Chapter 5

Approximation of a common point of a Families of Closed Convex Subsets on CAT(0) Spaces

In this chapter, we introduce the new iterative sequence for approximating a common point of closed convex subsets in a CAT(0) space. The previous results and our main theorem is presented in section 5.1. Its applications are also presented in section 5.2. The results in this chapter were appeared in the paper : W. Anakkamatee and S. Dhompongsa, *On the means of projections on CAT(0) spaces,* Journal of Nonlinear Analysis and Optimization, 4 (2013), 51-59 [3].

5.1 Strong Convergence Theorems

In 2010, Saejung [34] used the Halpern iteration scheme for computing a fixed point of a single-valued nonexpansive mapping:

Theorem 5.1. [34, Theorem 2.3] *Let C be a closed convex subset of a complete CAT(0) space X and let $t : C \to C$ be a nonexpansive mapping with a nonempty fixed point set $Fix(t)$. Suppose that $u, x_1 \in C$ are arbitrarily chosen and $\{x_n\}$ is iteratively generated by*

$$x_{n+1} = \alpha_n u \oplus (1 - \alpha_n) t x_n, \;\; n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C3) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$ *or* $\lim_{n\to\infty}(\alpha_n/\alpha_{n+1}) = 1$.

Then $\{x_n\}$ *converges to* $v \in Fix(t)$ *which is the nearest to* u.

They also proved the following.

Theorem 5.2. [34, Lemma 2.2] *Let* C *be a nonempty closed convex subset of a complete* CAT(0) *space* X, *let* $t : C \to C$ *be nonexpansive, fix* $u \in C$, *and for each* $s \in (0, 1)$ *let* z_s *be the point of* $[u, tz_s]$ *satisfying*

$$d(u, z_s) = sd(u, tz_s).$$

Then $Fix(t) \neq \emptyset$ *if and only if* $\{z_s\}$ *remains bounded as* $s \to 1$. *In this case, the following statements hold:*

(i) $\{z_s\}$ *converges to the unique fixed point* z *of* t *which is nearest to* u;

(ii) $d^2(u, z) \leq \mu_n d^2(u, u_n)$ *for all Banach limits* μ *and all bounded sequences* $\{u_n\}$ *with* $d(u_n, tu_n) \to 0$.

Recently, Dhompongsa et al. [16] proved the following strong convergence theorem for countably many number of single-valued nonexpansive mappings and a multivalued nonexpansive mapping.

Theorem 5.3. [16, Theorem 3.7] *Let* C *be a nonempty closed convex subset of a complete CAT(0) space* X. *Let* $\{t_n : C \to C\}$ *be a countable family of nonexpansive mappings and* $T : C \to K(C)$ *be a nonexpansive mapping with* $F \neq \emptyset$. *Suppose that* $Tp = \{p\}$ *for all* $p \in F$. *Let* t *and* $\{\lambda_n\}$ *be as in Lemma 2.50. Suppose that* $u, x_1 \in C$ *are arbitrarily chosen and* $\{x_n\}$ *is defined by*

$$x_{n+1} = \alpha_n u \oplus (1 - \alpha_n) \left(\frac{1}{2} w_n x_n \oplus \frac{1}{2} y_n \right), \tag{5.1}$$

such that $d(y_n, y_{n+1}) \leq d(x_n, x_{n+1})$ *for all* $n \in \mathbb{N}$, *where* $y_n \in Tx_n$ *and* $\{\alpha_n\}$ *is a sequence in* $(0, 1)$ *satisfying (C1), (C2) and (C3). Then* $\{x_n\}$ *converges to the unique point of* F *which is nearest to* u.

In the course of the proof of Theorem 5.3, the following results play important role.

Lemma 5.4. [36, Proposition 2] *Let a be a real number and let $(a_1, a_2, ...) \in \ell^\infty$ be such that $\mu_n(a_n) \leq a$ for all Banach limits μ and* $\limsup_n (a_{n+1} - a_n) \leq 0$*. Then* $\limsup_n a_n \leq a$*.*

Lemma 5.5. [4, Lemma 2.3] *Let $\{s_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ a sequence of real numbers in $[0,1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\eta_n\}$ a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} \eta_n < \infty$, and $\{\gamma_n\}$ a sequence of real numbers with* $\limsup_{n\to\infty} \gamma_n \leq 0$*. Suppose that*

$$s_{n+1} \leq (1-\alpha_n)s_n + \alpha_n\gamma_n + \eta_n \text{ for all } n \in \mathbb{N}.$$

Then $\lim_{n\to\infty} s_n = 0$*.*

We first consider the following convergence result.

Theorem 5.6. *Let C be a closed convex subset of a complete CAT(0) space X, $t : C \to C$ be a nonexpansive mapping such that $Fix(t) \neq \emptyset$ and M a positive real number. Suppose $\{\varepsilon_n\}$ and $\{\alpha_n\}$ are sequences in $(0,1)$ satisfying $\sum_{n=1}^{\infty} \varepsilon_n < \infty$, (C1), (C2) and (C3) respectively. Let $u, x_1 \in C$ be arbitrarily chosen and $\{x_n\}$ be defined by*

$$x_{n+1} = \alpha_n u \oplus (1-\alpha_n)u_n, \quad u_n \in C$$

such that

$$d(u_n, tn_n) \leq \varepsilon_n M \tag{5.2}$$

for all $n \in \mathbb{N}$. If the sequence $\{x_n\}$ is bounded, then it converges to the unique point of $Fix(t)$ which is nearest to u.

Proof. We follow the proof of Theorem 5.3. By (5.2),

$$\begin{aligned} d(u_n, u_{n+1}) &\leq d(u_n, tx_n) + d(tx_n, tx_{n+1}) + d(tx_{n+1}, u_{n+1}) \\ &\leq d(x_n, x_{n+1}) + M(\varepsilon_n + \varepsilon_{n+1}). \end{aligned}$$

From the definition of x_n, we have

$$\begin{aligned} d(x_{n+1}, x_n) &= d(\alpha_n u \oplus (1-\alpha_n)u_n, \alpha_{n-1}u \oplus (1-\alpha_{n-1})u_{n-1}) \\ &\leq d(\alpha_n u \oplus (1-\alpha_n)u_n, \alpha_n u \oplus (1-\alpha_n)u_{n-1}) \\ &\quad + d(\alpha_n u \oplus (1-\alpha_n)u_{n-1}, \alpha_{n-1}u \oplus (1-\alpha_{n-1})u_{n-1}) \\ &\leq (1-\alpha_n)d(u_n, u_{n-1}) + |\alpha_n - \alpha_{n-1}|d(u, u_{n-1}) \\ &\leq (1-\alpha_n)d(x_n, x_{n-1}) + |\alpha_n - \alpha_{n-1}|d(u, u_{n-1}) \\ &\quad + (1-\alpha_n)M(\varepsilon_n + \varepsilon_{n-1}). \end{aligned}$$

Putting in Lemma 5.5 $[s_n = d(x_n, x_{n-1}), \gamma_n = 0$ and $\eta_n = |\alpha_n - \alpha_{n-1}|d(u, u_{n-1}) + (1-\alpha_n)M(\varepsilon_n + \varepsilon_{n-1})]$ or $[s_n = d(x_n, x_{n-1}), \gamma_n = \left|1 - \frac{\alpha_{n-1}}{\alpha_n}\right|d(u, u_{n-1})$ and $\eta_n = (1-\alpha_n)M(\varepsilon_n+\varepsilon_{n-1})]$ according to $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$ or $\lim_{n\to\infty}(\alpha_n/\alpha_{n+1}) = 1$, respectively. Since $\sum_{n=1}^{\infty} \varepsilon_n < \infty$, we obtain

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0.$$

It follows from (C1) that

$$\begin{aligned} d(x_n, u_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, u_n) \\ &= d(x_n, x_{n+1}) + d(\alpha_n u \oplus (1-\alpha_n)u_n, u_n) \\ &\leq d(x_n, x_{n+1}) + \alpha_n d(u, u_n) \to 0. \end{aligned}$$

This implies

$$\begin{aligned} d(u_n, tu_n) &\leq d(u_n, tx_n) + d(tx_n, tu_n) \\ &\leq \varepsilon_n M + d(x_n, u_n) \to 0. \end{aligned}$$

Let $z_s \in [u, tz_s]$ satisfy $d(u, z_s) = sd(u, tz_s)$ for all $s \in (0,1)$. By Theorem 5.2, we have $v =: \lim_{s\to 1} z_s$ which is the unique point of $Fix(t)$ nearest to u and $\mu_n(d^2(u,v) - d^2(u, u_n)) \leq 0$ for all Banach limits μ. Moreover, since $d(u_n, u_{n+1}) \leq d(x_n, x_{n+1}) + M(\varepsilon_n + \varepsilon_{n+1}) \to 0$,

$$\limsup_{n\to\infty} \left(d^2(u,v) - d^2(u,u_n)\right) - \left(d^2(u,v) - d^2(u,u_{n+1})\right) = 0.$$

Therefore Lemma 5.4 implies

$$\limsup_{n\to\infty}\left(d^2(u,v)-(1-\alpha_n)d^2(u,u_n)\right)=\limsup_{n\to\infty}\left(d^2(u,v)-d^2(u,u_n)\right)\leq 0.$$

Consider the following estimates:

$$\begin{aligned}
d^2(x_{n+1},v) &= d^2(\alpha_n u\oplus(1-\alpha_n)u_n,v)\\
&\leq \alpha_n d^2(u,v)+(1-\alpha_n)d^2(u_n,v)-\alpha_n(1-\alpha_n)d^2(u,u_n)\\
&= (1-\alpha_n)d^2(u_n,v)+\alpha_n\left(d^2(u,v)-(1-\alpha_n)d^2(u,u_n)\right)\\
&\leq (1-\alpha_n)(d(u_n,tx_n)+d(tx_n,v))^2+\alpha_n\left(d^2(u,v)-(1-\alpha_n)d^2(u,u_n)\right)\\
&\leq (1-\alpha_n)(d^2(x_n,v)+2\varepsilon_n Md(x_n,v)+\varepsilon_n^2M^2)\\
&\quad +\alpha_n\left(d^2(u,v)-(1-\alpha_n)d^2(u,u_n)\right)\\
&= (1-\alpha_n)d^2(x_n,v)+\alpha_n\left(d^2(u,v)-(1-\alpha_n)d^2(u,u_n)\right)\\
&\quad +(1-\alpha_n)(2\varepsilon_n Md(x_n,v)+\varepsilon_n^2M^2)\\
&\leq (1-\alpha_n)d^2(x_n,v)+\alpha_n\left(d^2(u,v)-(1-\alpha_n)d^2(u,u_n)\right)\\
&\quad +(1-\alpha_n)(2\varepsilon_n MN+\varepsilon_n^2M^2),
\end{aligned}$$

where $N=\sup\{d(x_n,v):n\in\mathbb{N}\}$. We use Lemma 5.5 to conclude the proof. $\square$

Now, we are ready to present our first main result.

Theorem 5.7. *Let X be a complete CAT(0) space and $\{A_i : i\in\mathbb{N}\}$ be a family of closed convex subsets of X such that $\bigcap_{i=1}^{\infty}A_i\neq\emptyset$. Let $\{\lambda_n\}$ be a sequence in $(0,1)$ such that $\sum_{n=1}^{\infty}\lambda_n=1$, $\sum_{i=n}^{\infty}\lambda_i'\to 0$ as $n\to\infty$ where $\lambda_i'=\sum_{j=i+1}^{\infty}\lambda_j$. Suppose $\{\varepsilon_n\}$ and $\{\alpha_n\}$ are sequences in $(0,1)$ satisfying $\sum_{n=1}^{\infty}\varepsilon_n<\infty$, (C1), (C2) and (C3) respectively. Let $u,x_1\in X$ be arbitrarily chosen and set*

$$\begin{aligned}
r_n &= \sup_{i\in\mathbb{N}}\{dist(x_n,A_i)\},\\
\beta_n &\in \left(0,\frac{1}{2}\sqrt{4r_n^2+4\varepsilon_n^2}-r_n\right),\\
x_{n+1} &= \alpha_n u\oplus(1-\alpha_n)u_n,
\end{aligned}$$

where

$$u_n=\bigoplus_{i=1}^{\infty}\lambda_i u_n^{A_i},\ \ u_n^{A_i}\in A_i\cap B(x_n:dist(x_n,A_i)+\beta_n^2)$$

for all $n\in\mathbb{N}$. Then the sequence $\{x_n\}$ converges to the unique point of $\bigcap_{i=1}^{\infty}A_i$ which is nearest to u.

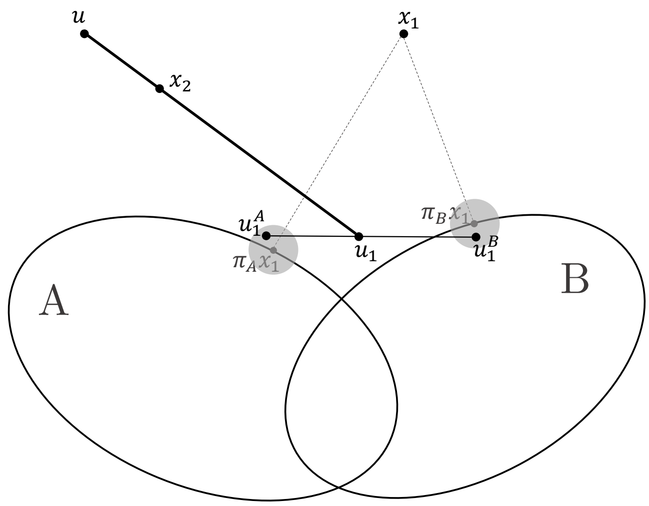

Figure 5.1: The first step of our iteration scheme for sets A and B

Proof. For each $i \in \mathbb{N}$, let $\pi_i : X \to A_i$ be the projection mapping. Using the law of cosine and the definition of β_n, we have

$$\begin{aligned} d^2(u_n^{A_i}, p_i x_n) &\le d^2(x_n, u_n^{A_i}) - d^2(x_n, \pi_i x_n) \\ &\le (d(x_n, \pi_i x_n) + \beta_n^2)^2 - d^2(x_n, \pi_i x_n) \\ &= 2\beta_n d(x_n, \pi_i x_n) + \beta_n^2 \\ &\le \beta_n(2r_n + \beta_n) \\ &< \left(\frac{1}{2}\sqrt{4r_n^2 + 4\varepsilon_n^2} - r_n\right)\left(\frac{1}{2}\sqrt{4r_n^2 + 4\varepsilon_n^2} + r_n\right) = \varepsilon_n^2. \end{aligned}$$

Hence $d(u_n^{A_i}, \pi_i z_n) < \varepsilon_n$ for all $n \in \mathbb{N}$. Let $\pi : X \to X$ be defined by

$$\pi x = \bigoplus_{i=1}^{\infty} \lambda_i \pi_i x$$

for each $x \in X$. From Lemma 2.50, π is nonexpansive and $Fix(\pi) = \bigcap_{i=1}^{\infty} Fix(\pi_i) = \bigcap_{i=1}^{\infty} A_i$. For each n, we can choose $m_n \in \mathbb{N}$ such that

$$d\left(\bigoplus_{i=1}^{\infty} \lambda_i u_n^{A_i}, \bigoplus_{i=1}^{m_n} \frac{\lambda_i}{\sum_{j=1}^{m_n} \lambda_j} u_n^{A_i}\right) + d\left(\bigoplus_{i=1}^{\infty} \lambda_i \pi_i x_n, \bigoplus_{i=1}^{m_n} \frac{\lambda_i}{\sum_{j=1}^{m_n} \lambda_j} \pi_i x_n\right) < \varepsilon_n.$$

Thus

$$
\begin{aligned}
d(u_n,\pi x_n) &\le d\left(\bigoplus_{i=1}^{\infty}\lambda_i u_n^{A_i},\bigoplus_{i=1}^{m_n}\frac{\lambda_i}{\sum_{j=1}^{m_n}\lambda_j}u_n^{A_i}\right)+d\left(\bigoplus_{i=1}^{m_n}\frac{\lambda_i}{\sum_{j=1}^{m_n}\lambda_j}u_n^{A_i},\bigoplus_{i=1}^{m_n}\frac{\lambda_i}{\sum_{j=1}^{m_n}\lambda_j}\pi_i x_n\right)\\
&\quad +d\left(\bigoplus_{i=1}^{m_n}\frac{\lambda_i}{\sum_{j=1}^{m_n}\lambda_j}\pi_i x_n,\bigoplus_{i=1}^{\infty}\lambda_i\pi_i x_n\right)\\
&< \sum_{i=1}^{m_n}\frac{\lambda_i}{\sum_{j=1}^{m_n}\lambda_j}d(u_n^{A_i}x_n,\pi_i x_n)+\varepsilon_n<2\varepsilon_n.
\end{aligned}
$$

Let $q\in\bigcap_{i=1}^{\infty}A_i$. Then

$$
\begin{aligned}
d(x_{n+1},q) &= d(\alpha_n u\oplus(1-\alpha_n)u_n,q)\\
&\le \alpha_n d(u,q)+(1-\alpha_n)d\left(\bigoplus_{i=1}^{\infty}\lambda_i u_n^{A_i},q\right)\\
&\le \alpha_n d(u,q)+(1-\alpha_n)d\left(\bigoplus_{i=1}^{\infty}\lambda_i u_n^{A_i},\bigoplus_{i=1}^{m_n}\frac{\lambda_i}{\sum_{j=1}^{m_n}\lambda_j}u_n^{A_i}\right)\\
&\quad +(1-\alpha_n)d\left(\bigoplus_{i=1}^{m_n}\frac{\lambda_i}{\sum_{j=1}^{m_n}\lambda_j}u_n^{A_i},q\right)\\
&\le \alpha_n d(u,q)+(1-\alpha_n)\left(\varepsilon_n+\sum_{i=1}^{m_n}\frac{\lambda_i}{\sum_{j=1}^{m_n}\lambda_j}(d(u_n^{A_i},\pi_i x_n)+d(\pi_i x_n,q))\right)\\
&\le \alpha_n d(u,q)+(1-\alpha_n)d(x_n,q)+2(1-\alpha_n)\varepsilon_n\\
&\le \max\{d(u,q),d(x_n,q)\}+2(1-\alpha_n)\varepsilon_n.
\end{aligned}
$$

By induction we have

$$d(x_{n+1},q)\le\max\{d(u,q),d(x_1,q)\}+2\sum_{n=1}^{\infty}(1-\alpha_n)\varepsilon_n<\infty \text{ for all } n\in\mathbb{N}.$$

This implies the sequence $\{x_n\}$ is bounded. The result now follows from Theorem 5.6. □

When the domain is bounded, we have the following result where the sequence $\{x_n\}$ is computable.

Theorem 5.8. *Let X be a complete CAT(0) space and $\{A_i : i\in\mathbb{N}\}$ be a family of closed convex subsets of X such that $\bigcap_{i=1}^{\infty}A_i\neq\emptyset$ and $\bigcup_{i=1}^{\infty}A_i$ is bounded. Let $\{\lambda_n\}$ be a sequence in $(0,1)$ such that $\sum_{n=1}^{\infty}\lambda_n=1$, $\sum_{i=n}^{\infty}\lambda_i'\to 0$ as $n\to\infty$ where $\lambda_i'=\sum_{j=i+1}^{\infty}\lambda_j$. Let $\{\varepsilon_n\}$ be a sequence in $(0,\frac{1}{2})$ and $\{\alpha_n\}$ be a sequence in $(0,1)$*

satisfying $\sum_{n=1}^{\infty} \varepsilon_n < \infty$*, (C1), (C2) and (C3) respectively. Let* $u, x_1 \in C$ *be arbitrarily chosen. For each* $n \in \mathbb{N}$*, choose* $k_n \in \mathbb{N}$ *such that* $\lambda_i' < \varepsilon_n$ *for all* $i \geq k_n$ *and set*

$$\begin{aligned} r_n &= \sup_{i\in\mathbb{N}}\{dist(x_n, A_i)\}, \\ \beta_n &\in \left(0, \frac{1}{2}\sqrt{4r_n^2 + 4\varepsilon_n^2}r_n\right), \\ x_{n+1} &= \alpha_n u \oplus (1-\alpha_n)u_n', \end{aligned}$$

where

$$u_n' = \bigoplus_{i=1}^{k_n} \frac{\lambda_i}{\sum_{j=1}^{k_n}\lambda_j} u_n^{A_i}, \;\; u_n^{A_i} \in A_i \cap B(z_n : dist(x_n, A_i) + \beta_n^2).$$

Then the sequence $\{x_n\}$ *converges to the unique point of* $\bigcap_{i=1}^{\infty} A_i$ *which is nearest to* u*.*

Proof. Let π_i and π be as in the proof of Theorem 5.7. Thus we have

$$d(u_n^{A_i}, \pi_i x_n) < \varepsilon_n$$

for all $n \in \mathbb{N}$. For each n, we can choose $m_n > k_n$ such that

$$d\left(\bigoplus_{i=1}^{\infty} \lambda_i \pi_i x_n, \bigoplus_{i=1}^{m_n} \frac{\lambda_i}{\sum_{j=1}^{m_n}\lambda_j} \pi_i x_n\right) < \varepsilon_n.$$

Since $\lambda_i' < \varepsilon_n < \frac{1}{2}$, we have

$$\begin{aligned} & d\left(\bigoplus_{i=1}^{k_n} \frac{\lambda_i}{\sum_{j=1}^{k_n}\lambda_j}\pi_i x_n, \bigoplus_{i=1}^{m_n} \frac{\lambda_i}{\sum_{j=1}^{m_n}\lambda_j}\pi_i x_n\right) \\ &\leq d\left(\bigoplus_{i=1}^{k_n} \frac{\lambda_i}{\sum_{j=1}^{k_n}\lambda_j}\pi_i x_n, \bigoplus_{i=1}^{k_n+1} \frac{\lambda_i}{\sum_{j=1}^{k_n+1}\lambda_j}\pi_i x_n\right) + \cdots \\ &\quad + d\left(\bigoplus_{i=1}^{m_n-1} \frac{\lambda_i}{\sum_{j=1}^{m_n-1}\lambda_j}\pi_i x_n, \bigoplus_{i=1}^{m_n} \frac{\lambda_i}{\sum_{j=1}^{m_n}\lambda_j}\pi_i x_n\right) \\ &\leq \frac{\lambda_{k_n+1}}{\sum_{j=1}^{k_n+1}\lambda_j} d\left(\bigoplus_{i=1}^{k_n} \frac{\lambda_i}{\sum_{j=1}^{k_n}\lambda_j}\pi_i x_n, \pi_{k_n+1}x_n\right) + \cdots \\ &\quad + \frac{\lambda_{m_n}}{\sum_{j=1}^{m_n}\lambda_j} d\left(\bigoplus_{i=1}^{m_n-1} \frac{\lambda_i}{\sum_{j=1}^{m_n-1}\lambda_j}\pi_i x_n, \pi_{m_n}x_n\right) \\ &\leq K\sum_{i=k_n+1}^{m_n} \frac{\lambda_i}{1-\lambda_i'} < 2K\sum_{i=k_n+1}^{m_n}\lambda_i < 2K\lambda_{k_n+1}' < 2K\varepsilon_n, \end{aligned}$$

where $K = \sup_{n\in\mathbb{N}} \left\{ \sup_{l\in\mathbb{N}} \left\{ d\left(\bigoplus_{i=1}^{l} \frac{\lambda_i}{\sum_{j=1}^{i}\lambda_j} \pi_i x_n, \pi_{l+1} x_n \right) \right\} \right\} < \infty$. Thus

$$d(u_n', \pi x_n) \leq \varepsilon_n (2K + 2).$$

The result now follows from Theorem 5.6. □

5.2 Applications

Let X be a complete CAT(0) space. For a function $h : X \to (-\infty, \infty]$, the $\alpha-$sublevel set is defined by

$$A_h^{\alpha} = \{x \in X : h(x) \leq \alpha\}.$$

The function h is said to be

- *lower semi-continuous* if A_h^{α} is closed for all $\alpha \in \mathbb{R}$;
- *uniformly continuous* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $d(h(x), h(y)) \leq \varepsilon$ for all $x, y \in X$ with $d(x, y) \leq \delta$;
- *convex* if for any $x, y \in X$, $u \in [x, y]$ and $t \in (0, 1)$ we have

$$h(u) \leq (1 - t)h(x) + th(y);$$

- *uniformly convex* if there exists $\lambda > 0$ such that for any $x, y \in X$ and $u \in [x, y]$ we have

$$h(u) \leq (1 - t)h(x) + th(y) - \lambda t(1 - t)d^2(x, y)$$

  where $t = \frac{d(x,u)}{d(x,y)}$.

Let $\{h_i : i \in \mathbb{N}\}$ be a family of lower semi-continuous and convex functions from X into $(-\infty, \infty]$. In [5], Bačák, Searston and Sims presented the application of the alternating projection method for approximating an minimizer of the function $H : X \to (-\infty, \infty]$ where $H = \sup_{i\in\mathbb{N}} h_i$ as follow:

Proposition 5.9. [5, Proposition 5.2] *Let X be a complete CAT(0) space and a mapping $F : X \to (-\infty, \infty]$ be of the form $F = \max\{f, g\}$, where $f, g : X \to (-\infty, \infty]$ are lower semi-continuous and convex functions. Let $\alpha > \inf_{x \in X} F(x) > -\infty$, and A_F^α be nonempty. Assume that f is both uniformly convex and uniformly continuous on bounded sets of X. Let $x_0 \in X$ be a starting point and $\{x_n\} \subset X$ be the sequence generated by*

$$x_{2n-1} = \pi_f(x_{2n-1}), \quad x_{2n} = \pi_g(x_{2n-1}), \; n \in \mathbb{N},$$

where π_f and π_g are projection mappings from X to A_f^α and A_g^α respectively. Then $\{x_n\}$ converges to $z \in A_F^\alpha$.

Two Propositions below also provide the strong convergence of the sequence $\{x_n\}$ to an (approximative) minimizer of H by using our iteration scheme.

Proposition 5.10. *Let X be a complete CAT(0) space and a mapping $H : X \to (-\infty, \infty]$ be of the form $H = \sup_{i \in \mathbb{N}} h_i$, where $h_i : X \to (-\infty, \infty]$ are lower semi-continuous and convex functions for all $i \in \mathbb{N}$. Let $\alpha > \inf_{x \in X} H(x) > -\infty$. Let $\{\lambda_n\}$ be a sequence in $(0,1)$ such that $\sum_{n=1}^\infty \lambda_n = 1$, $\sum_{i=n}^\infty \lambda_i' \to 0$ as $n \to \infty$ where $\lambda_i' = \sum_{j=i+1}^\infty \lambda_j$. Suppose $\{\varepsilon_n\}$ and $\{\alpha_n\}$ are sequences in $(0,1)$ satisfying $\sum_{n=1}^\infty \varepsilon_n < \infty$, (C1), (C2) and (C3) respectively. Let $u, x_1 \in X$ are arbitrarily chosen and set*

$$\begin{aligned} r_n &= \sup_{i \in \mathbb{N}} \{dist(x_n, A_{h_i}^\alpha)\}, \\ \beta_n &\in \left(0, \frac{1}{2}\sqrt{4r_n^2 + 4\varepsilon_n^2} - r_n\right), \\ x_{n+1} &= \alpha_n u \oplus (1-\alpha_n) u_n, \end{aligned}$$

where

$$u_n = \bigoplus_{i=1}^\infty \lambda_i u_n^i, \;\; u_n^i \in A_{h_i}^\alpha \cap B(x_n : dist(x_n, A_{f_i}^\alpha) + \beta_n^2)$$

for all $n \in \mathbb{N}$. Then the sequence $\{x_n\}$ converges to the unique point of A_H^α which is nearest to u.

Proof. Since $h_i : X \to (-\infty, \infty]$ are lower semi-continuous and convex functions, $A_{h_i}^\alpha$ is closed and convex for all $i \in \mathbb{N}$. The result then follows from Theorem 5.7. $\square$

Proposition 5.11. *Let X be a complete CAT(0) space and a mapping $H : X \to (-\infty, \infty]$ be of the form $H = \sup_{i\in\mathbb{N}} h_i$, where $h_i : X \to (-\infty, \infty]$ are lower semi-continuous and convex functions for all $i \in \mathbb{N}$. Let $\alpha > \inf_{x\in X} H(x) > -\infty$. Let $\{\lambda_n\}$ be a sequence in $(0,1)$ such that $\sum_{n=1}^{\infty} \lambda_n = 1$, $\sum_{i=n}^{\infty} \lambda'_i \to 0$ as $n \to \infty$ where $\lambda'_i = \sum_{j=i+1}^{\infty} \lambda_j$. Suppose $\{\varepsilon_n\}$ is a sequence in $(0, \frac{1}{2})$ and $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying $\sum_{n=1}^{\infty} \varepsilon_n < \infty$, (C1), (C2) and (C3) respectively. Let $u, x_1 \in C$ be arbitrarily chosen. For each $n \in \mathbb{N}$, choose $k_n \in \mathbb{N}$ such that $\lambda'_i < \varepsilon_n$ for all $i \geq k_n$ and set*

$$\begin{aligned} r_n &= \sup_{i\in\mathbb{N}}\{dist(x_n, A^{\alpha}_{h_i})\}, \\ \beta_n &\in \left(0, \frac{1}{2}\sqrt{4r_n^2 + 4\varepsilon_n^2} - r_n\right), \\ x_{n+1} &= \alpha_n u \oplus (1-\alpha_n)u'_n, \end{aligned}$$

where

$$u'_n = \bigoplus_{i=1}^{k_n} \frac{\lambda_i}{\sum_{j=1}^{k_n} \lambda_j} u^i_n, \quad u^i_n \in A^{\alpha}_{h_i} \cap B(x_n : dist(x_n, A^{\alpha}_{h_i}) + \beta_n^2).$$

If $\{x_n\}$ is bounded, then the sequence $\{x_n\}$ converges to the unique point of A^{α}_H which is nearest to u.

Proof. Here we apply Theorem 5.8. □