n \rightarrow \infty} \|u_n - v_n\| = 0$.

Proof. We suppose on the contrary that $\limsup_{n \rightarrow \infty} \|u_n - v_n\| \neq 0$. Since C and $[a, b]$ are compact, there exist subsequences $\{u_{n_k}\}$ of $\{u_n\}$, $\{v_{n_k}\}$ of $\{v_n\}$ and $\{\alpha_{n_k}\}$ of $\{\alpha_n\}$ such that $\lim_{k \rightarrow \infty} u_{n_k} = u$, $\lim_{k \rightarrow \infty} v_{n_k} = v$, $\lim_{k \rightarrow \infty} \alpha_{n_k} = \alpha$ for some $u, v \in C$ with $u \neq v$ and for some $\alpha \in [0, 1]$. From (i) and (ii) we have $\|u\| = \lim_{k \rightarrow \infty} \|u_{n_k}\| \leq c$ and $\|v\| = \lim_{k \rightarrow \infty} \|v_{n_k}\| \leq c$. Using the strictly convexity of X and (iii), we have $c = \lim_{k \rightarrow \infty} \|\alpha_{n_k} u_{n_k} + (1 - \alpha_{n_k})v_{n_k}\| = \|\alpha u + (1 - \alpha)v\| < \alpha\|u\| + (1 - \alpha)\|v\| \leq c$, a contradiction. Hence $\lim_{n \rightarrow \infty} \|u_n - v_n\| = 0$. $\square$

Let C be a nonempty subset of a Banach space X , $\{t_n : n \in \mathbb{N}\}$ a family of single-valued nonexpansive mappings on C , and $T : C \rightarrow FB(C)$ a multivalued nonexpansive mapping. Given a sequence of positive numbers $\{\gamma_n\}$ with $\sum_{n=1}^{\infty} \gamma_n < 1$. The sequence $\{x_n\}$ of the *modified Ishikawa iteration* is defined by $x_1 \in C$, and

$$\begin{aligned} y_n &= (1 - \beta_n)x_n + \beta_n z_n, \\ x_{n+1} &= (1 - \sum_{i=1}^n \gamma_i)x_n + \sum_{i=1}^n \gamma_i t_i y_n, \end{aligned} \tag{4.1}$$

where $z_n \in T x_n$. Along this thesis we use the notation F stand for the set of common fixed points of a family of single-valued nonexpansive mappings $\{t_n : n \in \mathbb{N}\}$ and a multivalued nonexpansive mapping T , i.e., $F := (\cap_{n=1}^{\infty} Fix(t_n)) \cap Fix(T)$.

Remark 4.2. Let $K(C)$ be the set of all nonempty compact subsets of C . It is well-known that every closed subset of a compact set is compact. Therefore $FB(C) = K(C)$ when C is compact.

Theorem 4.3. *Let C be a nonempty compact convex subset of a strictly convex Banach space X , $\{t_n : n \in \mathbb{N}\}$ a family of single-valued nonexpansive mappings on C , and $T : C \rightarrow K(C)$ a multivalued nonexpansive mapping. Suppose $F \neq \emptyset$ and $Tp = \{p\}$ for all $p \in F$. Given a sequence of positive numbers $\{\gamma_n\}$ with $\sum_{n=1}^{\infty} \gamma_n < 1$ and $\{\beta_n\} \subset [a, b] \subset (0, 1)$. Then the sequence $\{x_n\}$ defined by (4.1) converges strongly to some $v \in F$.*

Proof. We follow the proof of [37, Theorem 3.6] and split the proof into five steps.

Step 1. $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists for all $p \in F$:

We first note that $\|z_n - p\| = \text{dist}(z_n, Tp) \leq H(Tx_n, Tp) \leq \|x_n - p\|$. Consider the following estimates:

$$\begin{aligned}
 \|x_{n+1} - p\| &\leq \left(1 - \sum_{i=1}^n \gamma_i\right) \|x_n - p\| + \sum_{i=1}^n \gamma_i \|t_i y_n - p\| \\
 &\leq \left(1 - \sum_{i=1}^n \gamma_i\right) \|x_n - p\| + \sum_{i=1}^n \gamma_i \|y_n - p\| \\
 &\leq \left(1 - \sum_{i=1}^n \gamma_i\right) \|x_n - p\| + \left(\sum_{i=1}^n \gamma_i\right) \left((1 - \beta_n) \|x_n - p\| + \beta_n \|z_n - p\|\right) \\
 &\leq \|x_n - p\|.
 \end{aligned}$$

Therefore, $\{\|x_n - p\|\}$ is a bounded decreasing sequence in $\mathbb{R}$ and hence $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists.

Step 2. $\lim_{n \rightarrow \infty} \left\|x_n - \frac{\sum_{i=1}^n \gamma_i t_i y_n}{\sum_{i=1}^n \gamma_i}\right\| = 0$:

From Step 1, suppose $\lim_{n \rightarrow \infty} \|x_n - p\| = c$. We have

$$\begin{aligned}
 \left\| \frac{\sum_{i=1}^n \gamma_i t_i y_n}{\sum_{i=1}^n \gamma_i} - p \right\| &\leq \frac{1}{\sum_{i=1}^n \gamma_i} \left\| \sum_{i=1}^n \gamma_i t_i y_n - \sum_{i=1}^n \gamma_i p \right\| \\
 &\leq \|y_n - p\| \leq \|x_n - p\|.
 \end{aligned}$$

Thus

$$\limsup_{n \rightarrow \infty} \left\| \frac{\sum_{i=1}^n \gamma_i t_i y_n}{\sum_{i=1}^n \gamma_i} - p \right\| \leq \limsup_{n \rightarrow \infty} \|y_n - p\| \leq \limsup_{n \rightarrow \infty} \|x_n - p\| = c. \quad (4.2)$$

We also have

$$\begin{aligned}
 c &= \lim_{n \rightarrow \infty} \|x_{n+1} - p\| \\
 &= \lim_{n \rightarrow \infty} \left\| \left(1 - \sum_{i=1}^n \gamma_i\right) x_n + \sum_{i=1}^n \gamma_i t_i y_n - p \right\| \\
 &= \lim_{n \rightarrow \infty} \left\| \left(1 - \sum_{i=1}^n \gamma_i\right) (x_n - p) + \sum_{i=1}^n \gamma_i \left(\frac{\sum_{i=1}^n \gamma_i t_i y_n}{\sum_{i=1}^n \gamma_i} - p \right) \right\|.
 \end{aligned}$$

By Lemma 4.1, since $0 < \gamma_1 < \sum_{i=1}^n \gamma_i \leq \sum_{i=1}^{\infty} \gamma_i < 1$,

$$\lim_{n \rightarrow \infty} \left\| x_n - \frac{\sum_{i=1}^n \gamma_i t_i y_n}{\sum_{i=1}^n \gamma_i} \right\| = \lim_{n \rightarrow \infty} \left\| (x_n - p) - \left(\frac{\sum_{i=1}^n \gamma_i t_i y_n}{\sum_{i=1}^n \gamma_i} - p \right) \right\| = 0.$$

Step 3. $\lim_{n \rightarrow \infty} \|x_n - z_n\| = 0$:

From (4.1), we can see that

$$\|x_{n+1} - w\| \leq \left(1 - \sum_{i=1}^n \gamma_i\right) \|x_n - w\| + \sum_{i=1}^n \gamma_i \|y_n - w\|,$$

and hence $\|x_{n+1} - p\| - \|x_n - p\| \leq \sum_{i=1}^n \gamma_i (\|y_n - p\| - \|x_n - p\|)$. Therefore

$$\frac{\|x_{n+1} - p\| - \|x_n - p\|}{\sum_{i=1}^n \gamma_i} + \|x_n - p\| \leq \|y_n - p\|$$

and by (4.2) we obtain

$$\begin{aligned} c &= \liminf_{n \rightarrow \infty} \left\{ \left(\frac{\|x_{n+1} - p\| - \|x_n - p\|}{\sum_{i=1}^n \gamma_i} \right) + \|x_n - p\| \right\} \\ &\leq \liminf_{n \rightarrow \infty} \|y_n - p\| \leq \limsup_{n \rightarrow \infty} \|y_n - p\| \leq c. \end{aligned}$$

Thus $c = \lim_{n \rightarrow \infty} \|y_n - p\| = \lim_{n \rightarrow \infty} \|(1 - \beta_n)(x_n - p) + \beta_n(z_n - p)\|$. By Lemma 4.1, since $\beta_n \in [a, b] \subset (0, 1)$, $\lim_{n \rightarrow \infty} \|x_n - z_n\| = 0$.

Step 4. $\lim_{n \rightarrow \infty} \left\| x_n - \frac{\sum_{i=1}^{\infty} \gamma_i t_i x_n}{\sum_{i=1}^{\infty} \gamma_i} \right\| = 0$:

It follows from Step 3 that

$$\left\| \frac{\sum_{i=1}^n \gamma_i t_i x_n}{\sum_{i=1}^n \gamma_i} - \frac{\sum_{i=1}^n \gamma_i t_i y_n}{\sum_{i=1}^n \gamma_i} \right\| \leq \|x_n - y_n\| = \beta_n \|x_n - z_n\| \rightarrow 0 \text{ as } n \rightarrow \infty \quad (4.3)$$

and

$$\|t_i x_n\| \leq \|t_i x_n - p\| + \|p\| \leq \|x_n - p\| + \|p\| \leq \|x_1 - p\| + \|p\| := M$$

for all $i \in \mathbb{N}$. Therefore

$$\begin{aligned}
\left\| x_n - \frac{\sum_{i=1}^{\infty} \gamma_i t_i x_n}{\sum_{i=1}^{\infty} \gamma_i} \right\| &\leq \left\| x_n - \frac{\sum_{i=1}^n \gamma_i t_i x_n}{\sum_{i=1}^n \gamma_i} \right\| + \left\| \frac{\sum_{i=1}^n \gamma_i t_i x_n}{\sum_{i=1}^n \gamma_i} - \frac{\sum_{i=1}^{\infty} \gamma_i t_i x_n}{\sum_{i=1}^{\infty} \gamma_i} \right\| \\
&\leq \left\| x_n - \frac{\sum_{i=1}^n \gamma_i t_i x_n}{\sum_{i=1}^n \gamma_i} \right\| + \left\| \frac{\sum_{i=1}^n \gamma_i t_i x_n}{\sum_{i=1}^n \gamma_i} - \frac{\sum_{i=1}^n \gamma_i t_i x_n}{\sum_{i=1}^{\infty} \gamma_i} \right\| \\
&\quad + \frac{1}{\sum_{i=1}^{\infty} \gamma_i} \sum_{i=n+1}^{\infty} \gamma_i \|t_i x_n\| \\
&\leq \left\| x_n - \frac{\sum_{i=1}^n \gamma_i t_i x_n}{\sum_{i=1}^n \gamma_i} \right\| + \frac{\sum_{i=n+1}^{\infty} \gamma_i}{(\sum_{i=1}^n \gamma_i)(\sum_{i=1}^{\infty} \gamma_i)} \sum_{i=1}^n \gamma_i M \\
&\quad + \frac{\sum_{i=n+1}^{\infty} \gamma_i}{\sum_{i=1}^{\infty} \gamma_i} M \\
&= \left\| x_n - \frac{\sum_{i=1}^n \gamma_i t_i x_n}{\sum_{i=1}^n \gamma_i} \right\| + \frac{2 \sum_{i=n+1}^{\infty} \gamma_i}{\sum_{i=1}^{\infty} \gamma_i} M \\
&\leq \left\| x_n - \frac{\sum_{i=1}^n \gamma_i t_i y_n}{\sum_{i=1}^n \gamma_i} \right\| + \left\| \frac{\sum_{i=1}^n \gamma_i t_i y_n}{\sum_{i=1}^n \gamma_i} - \frac{\sum_{i=1}^n \gamma_i t_i x_n}{\sum_{i=1}^n \gamma_i} \right\| \\
&\quad + \frac{2 \sum_{i=n+1}^{\infty} \gamma_i}{\sum_{i=1}^{\infty} \gamma_i} M.
\end{aligned}$$

From Step 2 and (4.3) we obtain $\lim_{n \rightarrow \infty} \left\| x_n - \frac{\sum_{i=1}^{\infty} \gamma_i t_i x_n}{\sum_{i=1}^{\infty} \gamma_i} \right\| = 0$.

Step 5. $\lim_{n \rightarrow \infty} x_n = v \in F$:

Define a mapping $t : C \rightarrow C$ by

$$tx = \frac{\sum_{n=1}^{\infty} \gamma_n t_n x}{\sum_{n=1}^{\infty} \gamma_n}$$

for any $x \in C$. By Lemma 1.7, t is well-defined, nonexpansive and $Fix(t) = \bigcap_{n=1}^{\infty} Fix(t_n)$.

Since C is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges to v for some $v \in E$. Using Step 3 and Step 4, we have

$$\begin{aligned}
\|tv - v\| &\leq \lim_{k \rightarrow \infty} \left(\|tv - tx_{n_k}\| + \|tx_{n_k} - x_{n_k}\| + \|x_{n_k} - v\| \right) \\
&\leq \lim_{k \rightarrow \infty} \left(\|tx_{n_k} - x_{n_k}\| + 2\|x_{n_k} - v\| \right) = 0
\end{aligned}$$

and

$$\begin{aligned}
dist(v, Tv) &\leq \|v - x_{n_k}\| + dist(x_{n_k}, Tx_{n_k}) + H(Tx_{n_k}, Tv) \\
&\leq \|v - x_{n_k}\| + \|x_{n_k} - z_{n_k}\| + \|x_{n_k} - v\| \rightarrow 0 \text{ as } k \rightarrow \infty.
\end{aligned}$$

It follows that $v \in \text{Fix}(T) \cap \text{Fix}(t) = F$. Since $\lim_{n \rightarrow \infty} \|x_n - v\|$ exists by Step 1, $\lim_{n \rightarrow \infty} \|x_n - v\| = \lim_{k \rightarrow \infty} \|x_{n_k} - v\| = 0$. $\square$

The following example shows that the condition: “ $Tp = \{p\}$ for all $p \in F$ ” in Theorem 4.3 is necessary.

Example 4.4. We consider the space X of Example 3.9 in [15]. Let X be the Hilbert space $\mathbb{R}^2$ with the usual norm, and let $f : [0, 1] \rightarrow [0, 1]$ be a continuous strictly concave function such that $f(0) = \frac{1}{2}$, $f(1) = 1$ and $f'(x) \leq 1$ for all $x \in [0, 1]$. Let $\varepsilon_n = \sum_{i=1}^n (\frac{1}{2})^{i+1}$, $T : [0, 1]^2 \rightarrow K([0, 1]^2)$ be defined by $T(a, b) = [0, 1] \times [f(a), 1]$ and $t_n : [0, 1]^2 \rightarrow [0, 1]^2$ be defined by

$$t_n(a, b) = \begin{cases} (a, \varepsilon_n), & b < \varepsilon_n, \\ (a, b), & \text{otherwise.} \end{cases}$$

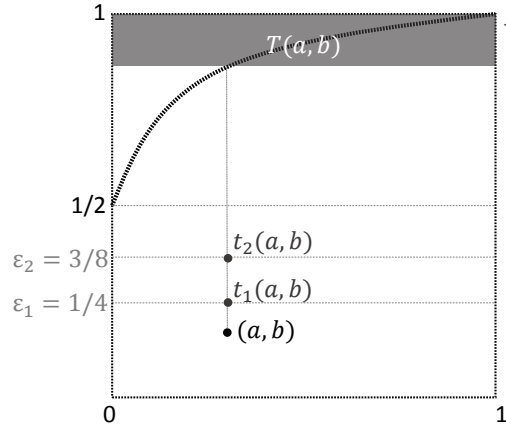


Figure 4.1: Example 4.4

It is straightforward showing that T and each t_n are nonexpansive. Set $x_1 = (1, 0) \in [0, 1]^2$ and for a subsequence $\{\gamma_n\}$ in $(0, 1)$ with $\sum_{n=1}^{\infty} \gamma_n < 1$. Let $\{x_n = (a_n, b_n)\}$ be a sequence in $[0, 1]^2$ defined as

$$\begin{aligned} y_n &= \frac{1}{2}x_n + \frac{1}{2}z_n, \\ x_{n+1} &= (1 - \sum_{i=1}^n \gamma_i)x_n + \sum_{i=1}^n \gamma_i t_i y_n, \end{aligned} \tag{4.4}$$

where

$$z_n = \begin{cases} (0, f(a_n)), & n \text{ is odd,} \\ (1, f(a_n)), & n \text{ is even.} \end{cases}$$

We will show that $\{x_n\}$ does not converge to a common fixed point of T and $\{t_n\}$.

Proof. Clearly, $\{z_n\}$ is a divergent sequence. We note that $\varepsilon_n \uparrow \frac{1}{2}$ and for each $y = (a, b) \in [0, 1]^2$ with $b \geq \frac{1}{2}$, we have $t_i y = y$ for all i . If we put $y_n = (c_n, d_n)$, then $d_n \geq \frac{1}{2}$ for all n . Suppose $\{x_n\}$ converges to v for some $v \in F = \{(a, b) \in [0, 1]^2 : b \geq f(a)\}$. Thus $\{z_n\}$ also converges to v , a contradiction. $\square$

Remark 4.5. With the same proof, Theorem 4.3 is valid when $\{x_n\}$ is of the following form: For a permutation Π on $\mathbb{N}$, define $\{x_n\}$ in C by $x_1 \in C$ and

$$\begin{aligned} y_n &= (1 - \beta_n)x_n + \beta_n z_n, \\ x_{n+1} &= (1 - \sum_{i=1}^n \gamma_{\Pi(i)})x_n + \sum_{i=1}^n \gamma_{\Pi(i)} t_{\Pi(i)} y_n, \end{aligned}$$

$z_n \in Tx_n$ and $\beta_n \in [a, b] \subset (0, 1)$.

Note also that the above result is equivalent to:

Let $\{I_n\}$ be a sequence of subsets of $\mathbb{N}$ satisfying $I_n \subset I_{n+1}$ for $n \in \mathbb{N}$ and $\bigcup_{n=1}^{\infty} I_n = \mathbb{N}$. Define $\{x_n\}$ in C by $x_1 \in C$ and

$$\begin{aligned} y_n &= (1 - \beta_n)x_n + \beta_n z_n, \\ x_{n+1} &= (1 - \sum_{i \in I_n} \gamma_i)x_n + \sum_{i \in I_n} \gamma_i t_i y_n, \end{aligned}$$

$z_n \in Tx_n$ and $\beta_n \in [a, b] \subset (0, 1)$. Then the sequence $\{x_n\}$ converges strongly to some $v \in F$.

Thus Theorem 4.3 contains Suzuki's result (Theorem 1.8).

4.2 CAT(0) Spaces

Let C be a nonempty subset of a complete CAT(0) space X , $\{t_n : n \in \mathbb{N}\}$ a family of single-valued nonexpansive mappings on C and $T : C \rightarrow FB(C)$ a multivalued nonexpansive mapping. Given $\{\gamma_n\}$ a sequence of positive numbers with $\sum_{n=1}^{\infty} \gamma_n < 1$ and $\sum_{i=n}^{\infty} \gamma'_i \rightarrow 0$ as $n \rightarrow \infty$ where $\gamma'_n = \sum_{i=n+1}^{\infty} \gamma_i$. The sequence $\{x_n\}$ of the *modified Ishikawa iteration* is defined by

$$\begin{aligned} y_n &= (1 - \beta_n)x_n \oplus \beta_n z_n, \\ x_{n+1} &= (1 - \sum_{i=1}^n \gamma_i)x_n \oplus (\sum_{i=1}^n \gamma_i) \bigoplus_{i=1}^n \frac{\gamma_i}{\sum_{i=1}^n \gamma_i} t_i y_n, \end{aligned} \tag{4.5}$$

where $x_1 \in C$, $z_n \in Tx_n$. We can prove the corresponding result in the framework of $CAT(0)$ spaces.

Theorem 4.6. *Let C be a compact convex subset of a complete $CAT(0)$ space X . Let $\{t_n : n \in \mathbb{N}\}$ be a family of single-valued nonexpansive mappings on C , and let $T : C \rightarrow K(C)$ be a multivalued nonexpansive mapping. Suppose $F \neq \emptyset$ and $Tp = \{p\}$ for all $p \in F$. Given $\{\gamma_n\}$ a sequence of positive numbers with $\sum_{n=1}^{\infty} \gamma_n < 1$, $\sum_{i=n}^{\infty} \gamma'_i \rightarrow 0$ as $n \rightarrow \infty$ where $\gamma'_n = \sum_{i=n+1}^{\infty} \gamma_i$ and $\{\beta_n\} \subset [a, b] \subset (0, 1)$. Then the sequence $\{x_n\}$ defined by (4.5) converges strongly to some $v \in F$.*

Proof. The proof follows along the lines with the proof of Theorem 4.3. Recall that $w_1x = t_1x$ and $w_nx = \bigoplus_{i=1}^n \frac{\gamma_i}{\sum_{i=1}^n \gamma_i} t_i x$ for all $n \geq 2$. Thus, by (4.5),

$$x_{n+1} = \left(1 - \sum_{i=1}^n \gamma_i\right) x_n \oplus \left(\sum_{i=1}^n \gamma_i\right) w_n y_n.$$

We note, from the proof of Lemma 3.8 in [16], that w_n is nonexpansive.

As before we consider the proof in 5 Steps. Because of the same details in some cases, we only present proofs for Step 2 - Step 4.

Step 2 and Step 3. $\lim_{n \rightarrow \infty} d(x_n, w_n y_n) = 0 = \lim_{n \rightarrow \infty} d(x_n, z_n)$:

Let $p \in F$. Using the nonexpansiveness of w_n we see that

$$d(w_n y_n, p) \leq d(y_n, p) \leq (1 - \beta_n) d(x_n, p) + \beta_n d(z_n, p) \leq d(x_n, p). \quad (4.6)$$

By (4.6) and (CN) inequality, we have

$$\begin{aligned} d^2(w_n y_n, p) &\leq d^2(y_n, p) \\ &\leq (1 - \beta_n) d^2(x_n, p) + \beta_n d^2(z_n, p) - \beta_n (1 - \beta_n) d^2(x_n, z_n) \\ &\leq d^2(x_n, p) - \beta_n (1 - \beta_n) d^2(x_n, z_n) \end{aligned}$$

Therefore

$$\begin{aligned}
d^2(x_{n+1}, p) &\leq \left(1 - \sum_{i=1}^n \gamma_i\right) d^2(x_n, p) + \left(\sum_{i=1}^n \gamma_i\right) d^2(w_n y_n, p) \\
&\quad - \sum_{i=1}^n \gamma_i \left(1 - \sum_{i=1}^n \gamma_i\right) d^2(x_n, w_n y_n) \\
&\leq d^2(x_n, p) - \beta_n \left(\sum_{i=1}^n \gamma_i\right) (1 - \beta_n) d^2(x_n, z_n) \\
&\quad - \sum_{i=1}^n \gamma_i \left(1 - \sum_{i=1}^n \gamma_i\right) d^2(x_n, w_n y_n).
\end{aligned}$$

Let $\gamma = \sum_{i=1}^{\infty} \gamma_i$. Since $0 < \gamma_1 \leq \sum_{i=1}^n \gamma_i \leq \gamma < 1$ and $\beta_n \in [a, b]$,

$$\gamma_1(1 - \gamma)d^2(x_n, w_n y_n) + a\gamma_1(1 - b)d^2(x_n, z_n) \leq d^2(x_n, p) - d^2(x_{n+1}, p).$$

This implies that

$$\sum_{n=1}^{\infty} \left[\gamma_1(1 - \gamma)d^2(x_n, w_n y_n) \right] + \sum_{n=1}^{\infty} \left[a\gamma_1(1 - b)d^2(x_n, z_n) \right] \leq d^2(x_1, p) < \infty,$$

and hence $\lim_{n \rightarrow \infty} d(x_n, w_n y_n) = 0 = \lim_{n \rightarrow \infty} d(x_n, z_n)$.

Step 4. $\lim_{n \rightarrow \infty} d(x_n, tx_n) = 0$, where $t = \bigoplus_{i=1}^{\infty} \frac{\gamma_i}{\sum_{i=1}^{\infty} \gamma_i} t_i$:

Since C is compact, there exists a subsequence $\{y_{n'}\}$ of $\{y_n\}$ such that $y_{n'} \rightarrow y$ as $n' \rightarrow \infty$ for some $y \in C$. Using the nonexpansiveness of $w_{n'}$ and t , we have

$$\begin{aligned}
d(w_{n'} y_{n'}, ty_{n'}) &\leq d(w_{n'} y_{n'}, w_{n'} y) + d(w_{n'} y, ty) + d(ty, ty_{n'}) \\
&\leq 2d(y_{n'}, y) + d(w_{n'} y, ty) \rightarrow 0 \text{ as } n' \rightarrow \infty.
\end{aligned}$$

Therefore $\lim_{n \rightarrow \infty} d(w_n y_n, ty_n) = 0$. From Step 2 and Step 3 we have

$$\begin{aligned}
d(x_n, tx_n) &\leq d(x_n, ty_n) + d(ty_n, tx_n) \\
&\leq d(x_n, ty_n) + d(y_n, x_n) \\
&\leq d(x_n, w_n y_n) + d(w_n y_n, ty_n) + \beta_n d(x_n, z_n) \rightarrow 0
\end{aligned}$$

as $n \rightarrow \infty$

□