CHAPTER 4

System Modelling

4.1 Parameter Identification for Rotor Modelling

As the rotor of the test rig consists of two disks of relatively high mass compared with the shaft and with the shaft simply supports at both ends, a lumped two-mass model is considered representative. External excitation can occur at both disks and so a two-input model is considered appropriate for each transverse plane (horizontal and vertical). The control force (u) is applied to the disk at plane 1 (disk 1). Mechanical contact can occur at plane 2 (disk 2) and so the contact force is considered as another input to the rotor model (f) as indicated in figure 4.1. Displacements of the disk at plane 1 (z_1) and plane 2 (z_2) are the outputs for the rotor model. Therefore the rotor model has 2 inputs and 2 outputs and it can be written in transfer function form

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} G11 & G12 \\ G21 & G22 \end{bmatrix} \begin{bmatrix} u \\ f \end{bmatrix} \tag{4.1}$$

The mechanical parameters for dynamic modelling of the rotor part are identified using the peak-picking method [36]. This method requires the vibration response data in frequency domain for the rotor which is obtained by using impact tests. Figure 4.2 and figure 4.3 are the plot of magnitude and phase of frequency response for the test rig's rotor. There

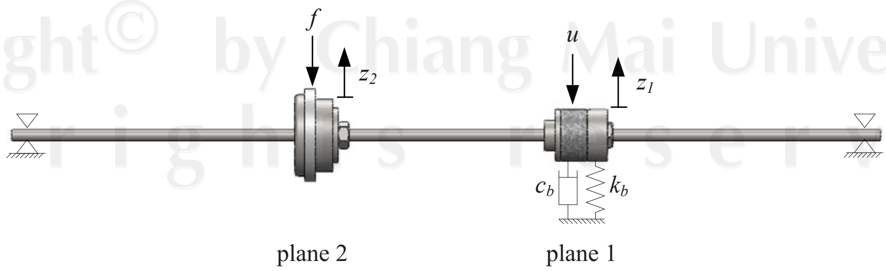


Figure 4.1: Rotor schematic for parameters identification

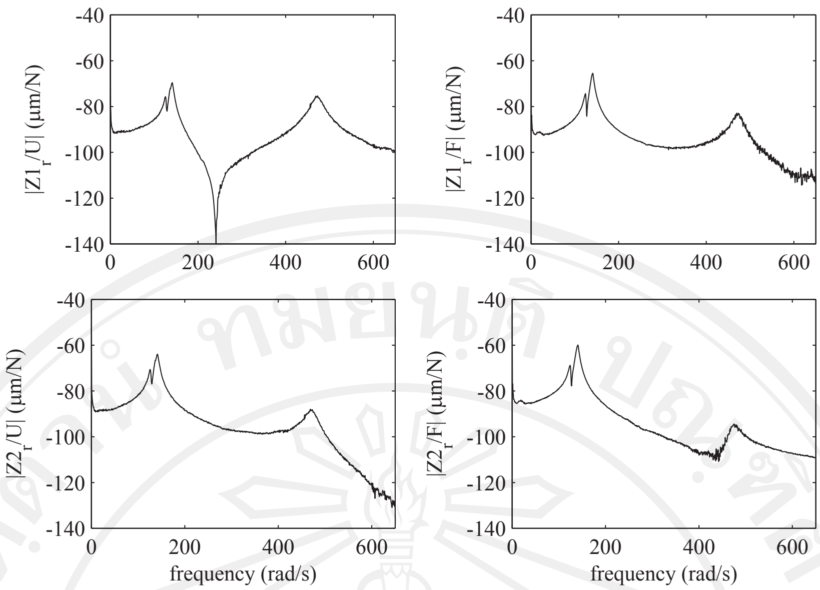


Figure 4.2: Magnitude of frequency response for test rig rotor (measured)

are 2 dominant peaks within the rotational frequency range (0-628 rad/s). Therefore, the frequency response function for each matrix element in (4.1) is considered as follows

$$\begin{aligned} G11(s) &= \frac{a11_1}{s^2+2\zeta_1\omega_{n1}s+\omega_{n1}^2} + \frac{a11_2}{s^2+2\zeta_1\omega_{n2}s+\omega_{n2}^2} \\ &= a11_1 g_1(s) + a11_2 g_2(s) \end{aligned} \tag{4.2}$$

where $g_1 = \frac{1}{s^2+2\zeta_1\omega_{n1}s+\omega_{n1}^2}$ and $g_2 = \frac{1}{s^2+2\zeta_1\omega_{n2}s+\omega_{n2}^2}$

To identify the parameters in (4.2), the real and imaginary part of each frequency response function are considered as shown in figure 4.4. The damping ratios of the system are calculated as

$$\begin{aligned} \zeta_1 &= \frac{\omega_B-\omega_A}{2\omega_{n1}} \\ \zeta_2 &= \frac{\omega_D-\omega_C}{2\omega_{n2}} \end{aligned}$$

The parameter $a11_1$ and $a11_2$ are identified by considering equivalence with a spring-mass-damper system, so that

$$\frac{a11_1/\omega_{n1}^2}{s^2/\omega_{n1}^2+2\zeta_1/\omega_{n1}s+1} + \frac{a11_2/\omega_{n2}^2}{s^2/\omega_{n2}^2+2\zeta_1/\omega_{n2}s+1} = \frac{1/k_1}{\frac{m_1}{k_1}s^2+\frac{c_1}{k_1}s+1} + \frac{1/k_2}{\frac{m_2}{k_2}s^2+\frac{c_2}{k_2}s+1}$$

The stiffness parameters are given by $k_1 = -1/2\zeta_1 D_1$ and $k_2 = -1/2\zeta_2 D_2$. Where D_1 and D_2 are the peak values of imaginary part as shown in figure 4.4. Parameters $a11_1$ and

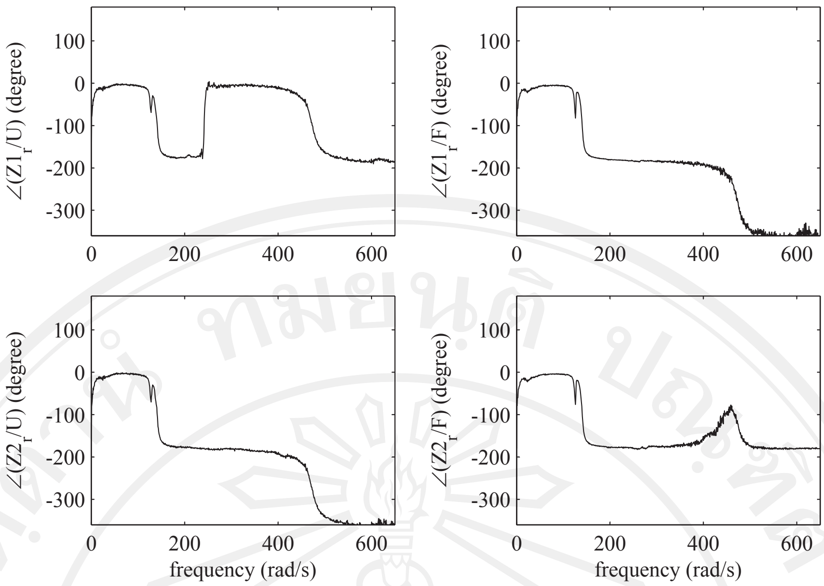


Figure 4.3: Phase of frequency response for test rig rotor (measured)

$a11_2$ can then be obtained as

$$\begin{aligned} a11_1 &= -2\zeta_1 D_1 \omega_{n1}^2 \\ a11_2 &= -2\zeta_1 D_2 \omega_{n2}^2 \end{aligned}$$

When coefficients a_1 and a_2 are obtained for all elements of the frequency response matrix in (4.1) then this can be expressed

$$\begin{bmatrix} G11(s) & G12(s) \\ G21(s) & G22(s) \end{bmatrix} = \begin{bmatrix} a11_1 & a12_1 \\ a21_1 & a22_1 \end{bmatrix} g_1(s) + \begin{bmatrix} a11_2 & a12_2 \\ a21_2 & a22_2 \end{bmatrix} g_2(s) \tag{4.3}$$

To convert to state-space model representation, the following diagonalized form is considered

$$\begin{bmatrix} G11(s) & G12(s) \\ G21(s) & G22(s) \end{bmatrix} = \begin{bmatrix} c1_1 & c1_2 \\ c2_1 & c2_2 \end{bmatrix} \begin{bmatrix} g_1(s) & 0 \\ 0 & g_2(s) \end{bmatrix} \begin{bmatrix} b1_1 & b2_1 \\ b1_2 & b2_2 \end{bmatrix} \tag{4.4}$$

Conversion from (4.3) to (4.4) requires that the columns of $\begin{bmatrix} a11 & a12 \\ a21 & a22 \end{bmatrix}_i$ are linearly dependent, however due to measurement sensors and other non-ideal effect this will not generally be true. Suppose $\mathbf{x}_i = \begin{bmatrix} a11 \\ a21 \end{bmatrix}_i$ and $\mathbf{y}_i = \begin{bmatrix} a12 \\ a22 \end{bmatrix}_i$ then the coefficients a_i

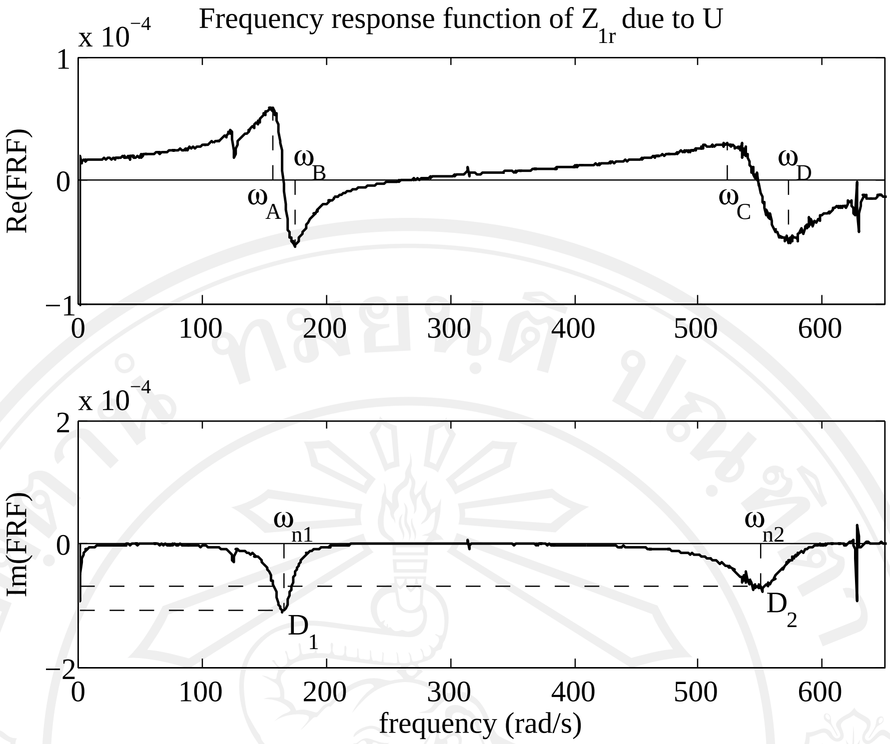


Figure 4.4: Real and Imaginary part of $G11(s)$ which is used to identify the rotor model parameters

must be approximated to

$$\begin{bmatrix} a11 & a12 \\ a21 & a22 \end{bmatrix}_i \approx \begin{bmatrix} c1 \\ c2 \end{bmatrix}_i \begin{bmatrix} b1 & b2 \end{bmatrix}_i \tag{4.5}$$

The aproximation errors in $\mathbf{x}_i$ and $\mathbf{y}_i$ are given by

$$\frac{\mathbf{e}_{\mathbf{x}i}}{|\mathbf{x}_i|} = \frac{\mathbf{x}_i - \mathbf{c}_i b1_i}{|\mathbf{x}_i|} \tag{4.6}$$

$$\frac{\mathbf{e}_{\mathbf{y}_i}}{|\mathbf{y}_i|} = \frac{\mathbf{y}_i - \mathbf{c}_i b2_i}{|\mathbf{y}_i|} \tag{4.7}$$

If $\mathbf{c}_i$ and $\mathbf{b}_i$ are taken as

$$\mathbf{c}_i = \frac{\mathbf{x}_i}{|\mathbf{x}_i|} + \text{sign}(\mathbf{x}_i^T \mathbf{y}_i) \frac{\mathbf{y}_i}{|\mathbf{y}_i|} \tag{4.8}$$

$$\mathbf{b}_i = \begin{bmatrix} \frac{|\mathbf{x}_i|}{2} & \text{sign}(\mathbf{x}_i^T \mathbf{y}_i) \frac{|\mathbf{y}_i|}{2} \end{bmatrix} \tag{4.9}$$

Then these solutions give an equal fractional error for each column (and thus each input)

$$\frac{\mathbf{e}_{\mathbf{x}i}}{|\mathbf{x}_i|} = \frac{1}{2} \left| \frac{\mathbf{x}_i}{|\mathbf{x}_i|} + \text{sign}(\mathbf{x}_i^T \mathbf{y}_i) \frac{\mathbf{y}_i}{|\mathbf{y}_i|} \right| \tag{4.10}$$

Table 4.1: Parameter values for rotor model identification

Parameter	Symbol	Value	Unit
Mass of disk 1	m_1	0.36	kg
Mass of disk 2	m_2	1.12	kg
First damping ratio	ζ_1	0.0023	
Second damping ratio	ζ_2	0.0024	
First natural frequency	ω_{n1}	139	rad/s
Second natural frequency	ω_{n2}	471	rad/s

$$\frac{\mathbf{e}_{\mathbf{y}_i}}{|\mathbf{y}_i|} = \frac{1}{2}\left|\frac{\mathbf{y}_i}{|\mathbf{y}_i|} + \text{sign}(\mathbf{x}_i^T\mathbf{y}_i)\frac{\mathbf{x}_i}{|\mathbf{x}_i|}\right| \tag{4.11}$$

Using (4.8) and (4.9), the system matrix $\mathbf{A}_r$, input matrices $\mathbf{B}_{ru}, \mathbf{B}_{rf}$ and output matrix $\mathbf{C}_r$ for the rotor model are

$$\begin{aligned}
\mathbf{A}_r &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\omega_{n1}^2 & -2\zeta_1\omega_{n1} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega_{n2}^2 & -2\zeta_2\omega_{n2} \end{bmatrix} \\
\mathbf{B}_{ru} &= \begin{bmatrix} 0 \\ b1_1 \\ 0 \\ b1_2 \end{bmatrix}, \quad \mathbf{B}_{rf} = \begin{bmatrix} 0 \\ b2_1 \\ 0 \\ b2_2 \end{bmatrix} \\
\mathbf{C}_r &= \begin{bmatrix} c1_1 & 0 & c1_2 & 0 \\ c2_1 & 0 & c2_2 & 0 \end{bmatrix}
\end{aligned}$$

As appropriate to the following state space form

$$\begin{aligned}
\dot{\mathbf{x}}_r &= \mathbf{A}_r\mathbf{x}_r + \mathbf{B}_{ru}u + \mathbf{B}_{rf}f \\
\mathbf{z}_r &= \mathbf{C}_r\mathbf{x}_r
\end{aligned} \tag{4.12}$$

Results for rotor model validation are shown in figure 4.5 for magnitude of frequency response and figure 4.6 for phase of frequency response. The frequency response function from the model are considered to give a good match with the experimental data over the running speed range of 0-100 Hz (0-628 rad/s). For the rotor structure, the value of the identified natural frequencies and damping ratios are shown in table 4.1.

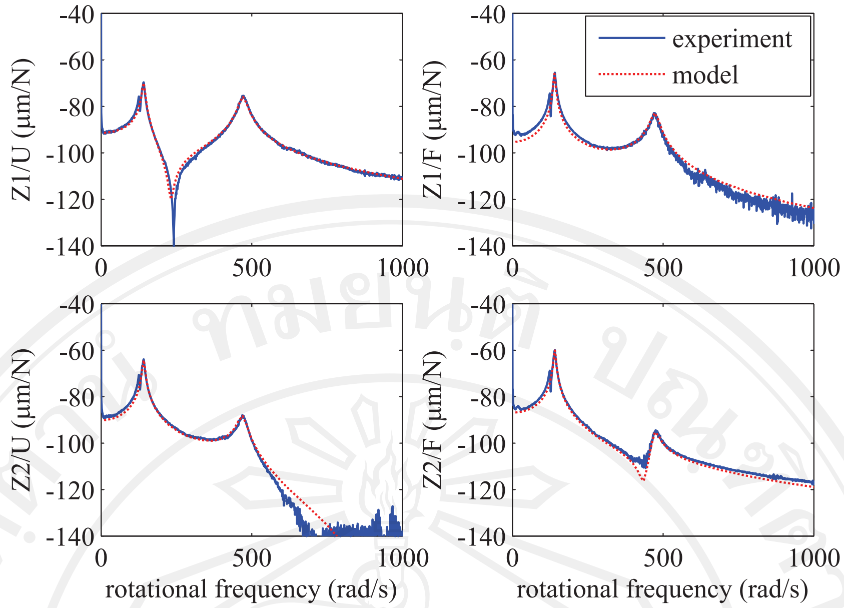


Figure 4.5: Rotor model validation for magnitude of frequency response function

4.2 Parameter Identification for Stator Modelling

The parameter identification method for the stator part will be described in this section. The stator part is considered as two identical spring-mass-damper systems for vertical and horizontal motion (considered uncoupled) as shown in figure 4.7. The equation of motion for the stator model is

$$\ddot{z}_s + 2\zeta_s\omega_{ns}\dot{z}_s + \omega_{ns}^2 z_s = -f \tag{4.13}$$

To perform an experimental identification, an impact force is applied to the stator in order to obtain the free vibration response of the stator. Figure 4.8 shows typical time response data measured by a non-contact probe.

The damped natural frequency ω_{ds} is calculated from the period of the vibration signal in figure 4.8. That is

$$\omega_{ds} = \frac{2\pi}{T} \tag{4.14}$$

where T is the period of the vibration signal. The natural frequency of the stator is obtained as $\omega_{ns} = \frac{\omega_{ds}}{\sqrt{1-\zeta_s^2}}$. The value for the damping ratio is obtained by fitting the decay rate of the vibration signal to $e^{-\zeta_s\omega_{ns}t}$, which is the dashed line in figure 4.8.

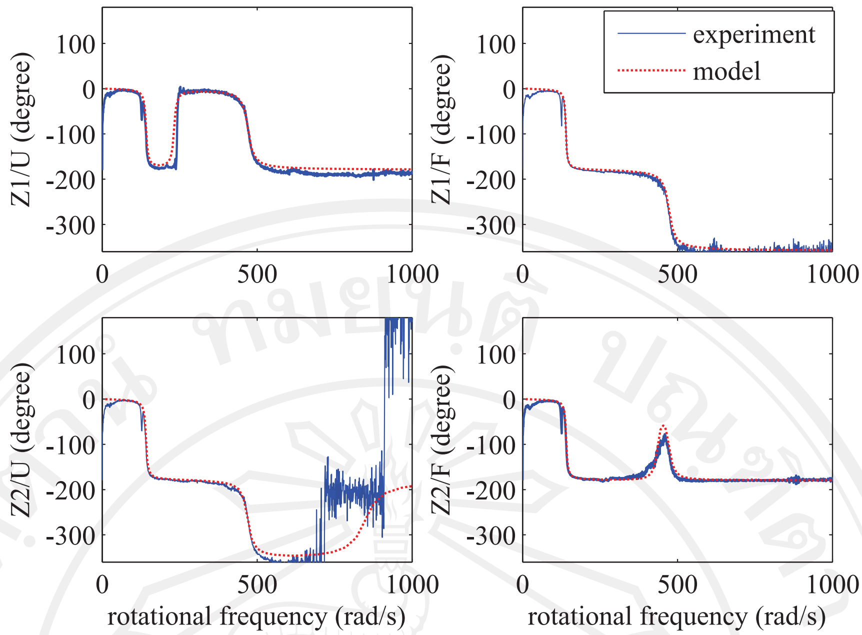


Figure 4.6: Rotor model validation for phase of frequency response function

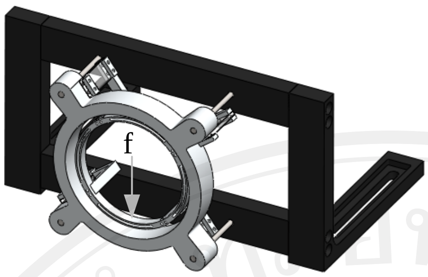
To obtained the state-space model of the stator, the system matrix $\mathbf{A}_s$, the input matrix $\mathbf{B}_s$ and the output matrix $\mathbf{C}_s$ are defined as

$$\begin{aligned}
\mathbf{A}_s &= \begin{bmatrix} 0 & 1 \\ -\omega_{ns}^2 & -2\zeta_s\omega_{ns} \end{bmatrix} \\
\mathbf{B}_s &= \begin{bmatrix} 0 \\ -1/m_s \end{bmatrix} \\
\mathbf{C}_s &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}$$

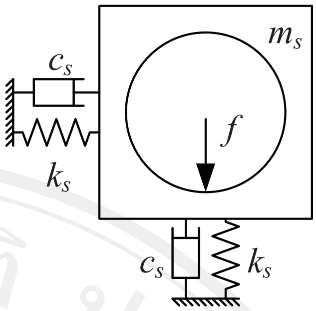
Where the model for the dynamics of the stator structure is considered in the form

$$\begin{aligned}
\dot{\mathbf{x}}_s &= \mathbf{A}_s\mathbf{x}_s + \mathbf{B}_s f \\
\mathbf{z}_s &= \mathbf{C}_s\mathbf{x}_s
\end{aligned} \tag{4.15}$$

Results of validation of the stator model, again assessed from free vibration response data, are shown in figure 4.9. For this setup of the stator structure, the value of the identified natural frequency and damping ratio are shown in table 4.2.



(a) Stator CAD model



(b) Equivalent spring-mass-damper system

Figure 4.7: Mechanical modelling of stator for parameter identification

Table 4.2: Parameter values for stator model identification

Parameter	Symbol	Value	Unit
Stator mass	m_s	0.46	kg
Damping ratio	ζ_s	0.020	
Natural frequency	ω_{ns}	754	rad/s

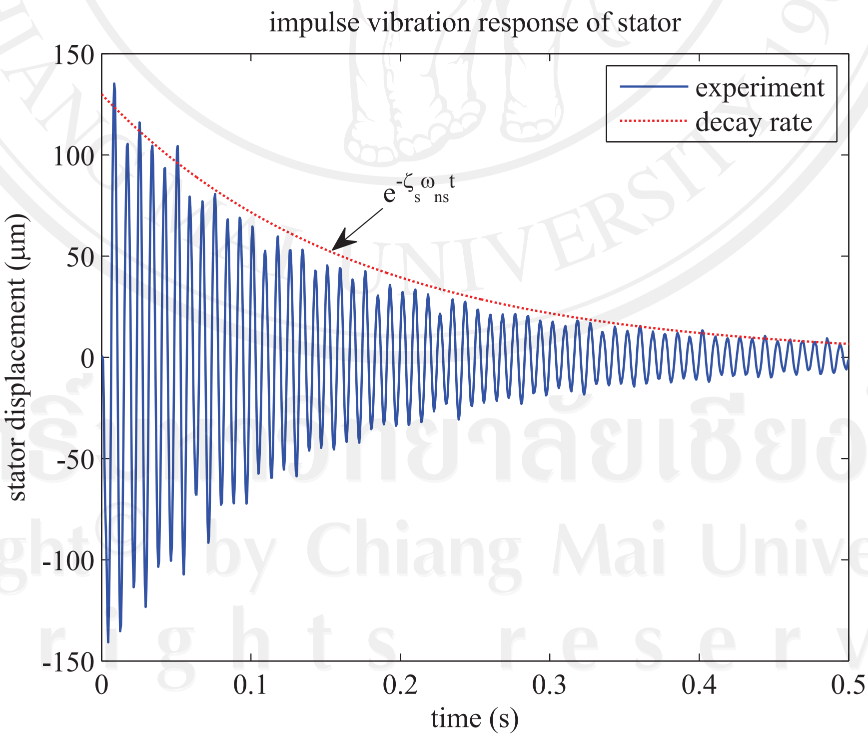


Figure 4.8: Impulse vibration response of stator measured during impact test

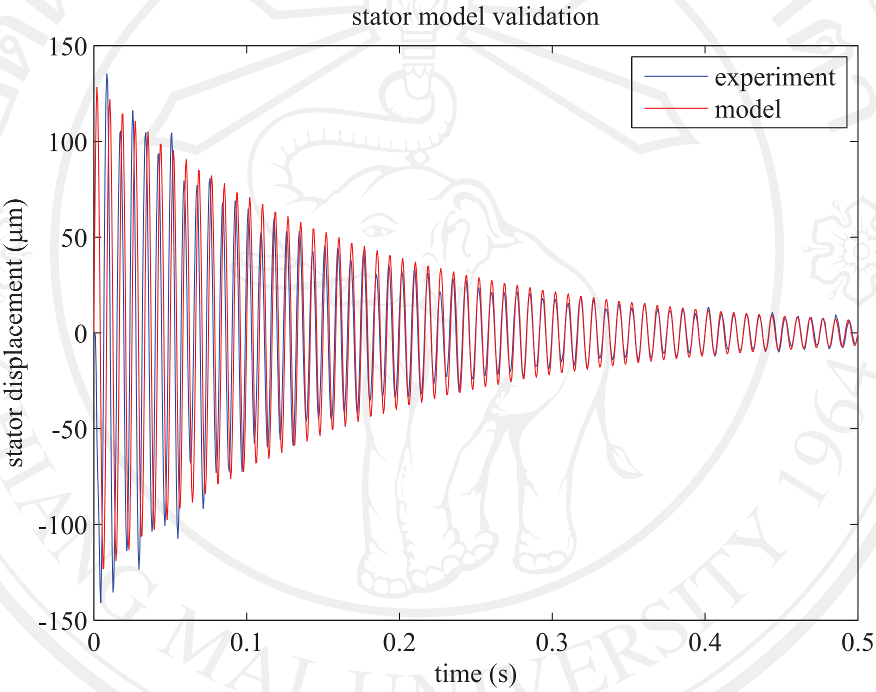


Figure 4.9: Stator model validation: impulse vibration response

4.3 Multi-Mode Rotor-Stator System Model

The combination of linear models for the rotor-stator system can be used in order to predict the stability region for possibility of amplitude jump and sustained rotor-stator interaction, as explained in chapter 2. Assuming that the system is radially isotropic then the mechanical properties in x and y direction are equivalent. The system matrices of the combined system then have the form

$$\mathbf{A}_{xx} = \mathbf{A}_{yy} = \begin{bmatrix} \mathbf{A}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_s \end{bmatrix} \tag{4.16}$$

The input matrices for control force input u at plane 1 is

$$\mathbf{B}_{u,xx} = \mathbf{B}_{u,yy} = \begin{bmatrix} \mathbf{B}_{ru} \\ \mathbf{0} \end{bmatrix} \tag{4.17}$$

The input matrices for contact force input f at plane 2 is

$$\mathbf{B}_{f,xx} = \mathbf{B}_{f,yy} = \begin{bmatrix} \mathbf{B}_{rf} \\ \mathbf{B}_s \end{bmatrix} \tag{4.18}$$

and the output matrices for relative displacement of rotor and stator in plane 2 are

$$\mathbf{C}_{xx} = \mathbf{C}_{yy} = \begin{bmatrix} \mathbf{C}2_r & -\mathbf{C}_s \end{bmatrix} \tag{4.19}$$

where $\mathbf{C}2_r = \begin{bmatrix} c2_1 & 0 & c2_2 & 0 \end{bmatrix}$.

Defining state variables $\mathbf{w}_x = \begin{bmatrix} \mathbf{x}_r \\ \mathbf{x}_s \end{bmatrix}$ and $\mathbf{w}_y = \begin{bmatrix} \mathbf{y}_r \\ \mathbf{y}_s \end{bmatrix}$, the complete state space model of the combined rotor-stator system is

$$\begin{aligned} \begin{bmatrix} \dot{\mathbf{w}}_x \\ \dot{\mathbf{w}}_y \end{bmatrix} &= \begin{bmatrix} \mathbf{A}_{xx} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{yy} \end{bmatrix} \begin{bmatrix} \mathbf{w}_x \\ \mathbf{w}_y \end{bmatrix} + \begin{bmatrix} \mathbf{B}_{u,xx} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_{u,yy} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix} \\ &\quad + \begin{bmatrix} \mathbf{B}_{f,xx} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_{f,yy} \end{bmatrix} \begin{bmatrix} f_x \\ f_y \end{bmatrix} \\ \begin{bmatrix} z_x \\ z_y \end{bmatrix} &= \begin{bmatrix} \mathbf{C}_{xx} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{yy} \end{bmatrix} \begin{bmatrix} \mathbf{w}_x \\ \mathbf{w}_y \end{bmatrix} \end{aligned} \tag{4.20}$$

Equation (4.20) can be written in compact form as

$$\begin{aligned} \dot{\mathbf{w}} &= \mathbf{A}\mathbf{w} + \mathbf{B}_u\mathbf{u} + \mathbf{B}_f\mathbf{f} \\ \mathbf{z} &= \mathbf{C}\mathbf{w} \end{aligned} \tag{4.21}$$

The linear dynamics involved in the rotor-stator interaction model can be described using (4.21). The remaining component of the complete interaction model must describe how the interaction/contact force (**f**) depends on the relative motion of the rotor and stator within the contact plane (**z**). This nonlinear interaction model will be explained in the following section.

4.4 Nonlinear Interaction Model

Here we focus on the case where there is a single plane where a finite radial clearance exists between the stator and rotor in their equilibrium state. The model for this nonlinear interaction can be written as

$$\mathbf{f} = \beta(\mathbf{z}) \tag{4.22}$$

For this relation, it is assumed that the nonlinear interaction force arises at an interface/ component having elastic properties and so **f** depends on local relative displacement (**z**) through a static non-linear mapping $\beta : \mathbb{R}^2 \to \mathbb{R}^2$.

To further specify β, it will be assumed that, within the plane of interaction, both rotor and surround are circular in cross section. A vector **q** is defined for the rotor radial deflection

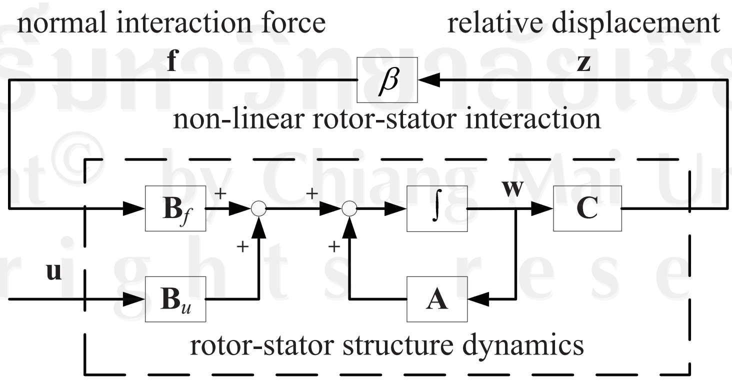


Figure 4.10: Block diagram of rotor system with nonlinear interaction model

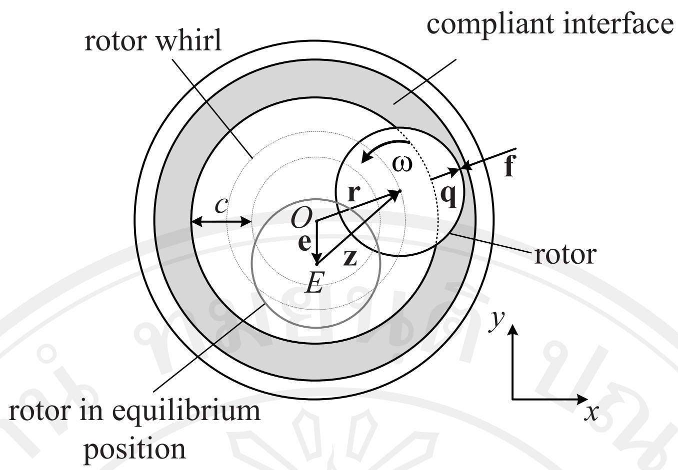


Figure 4.11: Schematic of rotor-stator interaction in contact plane

(penetration) beyond the clearance as show in figure 4.11.

$$\mathbf{q} = \begin{cases} 0, & \|\mathbf{r}\| \leq c \\ (1 - c/\|\mathbf{r}\|)\mathbf{r}, & \|\mathbf{r}\| > c \end{cases} \tag{4.23}$$

where $\mathbf{r} = \mathbf{z} + \mathbf{e}$, with $\mathbf{e}$ being the position of the rotor equilibrium point relative to the clearance circle center. Adopting a general compliant contact model, the interaction force may be defined in the form

$$\mathbf{f} = -\kappa(\|\mathbf{q}\|)\mathbf{q} \tag{4.24}$$

4.5 Summary

The mechanical parameters of the rotor and stator part were identified based on it's linear vibration behaviour. The method used to identify the rotor's parameters was the "peak-picking method" and the stator's parameters were obtained from measurements of the free vibration of stator due to an impact force. The state space model of the rotor and stator were obtained and combined. A general nonlinear interaction model was defined. The vibration stability prediction of rotor system presented in the next chapter is based on the rotor-stator model that has been defined in this chapter.