CHAPTER 2

Methodology

In this chapter, we introduce the methods that were used in this research, which consisted of ARMA-GARCH model, copulas and the criterion for selection of copula family, vine copulas and method to select vine tree structure, time-varying copulas, and maximal spanning tree. This research used the R-package fGarch by Wuertz and Chalabi (2013) to estimate the parameters of the ARMA-GARCH model. We used the R-package CDVine by Brechmann and Schepsmeier (2013) to analyze the constant copula since it provides a range of tools for bivariate copulas and vine copulas analysis. For the time-varying copula, we followed the method endorsed by Patton (2006a). The R-package VineCopula by Schepsmeier et al. (2013) was used to analyze the maximal spanning tree models and for bivariate copulas analysis.

2.1 Marginal distribution model

Different models are appropriate for different time series data. Therefore, we adopt ARMA(p,q) - GARCH(1,1) model (Bollerslev, 1986) with an appropriate distribution (*D*), residual distribution, for the marginal distribution of the log-difference, $\ln\frac{P_t}{P_{t-1}}$, of the data series: agricultural commodity price, energy price, and economic variable.

$$y_t = a_0 + \sum_{i=1}^{p} a_i y_{t-i} + \sum_{i=1}^{q} b_i \varepsilon_{t-i} + \varepsilon_t \quad (1)$$

$$\varepsilon_t = z_t\sqrt{h_t}, z_t \sim D \quad (2)$$

$$h_t = \omega_t + \alpha\varepsilon_{t-1}^2 + \beta h_{t-1} \quad (3)$$

In equation (1) is presented the ARMA(p,q) process, where y_{t-i} is an autoregressive term of y_t and ε_t is an error term. Equation (2) then defines this residual as the product between the conditional variance h_t and a random variable z_t. The residual ε_t will be standardized by $\frac{\varepsilon_t}{\sqrt{h_t}}$ to be a standardized residual z_t. The residual z_t will be assumed to follow an appropriate distribution (*D*). Equation (3) presents the GARCH(1,1) process where $\omega_t > 0,\ \alpha \geq 0,\ \beta \geq 0$ is sufficient to ensure that the conditional variance $h_t > 0$. The $\alpha\varepsilon_{t-1}^2$ represents the ARCH term and α refers to the short run persistence of shocks, while βh_{t-1} represents the GARCH term, and β refers to the contribution of shocks to long run persistence $(\alpha + \beta)$. The properties of the GARCH(1,1) model require stationary and persistence of the conditional variance, h_t, of the error term, ε_t. This study will use the second moment condition that was $\alpha + \beta < 1$ to check for these properties. In this study, the R-package fGarch by Wuertz and Chalabi (2013) was used to estimate the parameters of the GARCH(1,1) model.

For the next analysis, we transformed the standardized residuals from the ARMA(p,q)-GARCH(1,1) model to copula data $(F_1(x_1), F_2(x_2), F_3(x_3), \ldots, F_n(x_n))$ by using the empirical distribution function.

We can construct the pseudo-copula observations by using the empirical distribution functions to transform the standardized residual series into uniform [0, 1] as rank based. The empirical distribution functions can be written as equation (4), where $X_i \leq x$ is the order statistics and 1 is the indicator function.

$$F_n(x) = \frac{1}{n+1}\sum_{i=1}^{n} 1(X_i \leq x) \tag{4}$$

The transformed data were used in the Kolmogorov–Smirnov (K–S) test for uniformity [0, 1] and the Box–Ljung test for serial correlation. More details are available in Patton (2006a) and Manthos (2010). These tests are necessary to check for the marginal distribution models' misspecification before using the copula model.

2.2 Copulas and Dependence Measures

Copulas is a popular tool for modeling the multivariate dependence. The copula functions can offer us the flexibility to merge univariate distributions to get a joint distribution with an appropriate dependence structure. The fundamental theorem of copula is the Sklar's theorem, which was proposed by Sklar (1959). The standard reference book of the copula theory was made by Nelson (2006).

Definition of copula. A function $C : [0,1]^2 \rightarrow [0,1]$ is a copula if it satisfies the following:

For every u, v in [0,1].

$C(u, 0) = 0 = C(0, v)$ and

$C(u, 1) = u$ and $C(1, v) = v$

For every u_1, u_2, v_1, v_2 in [0,1] such that $u_1 \leq u_2$ and $v_1 \leq v_2$.

$C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0$

Sklar's Theorem. Let F be a n-dimensional distribution function with marginal distributions $F_1, F_2, \dots, F_n$. Then there exists a copula C for all $x = (x_1, \dots, x_n) \in [-\infty, \infty]^n$,

$$F(x) = C(F_1(x_1), \dots, F_n(x_n)) \tag{5}$$

If $F_1, \dots, F_n$ are continuous, then C is unique. Conversely, if C is a copula and $F_1, \dots, F_n$ are distribution functions, then the above function $F(x)$ in equation (5) is a joint distribution function with the marginal distribution $F_1, \dots, F_n$.

Given $F_1(x_1) = u_1, F_2(x_2) = u_2, \dots, F_n(x_n) = u_n$, then we obtain $x_1 = F_1{}^{-1}(u_1), x_2 = F_2{}^{-1}(u_2), \dots, x_n = F_n{}^{-1}(u_n)$. If F is known, the copula is an equation (5) that one can get from the form,

$$C(u_1, \dots, u_n) = F(F_1{}^{-1}(u_1), F_2{}^{-1}(u_2), \dots, F_n{}^{-1}(u_n)) \tag{6}$$

where $F_i{}^{-1}$are the inverse distribution functions of the marginals.

2.2.1 Dependence Measures

The copula model dealt with the multivariate distribution that describes the dependence structure. In this part will present three methods that are used to measure the dependence in copula model. The three methods are: 1) Spearman's rho 2) Kendall's tau and 3) tail dependence.

1) Spearman's rho

Spearman's rho (ρ_s) measures the distance between the distribution of random variables X and Y with continuous distribution functions F_X and F_Y,

$$\rho_s(X,Y) = \rho(F_X(X), F_Y(Y)) \quad (7)$$

In terms of copula, Spearman's rho $\rho_s(X,Y)$ can be expressed as (see Trivedi and Zimmer, 2005)

$$\rho_s(X,Y) = 12\int_0^1\int_0^1\{C(u_1,u_2) - u_1u_2\}du_1u_2 \quad (8)$$

2) Kendall's tau

Kendall's tau (τ) is also a rank correlation. It measures the difference between probability of concordance and the probability of discordance of two independent, identically distributed pairs of random variables. Let (X,Y) and (X',Y') be two independent pairs of random variables, Kendall's tau can be defined as

$$\tau(X,Y) = \Pr[\text{concordance}] - \Pr[\text{disconcordance}]$$

$$\tau(X,Y) = P[(X-X')(Y-Y') > 0] - P[(X-X')(Y-Y') < 0] \quad (9)$$

In terms of copula, Kendall's tau $\tau(X,Y)$ can be expressed as

$$\tau(X,Y) = 4\int_0^1\int_0^1 C(u_1,u_2)dC(u_1,u_2) - 1 \quad (10)$$

3) Tail Dependence

Tail dependence uses to explain the degree of dependence in the upper and lower tails of a bivariate distribution. As tail dependences can model the dependence of loss events across portfolio assets, the distributions of the tail dependences in the case of financial risk should be considered. The dependence of the tails of the bivariate copula was explained by Joe (1997).

Let X and Y be the random variables with marginal distribution functions F_X and F_Y. The upper tail dependence (T^U) of X and Y can be given as

$$T^U(X,Y) = \lim_{\alpha\to 1}\left(P(X > F_X{}^{-1}(\alpha) \middle| Y > F_Y{}^{-1}(\alpha)\right) \quad (11)$$

The upper tail dependence can be expressed in terms of copula as

$$T^U(X,Y) = \lim_{\alpha\to 1}\frac{1-2\alpha+C(\alpha,\alpha)}{1-\alpha} \quad (12)$$

If $T^U(X,Y) \in (0,1]$, the joint distribution of X and Y, shows upper tail, indicating that the probability of the joint occurrence of the extreme values is positive; if $T^U(X,Y) = 0$, then there is no upper tail dependence.

Similarly, the lower tail dependence (T^L) of X and Y can be given as

$$T^L(X,Y) = \lim_{\alpha\to 0}\left(P(X < F_X{}^{-1}(\alpha) \middle| Y < F_Y{}^{-1}(\alpha)\right) \quad (13)$$

The lower tail dependence can be expressed in terms of copula as

$$T^L(X,Y) = \lim_{\alpha\to 0}\frac{C(\alpha,\alpha)}{\alpha} \quad (14)$$

If $T^L(X,Y) \in (0,1]$, the joint distribution of X and Y shows lower tail dependence, indicating that the probability of the joint occurrence of the extreme values is negative; if $T^L(X,Y) = 0$, then there is no lower tail dependence.

2.2.2 Copula Families

This research adopted various constant copulas to measure the dependence between two marginal distributions. The R-package CDVine by Brechmann and Schepsmeier (2013) and the R-package VineCopula by Schepsmeier et al. (2013) were used to estimate of the copula parameters. Table 2.1 and Table 2.2 presented the characteristics of some copula families that were used in this study.

Table 2.1 Pair-copula function and parameter range of copula families

Name	Pair-copula function	Parameter range
Gaussian	$C(u_1,u_2;\rho) \;=\; \Phi_G(\Phi^{-1}(u_1),\Phi^{-1}(u_2);\,\rho)$ $= \int_{-\infty}^{\Phi^{-1}(u_1)}\int_{-\infty}^{\Phi^{-1}(u_2)} \frac{1}{2\pi\sqrt{(1-\rho^2)}} \times \left\{\frac{-(s^2-2\rho st+t^2)}{2(1-\rho^2)}\right\} dsdt$	$\rho \in (-1,1)$
Student's T	$C^T(u_1,u_2;\rho,\nu) = \int_{-\infty}^{T_\nu^{-1}(u_1)}\int_{-\infty}^{T_\nu^{-1}(u_2)} \frac{1}{2\pi\sqrt{1-\rho^2}} \times \left\{1 + \frac{(s^2-2\rho sT+T^2)}{\nu(1-\rho^2)}\right\}^{-(\frac{\nu+2}{2})} dsdT$	$\rho \in (-1,1)$, $\nu > 2$
Clayton	$C(u_1,u_2;\theta) = (u_1^{-\theta}+u_2^{-\theta}-1)^{-\frac{1}{\theta}}$	$\theta \in (0,\infty)$
Gumbel	$C(u_1,u_2;\theta) = exp\left(-\left[(-logu_1)^\theta+(-logu_2)^\theta\right]^{\frac{1}{\theta}}\right)$	$\theta \in [1,\infty)$
Frank	$C(u_1,u_2;\theta) = -\frac{1}{\theta}log\left(1+\frac{(e^{-\theta u_1}-1)(e^{-\theta u_2}-1)}{e^{-\theta}-1}\right)$	$\theta \in (-\infty,\infty)/\{0\}$
Joe	$C(u_1,u_2;\theta) = 1-\left[(1-u_1)^\theta+(1-u_2)^\theta-(1-u_1)^\theta(1-u_2)^\theta\right]^{\frac{1}{\theta}}$	$\theta \in [1,\infty)$
Rotated Clayton 90°	$C(u_1,u_2;\theta) = u_2-[(1-u_1)^{-\theta}+u_2^{-\theta}-1]^{-\frac{1}{\theta}}$	$\theta \in (-\infty, 0)$
Rotated Gumbel 90°	$C(u_1,u_2;\theta) = u_2-exp\left(-\left[(-log(1-u_1))^\theta+(-logu_2)^\theta\right]^{\frac{1}{\theta}}\right)$	$\theta \in (-\infty, -1]$
Rotated Joe 90°	$C(u_1,u_2;\theta) = u_2-1-\left[{u_1}^\theta+(1-u_2)^\theta-{u_1}^\theta(1-u_2)^\theta\right]^{\frac{1}{\theta}}$	$\theta \in (-\infty, -1)$
Rotated Clayton 180°	$C(u_1,u_2;\theta) = u_1+u_2-1+\left[(1-u_1)^{-\theta}+(1-u_2)^{-\theta}-1\right]^{-\frac{1}{\theta}}$	$\theta \in (0,\infty)$
Rotated Gumbel 180°	$C(u_1,u_2;\theta) = u_1+u_2-1+exp\left(-\left[(-\log(1-u_1))^\theta+(-log(1-u_2))^\theta\right]^{\frac{1}{\theta}}\right)$	$\theta \in [1,\infty)$
Rotated Joe 180°	$C(u_1,u_2;\theta) = u_1+u_2-\left({u_1}^\theta+{u_2}^\theta-{u_1}^\theta{u_2}^\theta\right)^{\frac{1}{\theta}}$	$\theta \in [1,\infty)$

Source: More descriptions of copula functions were presented in Trivedi and Zimmer (2005), Nelson (2006), Fisher (2003), and Brechmann and Schepsmeier (2013).

Table 2.1 Pair-copula function and parameter range of copula families (cont.)

Name	Pair-copula function	Parameter range
BB1	$C(u_1,u_2;\theta,\delta)=\left\{1+\left[(u_1{}^{-\theta}-1)^{\delta}+(u_2{}^{-\theta}-1)^{\delta}\right]^{\frac{1}{\delta}}\right\}^{-\frac{1}{\theta}}$	$\theta \in (0, \infty)$, $\delta \in [1, \infty)$
Rotated BB1 90°	$C(u_1,u_2;\theta,\delta)=u_2-\left\{1+\left[((1-u_1)^{-\theta}-1)^{\delta}+(u_2{}^{-\theta}-1)^{\delta}\right]^{\frac{1}{\delta}}\right\}^{-\frac{1}{\theta}}$	$\theta \in (-\infty, 0)$, $\delta \in (-\infty, -1]$
Rotated BB1 180°	$C(u_1,u_2;\theta,\delta)=u_1+u_2-1+\left\{1+\left[((1-u_1)^{-\theta}-1)^{\delta}+((1-u_2)^{-\theta}-1)^{\delta}\right]^{\frac{1}{\delta}}\right\}^{-\frac{1}{\theta}}$	$\theta \in (0, \infty)$, $\delta \in [1, \infty)$
BB8	$C(u_1,u_2;\theta,\delta)=\delta^{-1}[1-\{1-[1-(1-\delta)^{\theta}]^{-1}\left[1-(1-\delta u_1)^{\theta}\right][1-(1-\delta u_2)^{\theta}]\}^{\frac{1}{\theta}}$	$\theta \in [1, \infty)$, $\delta \in (0, 1]$
Rotated BB8 180°	$C(u_1,u_2;\theta,\delta)=u_1+u_2-1+\delta^{-1}[1-\{1-[1-(1-\delta)^{\theta}]^{-1}\left[1-(1-\delta(1-u_1))^{\theta}\right][1-(1-\delta(1-u_2))^{\theta}]\}^{\frac{1}{\theta}}$	$\theta \in [1, \infty)$, $\delta \in (0, 1]$

Source: More descriptions of copula functions were presented in Trivedi and Zimmer (2005), Nelson (2006), Fisher (2003), Brechmann and Schepsmeier (2013), Manner (2007), and Joe (1997).

Table 2.2 Kendall's tau and tail dependence of copula families

Name	Kendall's tau	Tail dependence (lower, upper)
Gaussian	$\frac{2}{\pi}\arcsin(\rho)$	0
Student's T	$\frac{2}{\pi}\arcsin(\rho)$	$2T_{\nu+1}\left(-\sqrt{\nu+1}\sqrt{\frac{1-\rho}{1+\rho}}\right)$
Clayton	$\frac{\theta}{\theta+2}$	$\left(2^{-1/\theta},0\right)$
Gumbel	$1-\frac{1}{\theta}$	$(0,2-2^{1/\theta})$
Frank	$1-\frac{4}{\theta}+4\frac{D_1(\theta)}{\theta}$	(0, 0)
Joe	$1+\frac{4}{\theta^2}\int_0^1 t\log(t)\times(1-t)^{2(1-\theta)/\theta}dt$	$(0,2-2^{1/\theta})$
Rotated Clayton 90°	$\frac{\theta}{2-\theta}$	-

Source: Kendall's tau and tail dependence were presented in Brechmann and Schepsmeier (2013).

Note: $D_1(\theta) = \int_0^\theta \frac{c/\theta}{\exp(x)-1} dx$ is the Debye function.

Table 2.2 Kendall's tau and tail dependence of copula families (cont.)

Name	Kendall's tau	Tail dependence (lower, upper)
Rotated Gumbel 90°	$-1-\frac{1}{\theta}$	-
Rotated Joe 90°	$-1-\frac{4}{\theta^2}\int_0^1 t\log(t)\times(1-t)^{-2(+\theta)/\theta}dt$	-
Rotated Clayton 180°	$\frac{\theta}{\theta+2}$	$(0, 2^{-1/\theta})$
Rotated Gumbel 180°	$1-\frac{1}{\theta}$	$(2-2^{1/\theta}, 0)$
Rotated Joe 180°	$1+\frac{4}{\theta^2}\int_0^1 t\log(t)\times(1-t)^{2(1-\theta)/\theta}dt$	$(2-2^{1/\theta}, 0)$
BB1	$1-\frac{2}{\delta(\theta+2)}$	$(2^{-1/(\theta\delta)}, 2-2^{1/\theta})$
Rotated BB1 90°	$-1-\frac{2}{\delta(2-\theta)}$	-
Rotated BB1 180°	$1-\frac{2}{\delta(\theta+2)}$	$(2-2^{1/\delta}, 2^{-1/(\theta\delta)})$
BB8	$1+4\int_0^1 -\log(((1-t\delta)^\theta-1)/((1-\delta)^\theta-1)) * (1-t\delta-(1-t\delta)^{-\theta}+(1-t\delta)^{-\theta}t\delta)/(\theta\delta)dt$	$(2-2^{1/\theta}$ if $\delta=1$ otherwise 0, 0)
Rotated BB8 180°	$1+4\int_0^1 -\log(((1-t\delta)^\theta-1)/((1-\delta)^\theta-1)) * (1-t\delta-(1-t\delta)^{-\theta}+(1-t\delta)^{-\theta}t\delta)/(\theta\delta)dt$	(0, $2-2^{1/\theta}$ if $\delta=1$ otherwise 0)

Source: Kendall's tau and tail dependence were presented in Brechmann and Schepsmeier (2013).

Note: $D_1(\theta) = \int_0^\theta \frac{c/\theta}{\exp(x)-1} dx$ is the Debye function.

2.2.3 Estimation of Copula Parameters

This research used the method of maximum pseudo-log likelihood (MPL) by Genest et al. (1995) for estimation because the marginal distribution functions F_X and F_Y of the random vectors are unknown. Therefore, the pseudo copula observations were constructed by using the empirical distribution functions to transform the standardized residual series into uniform [0, 1].

Under the assumption that the marginal distributions F_X and F_Y are continuous, the copula C_θ is a bivariate distribution with density c_θ and pseudo-observations $F_n(X_i)$ and

$F_n(Y_i)$, $i = 1,2,\ldots,n$. The pseudo-log likelihood function of parameter, θ, can be expressed as

$$L(\theta) = \sum_{i=1}^{n} log[c_\theta(F_n(X_i), F_n(Y_i))] \tag{15}$$

where the empirical distributions are

$$F_n(x) = \frac{1}{n+1} \sum_{i=1}^{n} 1(X_i \leq x) \tag{16}$$

$$F_n(y) = \frac{1}{n+1} \sum_{i=1}^{n} 1(Y_i \leq y) \tag{17}$$

and the density function of copula is

$$c_\theta = \frac{\partial^2 C_\theta(F_n(x), G_n(y))}{\partial x \partial y} \tag{18}$$

2.2.4 Criterion for copula selection

Selecting a family of copulas is based upon information criteria such as Akaike Information Criterion (AIC) by Akaike (1973) and Bayesian Information Criterion (BIC) by Schwarz (1978). For examining whether the dependence structure of the data series is appropriate for a chosen family of copulas, we used a goodness-of-fit test based on Kendall’s tau by Genest and Rivest (1993), and Wang and Wells (2000), which was contained in the R-package CDVine. A second goodness-of-fit test based on White’s information matrix equality (White, 1982) as introduced by Huang and Prokhorov (2011), which was contained in the R-package VineCopula.

1) Aikaike Information Criterion (AIC)

$$AIC = -2\log[L(\theta|X)] + 2k \tag{19}$$

where $[L(\theta|X)]$ is the log likelihood function of the maximum likelihood estimation, θ , given the observed data X, and k is the number of the parameter θ in the model. The models with the smaller AIC value indicating a better fit.

2) Bayesian Information Criterion (BIC)

$$BIC = -2\log[L(\theta|X)] + k\log(n) \tag{20}$$

where k is the number of the parameter θ in the model, n is the sample size. The models with the smaller BIC value indicating a better fit.

3) A Goodness-of-fit test based on Kendall's tau process for bivariate copula

A goodness-of-fit test based on Kendall's process was presented by Genest and Rivest (1993) and Wang and Wells (2000). This method uses to test the appropriateness of copula model under null hypothesis that the empirical copula C belong to a parametric class $\mathcal{C}$ of any copulas, $H_0: C \in \mathcal{C}$. A copula goodness-of-fit test provides the Cramér-von Mises (CvM) test statistic and Kolmogorov-Smirnov (KS) test statistic and estimated p-values by bootstrapping (Brechmann and Schepsmeier, 2013).

Cramér-von Mises test statistic takes form (see Genest and Favre 2007):

$$S_{n\xi} = n\int_{\xi}^{1}\left\{K_n(w) - K_{\theta_n}(w)\right\}^2 dw \tag{21}$$

where $\xi \in (0,1)$, $K_n(w)$ is an empirical distribution function (non-parametric estimate) of variable $W_1, \ldots, W_n$, and $K_{\theta_n}(w)$ is a theoretical distribution function (parametric estimate) of $W = C_{\theta_n}(U_1, U_2)$, where the pair (U_1, U_2) drawn from C_{θ_n}; $K_{\theta_n}(w) = P\left\{C_{\theta_n}(U_1, U_2) \le w\right\}$.

Kolmogorov-Smirnov test statistics takes form (see Genest and Favre 2007):

$$T_n = \sup_{0\le w\le 1}|K_n(w)| \tag{22}$$

where $K_n(w) = \sqrt{n}\{K_n(w) - K_{\theta_n}(w)\}$ is Kendall's process, Genest and Rivest (1993) mentioned that this function is called Kendall's process since the proposed estimator K_n

can be viewed as a decomposition of Kendall's tau. Please see more detail and process in Genest et al. (2006) and Genest and Favre (2007).

4) A goodness-of-fit test based on White's information matrix equality

This goodness-of-fit test adopts the information matrix equality of White (1982) and was presented by Huang and Prokhorov (2011). The contribution of this method is that under a correct model specification the Fisher Information can be equivalently calculated as minus the expected Hessian matrix, $\mathbf{H}(\theta)$, or as the expected outer product of the score function, $\mathbb{C}(\theta)$, Schepsmeier et al. (2013). The null hypothesis is that a copula family is correctly specified if,

$$H_0: \mathbf{H}(\theta) + \mathbb{C}(\theta) = 0 \tag{23}$$

against the alternative hypothesis,

$$H_a: \mathbf{H}(\theta) + \mathbb{C}(\theta) \neq 0 \tag{24}$$

where $\mathbf{H}(\theta)$ is the expected Hessian matrix and $\mathbb{C}(\theta)$ is the expected outer product of the score function (not to confuse with copula C), please see more detail and process in Huang and Prokhorov (2011).

2.3 Vine Copulas

Modeling dependencies in high dimension by the standard multivariate copulas are inflexible because they do not allow for different dependency structures between pairs of variables (Kramer and Schepsmeier, 2011). Vine copulas can cross over this restriction; vine copulas are a flexible tool for illustrating the multivariate copulas through graphical models. The multivariate copulas are constructed from a cascade of bivariate copulas (called pair-copulas), as a result of which we are able to select bivariate copulas from a wide range of families. The principles of vine copulas were propounded by Joe (1996) and extended by Bedford and Cooke (2001, 2002). For statistical inference techniques of two classes of C-vine and D-vine were described by Aas et al. (2009). Brechmann and Schepsmeier (2013) stated that a vine structure can be chosen manually or through expert knowledge, or be given by the data itself.

This research used the three dimensions of C-vine copula and D-vine copula to analyze the dependence between variables. The modeling of the C-vine copula and D-vine copula are as follows: first an appropriate C-vine tree structure and D-vine tree structure have to be specified; next, adequate copula families have to be selected and estimated (Brechmann and Schepsmeier, 2013).

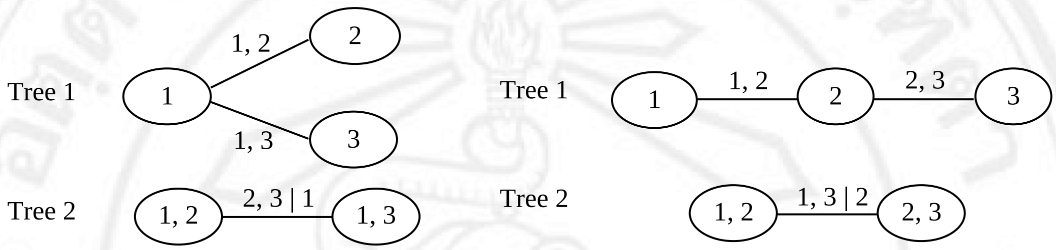

Figure 2.1 The structures of the C-vine (left) and the D-vine (right) copulas

Figure 2.1 presents the three dimensions, which are what we used in this paper. Let $X = (X_1, X_2, X_3) \sim F$ with marginal distribution functions F_1, F_2, F_3 and their density functions f_1, f_2, f_3, which have been proposed as follows (see Aas et al., 2009).

The density function of C-vine copula

$$f(x_1, x_2, x_3) = f(x_1) \cdot f(x_2) \cdot f(x_3) \cdot c_{1,2}\big(F_1(x_1), F_2(x_2)\big) \cdot c_{1,3}\big(F_1(x_1), F_3(x_3)\big) \cdot c_{2,3|1}\Big(F_{2|1}(x_2|x_1), F_{3|1}(x_3|x_1)\Big) \tag{25}$$

where $c_{1,2}$, $c_{1,3}$, and $c_{2,3|1}$ denote the densities of bivariate copulas $C_{1,2}$, $C_{1,3}$, and $C_{2,3|1}$, respectively. $F_{2|1}$ and $F_{3|1}$ are the marginal conditional distributions that can be derived from formula (27).

The density function of D-vine copula

$$f(x_1, x_2, x_3) = f(x_1) \cdot f(x_2) \cdot f(x_3) \cdot c_{1,2}\big(F_1(x_1), F_2(x_2)\big) \cdot c_{2,3}\big(F_2(x_2), F_3(x_3)\big) \tag{26}$$

$$\cdot\ c_{1,3|2}\left(F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2)\right)$$

where $c_{1,2}$, $c_{2,3}$, and $c_{1,3|2}$ denote the densities of bivariate copulas $C_{1,2}$, $C_{2,3}$, and $C_{1,3|2}$, respectively. $F_{1|2}$ and $F_{3|2}$ are the marginal conditional distributions that can be derived from formula (27).

The vine copulas involve marginal conditional distributions. The general form of a conditional distribution function is $F(x|v)$, given by

$$F(x|v) = \frac{\partial C_{xv_j|v_{-j}}(F(x|v_{-j}), F(v_j|v_{-j}))}{\partial F(v_j|v_{-j})} \tag{27}$$

where v denotes all the conditional variables and $C_{xv_j|v_{-j}}$ is a bivariate copula distribution function. For v is univariate, the marginal condition distribution, for example, $F_{1|2}$ can be presented as

$$F_{1|2}(x_1 \mid x_2) = \frac{\partial C_{12}(F_1(x_1), F_2(x_2))}{\partial F_2(x_2)} \tag{28}$$

In the R-package CDVines, the maximum likelihood was used to estimate the parameters of the copulas. The log-likelihood of the C-vine copula and D-vine copula with three dimensions in equation (25) and (26) can be written as

The log-likelihood of C-vine copula is

$$\sum_{t=1}^{T} \log(c_{1,2}(F_1(x_{1t}), F_2(x_{2t})) \cdot c_{1,3}(F_1(x_{1t}), F_3(x_{3t})) \cdot c_{2,3|1}\left(F_{2|1}(x_{2t}|x_{1t}), F_{3|1}(x_{3t}|x_{1t})\right)) \tag{29}$$

The log-likelihood of the D-vine copula is

$$\sum_{t=1}^{T} \log(c_{1,2}(F_1(x_{1t}), F_2(x_{2t})) \cdot c_{2,3}(F_2(x_{2t}), F_3(x_{3t})) \cdot c_{1,3|2}\left(F_{1|2}(x_{1t}|x_{2t}), F_{3|2}(x_{3t}|x_{2t})\right)) \tag{30}$$

2.4 Method to select vine tree structure

Brechmann and Schepsmeier (2013) stated that modeling of the C-vine copula and D-vine copula, an appropriate C-vine tree structure and D-vine tree structure have to be specified, and suggested that a vine structure can be chosen manually or through expert knowledge, or be given by the data itself.

For this research, the C-vine tree structures were chosen by knowledge, as presented in Chapter 3 and Chapter 4. For D-vine tree structures be given by the data itself. To construct a D-vine structure, we need to select the order of the variables in the first tree, as the first step. There are many approaches to ordering the sequences of variables, such as the empirical Kendall's tau, the Spearman's rho, the distance measure (Kramer and Schepsmeier, 2011), and the degree of freedom parameters of the Student's T copula (Aas et al., 2009). This research used the empirical Kendall's tau and the distance measure, as shown in Chapter 5.

2.4.1 Empirical Kendall's tau

The Kendall's rank correlation, or the empirical Kendall's tau ($\bar{\tau}_n$), as in equation (31), is used to measure the degree of dependence in each pair of the transformed standardized residuals of the data set. A high value of $\bar{\tau}_n$ means that there is high dependency between the two variables. The strongest dependencies, in terms of absolute empirical values of pairwise Kendall's tau, are used as the first pair in the first, and is subsequently followed by the next. The selection of the D-vine structure is based on the one that maximizes the sum of the corresponding absolute value of $\bar{\tau}_n$ in the first tree.

$$\bar{\tau}_n = \frac{P_n - Q_n}{\binom{n}{2}} = \frac{4}{n(n-1)} P_n - 1, \quad (31)$$

where P_n and Q_n are the number of concordant and discordant pairs, respectively. The two pairs, (X_i, Y_i) and (X_j, Y_j), can be said to be concordant when $(X_i - X_j)(Y_i - Y_j) > 0$, and discordant when $(X_i - X_j)(Y_i - Y_j) < 0$ (Genest and Favre, 2007).

2.4.2 Distance Measure

There are many approaches to measuring the distance between probability distributions or data set. This study used the approach to distance measure which is closely related to divergence measures based on the idea of information-theoretic entropy first presented by Shannon (1948). This divergence measure is symmetric and is referred to as the non-directional divergence measure. It qualifies as distance measure (Ullah, 1996). The formula can be written as given in equation (32).

$$I(f_1, f_2) = K(f_1, f_2) = \int (f_1 - f_2) \log \frac{f_1}{f_2} dy, \tag{32}$$

where $I(f_1, f_2)$ is the distance measure between the probability functions f_1 and f_2 of the standardized residual. A low value of $I(f_1, f_2)$ means that there is high association, or high affinity between f_1 and f_2. For ordering variables, the lowest $I(f_1, f_2)$ is used as the first pair in the first tree, and is subsequently followed by the next. The selection of the D-vine structure is based on the one that minimizes the sum of the corresponding absolute value of $I(f_1, f_2)$ in the first tree.

2.5 Time-varying Copulas

For the time-varying copula, this research used the ARMA(1,10) process that was presented by Patton (2006a).

2.5.1 Time-varying Gaussian copula

$$\rho_t = \tilde{\Lambda}\left(\omega_\rho + \beta_\rho \cdot \rho_{t-1} + \alpha \cdot \frac{1}{10} \sum_{j=1}^{10} \Phi^{-1}(u_{1,t-j}) \cdot \Phi^{-1}(u_{2,t-j})\right) \tag{33}$$

where Φ^{-1} is the inverse of the standard normal *c.d.f.*, $\tilde{\Lambda}(x) \equiv (1 - e^{-x})(1 + e^{-x})^{-1}$ is the modified logistic transformation, used to hold ρ_t in the range $(-1,1)$ at all times, ρ_{t-1} is a regressor to measure any persistence in the dependence parameter, and 10 is used to average the transformed variables $\Phi^{-1}(u_{1,t-j})$ and $\Phi^{-1}(u_{2,t-j})$ over the previous 10 lags (ARMA(1,10) process), to capture the variation in the dependence (Patton, 2006a).

2.5.2 Time-varying Student's T

$$\rho_t = \tilde{\Lambda}\left(\omega_\rho + \beta_\rho \cdot \rho_{t-1} + \alpha \cdot \frac{1}{10} \sum_{j=1}^{10} T^{-1}(u_{1,t-j}; \nu) \cdot T^{-1}(u_{2,t-j}; \nu)\right) \tag{34}$$

where $T^{-1}(\cdot, \nu)$ is the inverse *c.d.f.* of a student's T random variable with degree of freedom. $\tilde{\Lambda}(x) \equiv (1-e^{-x})(1+e^{-x})^{-1}$ is the modified logistic transformation as given by Patton (2006b).

2.5.3 Time-varying for non-Gaussian copula

Patton (2006a) presented the modeling of the tail dependence parameters, the symmetrized Joe-Clayton (SJC) copula, where the upper tail τ^U and the lower tail τ^L were related to the parameters of the copulas. Thus, in addition to specifying the tail dependence parameters over the sample, the equation that follows also specifies the parameters of the copula. Manner and Reznikova (2012) presented an equation which is based on Patton (2006a), as

$$\theta_t = \tilde{\Lambda}\left(\omega + \beta\Lambda^{-1}\theta_{t-1} + \alpha \cdot \frac{1}{10}\sum_{j=1}^{10}\left|u_{1,t-j} - u_{2,t-j}\right|\right) \tag{35}$$

where $\tilde{\Lambda}$(x) is a transformation function to always keep the parameters in their intervals. $\tilde{\Lambda}(x) \equiv (1+e^{-x})^{-1}$ is the logistic transformation, used to hold the tail dependence in the range (0,1), $\tilde{\Lambda}$(x) $\equiv e^x$ for the Clayton copula, and $\tilde{\Lambda}$(x) $\equiv e^x + 1$ for the Gumbel copula. $\beta\Lambda^{-1}\theta_{t-1}$ is an autoregressive term, and the last term on the right-hand side of the equation is the mean absolute difference between u_1 and u_2 over the previous 10 observations. This is a forcing variable, and under perfect positive dependence it will be close to zero, in case of perfect negative dependence it will equal to 0.5, and in case of independence it will equal to 0.33 (Patton, 2006a). In addition, we used $\tilde{\Lambda}$(x) $\equiv e^x + 1$ for the Joe copula, same as that for the Gumbel copula.

For the time-varying for rotated non-Gaussian copula, Patton (2002) suggested that the rotated copulas can be formed thus: If (U_1, U_2) are distributed as the copula *C*, then $(1-U_1, 1-U_2)$ will be distributed as the rotated *C* copula. Thus, with regard to estimating time-varying for the rotated non-Gaussian copula, we will transform the input arguments and use the same function as in equation (35).

2.6 Maximum Spanning Tree

The maximum spanning trees were used to select R-vine tree structures. In the R-package VineCopula (Schepsmeier et al., 2013), the maximum spanning trees with absolute values of pairwise Kendall's tau as weights, i.e., the following optimization problem is solved for each tree:

$$\max \sum_{edges\ e_{ij}\ in\ spanning\ tree} |\hat{\tau}_{ij}|$$

where $\hat{\tau}_{ij}$ is the pairwise empirical Kendall's tau and a spanning tree is a tree on all nodes. More details are available in Diestel (2010).