CHAPTER 5

Prediction of Vibrational Stability

Two stability prediction approaches, which are Nyquist and Lyapunov stability based approach, are presented in this chapter. The Nyquist based approach was previously shown in Chapter 2. However, same aspects of its application with the multi-mode rotordynamic model developed in Chapter 4 will be explained in this chapter. Both approaches can be based on a nonlinear rotor system model of the form presented in Chapter 4.

5.1 Stability Condition for Forward Whirl using Frequency Response Function

The basic theory for predicting vibration response mode involving rotor-stator interaction using frequency response functions was described in Chapter 2. In this section, the theory will be applied to the test rig model developed in Chapter 4.

When the system is rotationally symmetry (radial isotropic), the possibility of an alternative vibration response involving coupled forward whirl of rotor and stator can be calculated directly from frequency response data for the rotor-stator structure as explained in section 2.2. For an initial linear response with orbit radius (orbit-clearance ratio, $\rho = \frac{\|z\|}{c}$) $\rho < 1$, the magnitude of rotor-stator interaction force for an alternative vibration response involving constant rub is given by [9, 37]

$$\frac{\|\mathbf{f}\|}{c} = \frac{-\cos\angle H(\omega) + \sqrt{\rho^2 - \sin^2\angle H(\omega)}}{|H(\omega)|} \tag{5.1}$$

This equation is equivalent to (2.9). For a multi-mode rotordynamic system, the dynamic compliance of the rotor-stator structure $H(\omega)$ will be calculated from a state space model here, but is equivalent to $G(\omega) + 1/\kappa$, as in (2.8). Considering the linear submodel for the

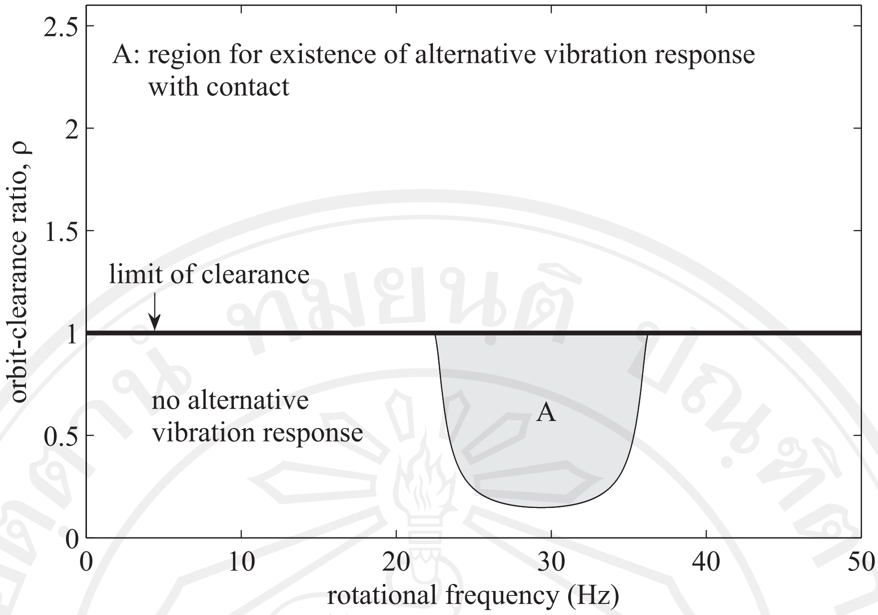

Figure 5.1: Whirl mode map for test rig showing region for possibility of an alternative vibration response involving rotor-stator interaction

complete test rig described in section 4.3 then

$$H(\omega) = \left[\begin{array}{cc} 1 & 0 \end{array}\right] \mathbf{T}(j\omega) \left[\begin{array}{cc} 1 & -j \end{array}\right]^T + k^{-1} \tag{5.2}$$

where $\mathbf{T}(j\omega) = \mathbf{C}(j\omega\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}_f$ and $\mathbf{A}$, $\mathbf{B}_f$ and $\mathbf{C}$ are defined according to (4.21).

An alternative vibration response involving rotor-stator interaction exists only if $\|\mathbf{f}\|$ is real and positive. The conditions for existence of the alternative vibration response are

$$\rho \geq \sin \angle H(\omega) \tag{5.3}$$

and

$$\cos \angle H(\omega) < 0 \tag{5.4}$$

As seen in section 2.2, for a typical rotor system, equation 5.4 holds for a finite speed interval above each critical speed. It is useful to consider these two conditions together to plot a "whirl mode map" which shows operating regions where an alternative vibration response involving rotor-stator interaction is possible [37]. The whirl mode map for the test rig model is shown in figure 5.1

5.2 Stability Condition for Backward Whirl using Frequency Response Function

In this section, the theories presented in subsection 2.3.2 are applied to the test rig model in order to investigate the backward whirl solution. The backward whirl solutions are obtained with assumptions

1) The external disturbance is negligible compared with the contact force ($\mathbf{Z}_0 = 0$).
2) The backward whirl solution is a circular orbit ($\mathbf{z}_2 = \mathbf{Z}_2 e^{-i\Omega t}$).
3) The stator is rigidly fixed ($k_s \to \infty$).

The inequality (2.21) can be used to predict the region for possibility of backward whirl solution. This inequality shows that the condition for existence of backward whirl does not depend on frequency response function, as in the forward whirl case, but it is dependent on the friction coefficient between rotor and stator surfaces and the damping ratio of the rotor. For the rotor test rig, the equivalent damping ratio is 0.0227. The region for possibility of backward whirl for the rotor test rig is shown in figure 5.2.

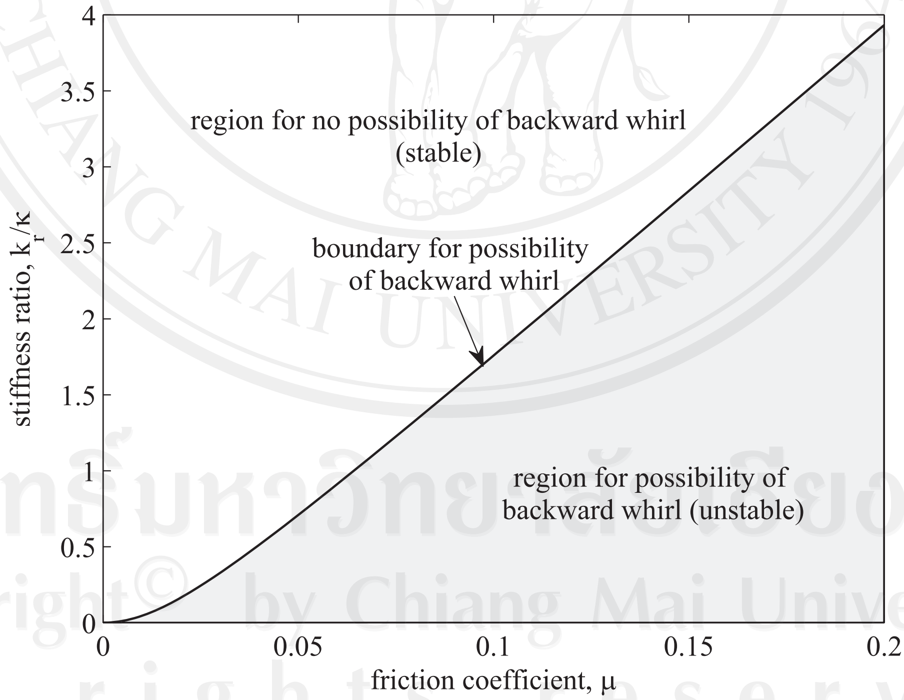

Figure 5.2: Boundary for possibility of backward whirl solution of the test rig predicted by using equation (2.21).

5.3 Stability Conditions for Forward Whirl using Lyapunov Based Approach

This section will consider the application of Lyapunov stability theory to the non-linear rotor vibration behaviours which are the focus of this thesis.

5.3.1 Lyapunov stability theory

Stability theory plays an important role in system and control theory. Stability of an equilibrium points is usually characterized in the sense of Lyapunov. An equilibrium point is stable if all solutions starting at nearby points stay nearby. Otherwise, it is unstable. It is asymptotically stable if all solutions starting at nearby points not only stay nearby, but also tend to the equilibrium point as time approaches infinity. The basic theorems of Lyapunov's method for autonomous systems are as follows [38].

Consider the autonomous system

$$\dot{x} = f(x) \tag{5.5}$$

where $f : \mathbb{D} \rightarrow \mathbb{R}^n$ is a local Lipschitz map from a domain $\mathbb{D} \subset \mathbb{R}^n$ into $\mathbb{R}^n$. Suppose $\bar{x} \in \mathbb{D}$ is an equilibrium point of (5.5) that is $f(\bar{x}) = 0$. Without loss of generality, all definitions and theorems for the case when the equilibrium point is at the origin of $\mathbb{R}^n$, that is $\bar{x} = 0$, because any equilibrium point can be shifted to the origin via a change of variables. Suppose $\bar{x} \neq 0$ and consider a change of variables $y = x - \bar{x}$. The derivative of y is given by

$$\dot{y} = \dot{x} = f(x) = f(y + \bar{x}) = g(y), \quad \text{where } g(0) = 0 \tag{5.6}$$

The system with the new variable y has equilibrium at the origin.

Definition 5.1 The equilibrium point $x = 0$ of (5.5) is

- stable if, for each $\epsilon > 0$, there is $\delta_\epsilon = \delta_\epsilon(\epsilon)$ such that

$$\|x(0)\| < \delta_\epsilon \Rightarrow \|x(t)\| < \epsilon, \quad \forall t \geq 0$$

- unstable if it is not stable.

- asymptotically stable if it is stable and δ_ϵ can be chosen such that

$$\|x(0)\| < \delta_\epsilon \Rightarrow \lim_{x \to \infty} x(t) = 0$$

Theorem 5.1 Let $x = 0$ be an equilibrium point for (5.5) and $\mathbb{D} \subset \mathbb{R}^n$ be a domain containing $x = 0$. Let $V : \mathbb{D} \to \mathbb{R}^n$ be a continuously differentiable function such that

$$V(0) = 0 \text{ and } V(x) > 0 \text{ in } \mathbb{D} - \{0\}$$

$$\dot{V}(x) \leq 0 \text{ in } \mathbb{D}$$

then $x = 0$ is stable. Moreover, if

$$\dot{V}(x) < 0 \text{ in } \mathbb{D} - \{0\}$$

then $x = 0$ is asymptotically stable.

Theorem 5.2 (Barbashin-Krasovskii theorem) Let $x = 0$ be an equilibrium point for (5.5). Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function such that

$$V(0) = 0 \text{ and } V(x) > 0, \quad \forall x \neq 0$$

$$\|x\| \to \infty \Rightarrow V(x) \to \infty$$

$$\dot{V}(x) < 0, \quad \forall x \neq 0$$

then $x = 0$ is globally asymptotically stable.

5.3.2 Linear matrix inequalities

A linear matrix inequality (LMI) is any constraint of the form

$$\mathbf{A}(\mathbf{x}) = \mathbf{A}_0 + \mathbf{A}_1 x_1 + \mathbf{A}_2 x_2 + ... + \mathbf{A}_N x_N < 0 \tag{5.7}$$

$\mathbf{A}_0, ..., \mathbf{A}_N$ are given symmetric matrix and $\mathbf{x} = [x_1, ..., x_N]$ is a vector of unknown scalars which are variables to be optimized. Equation (5.7) states that $\mathbf{A}(\mathbf{x})$ is negative definite. It means the largest eigenvalue of $\mathbf{A}(\mathbf{x})$ is negative.

Convexity has an important consequence in the solving LMI problem. Generally, LMI has no analytical solution. However, if it is a convex function, existence of the numerical solution is guarantee. The LMI (5.7) is a convex function on x since $\mathbf{A}(y) < 0$ and $\mathbf{A}(z) < 0$ imply that $\mathbf{A}(\frac{y+z}{2}) < 0$. In most control application, LMIs do not have canonical form (5.7), but rather are in the form

$$\mathbf{L}(x_1, ..., x_n) < \mathbf{R}(x_1, ..., x_n) \Longleftrightarrow \mathbf{L}(x_1, ..., x_n) - \mathbf{R}(x_1, ..., x_n) < 0 \tag{5.8}$$

where $\mathbf{L}(.)$ and $\mathbf{R}(.)$ are affine functions of matrix variables $x_1, ..., x_n$.

Consider a dynamic system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{5.9}$$

If a positive symmetric matrix **P** ($\mathbf{P} > 0$ and $\mathbf{P} = \mathbf{P}^T$) is given, a Lyapunov function can be written as

$$V(\mathbf{x}) = \mathbf{x}^T\mathbf{P}\mathbf{x} \tag{5.10}$$

and a derivative of the Lyapunov function is

$$\dot{V}(\mathbf{x}) = \mathbf{x}^T(\mathbf{P}\mathbf{A} + \mathbf{A}^T\mathbf{P})\mathbf{x} \tag{5.11}$$

The theorem 5.2 states that the system is globally asymptotically stable when $\dot{V}(\mathbf{x} < 0$ for all $\mathbf{x} \neq 0$, therefore the system (5.9) is globally asymptotically stable when

$$\mathbf{P}\mathbf{A} + \mathbf{A}^T\mathbf{P} < 0 \tag{5.12}$$

Equation (5.12) is called the "Lyapunov inequality", which is a standard LMI problem. The feasible solution for **P** must be obtained in order to prove that the system (5.12) is globally asymptotically stable. The stability of contact-free orbit for the rotor-stator system presented in Chapter 4 can also be formulated as LMI problems shown in the following subsections.

5.3.3 Stability of a fixed equilibrium

Consider the state space model of the test rig without control input ($\mathbf{u} = 0$)

$$\begin{aligned} \dot{\mathbf{w}} &= \mathbf{A}\mathbf{w} + \mathbf{B}_f\mathbf{f} \\ \mathbf{z} &= \mathbf{C}\mathbf{w}, \quad \mathbf{f} = \beta(\mathbf{z}) \end{aligned} \tag{5.13}$$

This system is *globally asymptotically stable* at an equilibrium $\mathbf{w} = 0$, if there is the Lyapunov function $V(\mathbf{w}) > 0$ that satisfies $\dot{V}(\mathbf{w}) < 0 \quad \forall \mathbf{w} \neq 0$. A Lyapunov function will be considered for the system that has a quadratic term involving the system states and a 2-D Lur'e-type term associated with the elastic energy storage function for the non-linear rotor-stator interaction. A small change in the rotor position relative to the stator in the plane of contact $d\mathbf{z}$ is considered. The change in stored elastic energy is associated with the action of the contact force $\mathbf{f}$ moving along a path in normal contact direction and the displacement $d\mathbf{z}$ deflecting the surrounding (figure 4.11). The work done on the surround is

$$dW = -\mathbf{f}^T d\mathbf{z} \tag{5.14}$$

If $\mathbf{P} = \mathbf{P}^T \in \mathbb{R}^{n\times n}$ is positive definite and ν is positive scalar. A Lyapunov function candidate involving free parameters $\mathbf{P}$ and ν may be defined as

$$V(\mathbf{w}) = \mathbf{w}^T\mathbf{P}\mathbf{w} + 2\nu \int_0^{\mathbf{z}} dW \tag{5.15}$$

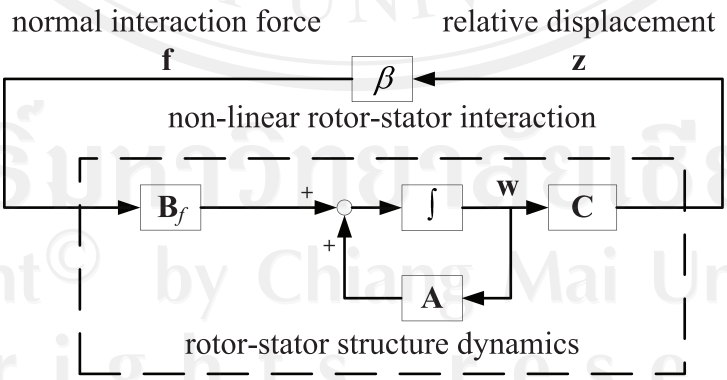

Figure 5.3: Schematic diagram of rotordynamic system

Noting that $\dot{W} = -\mathbf{f}^T\dot{\mathbf{z}}$, then

$$\dot{V}(\mathbf{w}) = \dot{\mathbf{w}}^T\mathbf{P}\mathbf{w} + \mathbf{w}^T\mathbf{P}\dot{\mathbf{w}} - 2\nu\mathbf{f}^T\dot{\mathbf{z}} \tag{5.16}$$

With the system (5.13), $\dot{\mathbf{z}} = \mathbf{C}(\mathbf{A}\mathbf{w} + \mathbf{B}_f\mathbf{f})$ and $\mathbf{C}\mathbf{B}_f = 0$, so (5.16) becomes

$$\dot{V}(\mathbf{w}) = \mathbf{w}^T(\mathbf{P}\mathbf{A} + \mathbf{A}^T\mathbf{P})\mathbf{w} + 2\mathbf{f}^T(\mathbf{B}^T\mathbf{P} - \nu\mathbf{C}\mathbf{A})\mathbf{w} \tag{5.17}$$

The stability condition for the system $\dot{V}(\mathbf{w}) < 0$ is

$$\mathbf{w}^T(\mathbf{P}\mathbf{A} + \mathbf{A}^T\mathbf{P})\mathbf{w} + 2\mathbf{f}^T(\mathbf{B}^T\mathbf{P} - \nu\mathbf{C}\mathbf{A})\mathbf{w} < 0 \tag{5.18}$$

This condition must be satisfied for all possible state trajectories with $\mathbf{f} \neq 0$ given by the nonlinear interaction model (4.23) and (4.24). Consider the geometry in figure 5.4, the deflection (penetration) of the surround $\mathbf{q}$ will be oriented at an angle α to the rotor displacement from the equilibrium position E. Provided the equilibrium position E is within the clearance circle ($\|\mathbf{e}\| < c$) then $|\mathbf{q}| < |\mathbf{z}|\cos\alpha$ and so the following orientation constraint will always hold.

$$\mathbf{q}^T\mathbf{q} < \mathbf{q}^T\mathbf{z} \tag{5.19}$$

From the relation of the interaction force and the radial penetration in (4.24), substitute

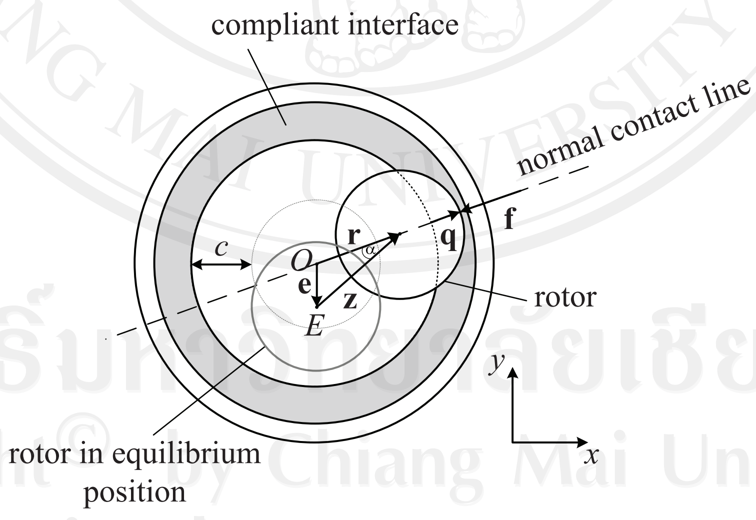

Figure 5.4: The geometry for rotor-stator interaction occurring at contact plane

$\mathbf{q} = -\kappa^{-1}\mathbf{f}$ as from (4.24) and $\mathbf{z} = \mathbf{C}\mathbf{w}$ in (5.19) and then the constraint will be

$$\begin{aligned} \kappa^{-1}\mathbf{f}^T\mathbf{f} &< -\mathbf{f}^T\mathbf{C}\mathbf{w} \\ \mathbf{f}^T\mathbf{C}\mathbf{w} + \kappa^{-1}\mathbf{f}^T\mathbf{f} &< 0 \end{aligned} \tag{5.20}$$

As this constraint is always true, the stability conditions (5.18) has to be satisfied only when the constraint (5.20) holds. To establish sufficient condition for stability, inequalities (5.18) and (5.20) can be combined into a single inequality via the '$\mathcal{S}$-procedure' [39]. A sufficient condition for stability is that a positive scalar σ exists such that

$$\mathbf{w}^T(\mathbf{P}\mathbf{A} + \mathbf{A}^T\mathbf{P})\mathbf{w} + 2\mathbf{f}^T(\mathbf{B}_f^T\mathbf{P} - \nu\mathbf{C}\mathbf{A} - \sigma\mathbf{C})\mathbf{w} - 2\sigma\kappa^{-1}\mathbf{f}^T\mathbf{f} < 0 \tag{5.21}$$

Equation (5.21) can be written more concisely as

$$\begin{bmatrix} \mathbf{w} \\ \mathbf{f} \end{bmatrix}^T \begin{bmatrix} \mathbf{P}\mathbf{A} + \mathbf{A}^T\mathbf{P} & \mathbf{P}\mathbf{B}_f - \nu\mathbf{A}^T\mathbf{C}^T - \sigma\mathbf{C}^T \\ \mathbf{B}_f^T\mathbf{P} - \nu\mathbf{C}\mathbf{A} - \sigma\mathbf{C} & -2\sigma k^{-1}\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \mathbf{f} \end{bmatrix} < 0, \qquad \begin{bmatrix} \mathbf{w} \\ \mathbf{f} \end{bmatrix} \neq 0$$

This holds only if the matrix inside the quadratic form is negative definite, which gives the following condition for stability of the fixed point $\mathbf{w} = 0$

A sufficient condition for globally stability of a fixed equilibrium: If there exist $\mathbf{P} = \mathbf{P}^T > 0$, $\nu \geq 0$ and $\sigma \geq 0$ such that

$$\begin{bmatrix} \mathbf{P}\mathbf{A} + \mathbf{A}^T\mathbf{P} & \mathbf{P}\mathbf{B}_f - \nu\mathbf{A}^T\mathbf{C}^T - \sigma\mathbf{C}^T \\ \mathbf{B}_f^T\mathbf{P} - \nu\mathbf{C}\mathbf{A} - \sigma\mathbf{C} & -2\sigma k^{-1}\mathbf{I} \end{bmatrix} < 0 \tag{5.22}$$

then the system (5.13) is *globally asymptotically stable* at the equilibrium point $\mathbf{w} = 0$.

5.3.4 Stability of a steady orbit

Suppose that there is a disturbance $\mathbf{d}$ acting on the rotor (nominally unbalance force). The rotor vibration response $\mathbf{w}(t) = \mathbf{w}_d(t)$ arises due to the disturbance and which is contact free ($\mathbf{f} = 0$). Defining $\mathbf{W}(t) = \mathbf{w}(t) - \mathbf{w}_d(t)$, the dynamic system is given by

$$\begin{aligned} \dot{\mathbf{W}} &= \mathbf{A}\mathbf{W} + \mathbf{B}_f\mathbf{f} \\ \mathbf{z} &= \mathbf{C}\mathbf{W} + \mathbf{z}_d \quad \mathbf{f} = \beta(\mathbf{z}) \end{aligned} \tag{5.23}$$

where $\mathbf{z}_d = \mathbf{C}\mathbf{w}_d$

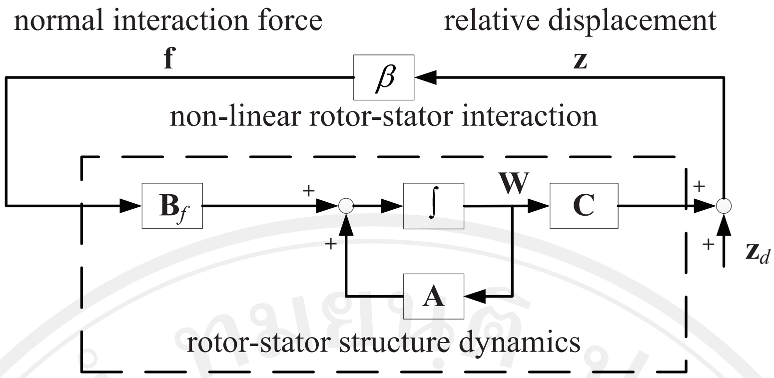

Figure 5.5: Schematic diagram of rotordynamic system with steady orbit due to steady initial disturbance

Although the system (5.23) is a time-varying because $\mathbf{z}(t)$ depends on $\mathbf{z}_d(t)$, Lyapunov's direct method can still be readily applied [38]. To establish stability of $\mathbf{w}_d(t)$, meaning that $\mathbf{W}(t) \rightarrow 0$ when $t \rightarrow \infty$ for all initial $\mathbf{W}(0) = \mathbf{W}_0$, a Lyapunov function is defined in the form

$$V(t, \mathbf{W}) = \mathbf{W}^T\mathbf{P}\mathbf{W} - 2\nu \int_{\mathbf{z}_d}^{\mathbf{z}} \mathbf{f}\, d\mathbf{z} \tag{5.24}$$

Hence

$$\dot{V}(t, \mathbf{W}) = \dot{\mathbf{W}}^T\mathbf{P}\mathbf{W} + \mathbf{W}^T\mathbf{P}\dot{\mathbf{W}} - 2\nu\mathbf{f}^T\dot{\mathbf{z}} \tag{5.25}$$

The variable $\dot{\mathbf{z}}$ depends on both $\mathbf{W}$ and $\mathbf{z}_d(t)$. Under the assumption that the response in the contact plane involves a circular orbit, $\mathbf{z}_d(t)$ may be written in form $\mathbf{T}_\omega(t)\mathbf{z}_0$ where $\mathbf{z}_0$ is a static vector and $\mathbf{T}_\omega(t)$ is a rotation matrix, which is defined by

$$\mathbf{T}_\omega(t) = \begin{bmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{bmatrix}$$

Noting that $\mathbf{T}_\omega^T\mathbf{T}_\omega = \mathbf{I}$ and $\dot{\mathbf{T}}_\omega = \omega\Pi^T\mathbf{T}_\omega$ with $\Pi = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Defining transformed variables $\bar{\mathbf{z}} = \mathbf{T}_\omega^T\mathbf{z}$ and $\bar{\mathbf{f}} = \mathbf{T}_\omega^T\mathbf{f}$. For the form of β defined by (4.23) and (4.24), it follows that

$$\bar{\mathbf{f}}^T\dot{\bar{\mathbf{z}}} = \mathbf{f}^T\dot{\mathbf{z}} + \omega\mathbf{f}^T\Pi\mathbf{z} \tag{5.26}$$

Under assumption $\mathbf{e} = 0$, the second term is always zero. The variable $\bar{\mathbf{z}}$ may be also written as $\bar{\mathbf{z}} = \mathbf{T}_\omega^T \mathbf{C}\mathbf{W} + \mathbf{z}_0$ and $\dot{\bar{\mathbf{z}}} = \mathbf{T}_\omega^T \mathbf{C}\dot{\mathbf{W}} + \omega \mathbf{T}_\omega^T \Pi \mathbf{C}\mathbf{W}$. Thus

$$\begin{aligned} \mathbf{f}^T \dot{\mathbf{z}} &= \bar{\mathbf{f}}^T \dot{\bar{\mathbf{z}}} \\ &= \mathbf{f}^T \mathbf{T}_\omega \dot{\bar{\mathbf{z}}} \\ &= \mathbf{f}^T \mathbf{T}_\omega \left(\mathbf{T}_\omega^T \mathbf{C}\dot{\mathbf{W}} + \omega \mathbf{T}_\omega^T \Pi \mathbf{C}\mathbf{W} \right) \end{aligned} \tag{5.27}$$

With $\dot{\mathbf{W}} = \mathbf{A}\mathbf{W} + \mathbf{B}_f \mathbf{f}$ and $\mathbf{C}\mathbf{B}_f = 0$, then (5.27) becomes

$$\mathbf{f}^T \dot{\mathbf{z}} = \mathbf{f}^T \left(\mathbf{C}\mathbf{A} + \omega \Pi \mathbf{C} \right) \mathbf{W} \tag{5.28}$$

Similar to the sufficient condition for global asymptotic stability of a fixed equilibrium, the stability condition $\dot{V}(t, \mathbf{W}) < 0$ can be combined with constraint (5.20) to give the condition

$$\mathbf{W}^T (\mathbf{P}\mathbf{A} + \mathbf{A}^T \mathbf{P})\mathbf{W} + 2\mathbf{f}^T \left[\mathbf{B}_f^T \mathbf{P} - \nu \left(\mathbf{C}\mathbf{A} + \omega \Pi \mathbf{C} \right) - 2\sigma \mathbf{C} \right] \mathbf{W} - 2\sigma \kappa^{-1} \mathbf{f}^T \mathbf{f} < 0 \tag{5.29}$$

It can be written in form

$$\begin{bmatrix} \mathbf{W} \\ \mathbf{f} \end{bmatrix}^T \bar{\mathbb{N}} \begin{bmatrix} \mathbf{W} \\ \mathbf{f} \end{bmatrix} < 0$$

where

$$\bar{\mathbb{N}} = \begin{bmatrix} \mathbf{P}\mathbf{A} + \mathbf{A}^T \mathbf{P} & \mathbf{P}\mathbf{B}_f - \nu \left(\mathbf{A}^T \mathbf{C}^T + \omega \mathbf{C}^T \Pi^T \right) - \sigma \mathbf{C}^T \\ \mathbf{B}_f^T \mathbf{P} - \nu \left(\mathbf{C}\mathbf{A} + \omega \Pi \mathbf{C} \right) - \sigma \mathbf{C} & -2\sigma k^{-1} \mathbf{I} \end{bmatrix}$$

which provides the following stability condition

A sufficient condition for global asymptotic stability of a steady orbit: If there exist $\mathbf{P} = \mathbf{P}^T > 0$, $\nu \geq 0$ and $\sigma \geq 0$ such that

$$\begin{bmatrix} \mathbf{P}\mathbf{A} + \mathbf{A}^T \mathbf{P} & \mathbf{P}\mathbf{B}_f - \nu \left(\mathbf{A}^T \mathbf{C}^T + \omega \mathbf{C}^T \Pi^T \right) - \sigma \mathbf{C}^T \\ \mathbf{B}_f^T \mathbf{P} - \nu \left(\mathbf{C}\mathbf{A} + \omega \Pi \mathbf{C} \right) - \sigma \mathbf{C} & -2\sigma k^{-1} \mathbf{I} \end{bmatrix} < 0 \tag{5.30}$$

then the system (5.23) is *globally asymptotically stable* at the equilibrium point $\mathbf{w} = \mathbf{w}_d$.

5.4 Stability Condition to Avoid Backward Whirl using Lyapunov Based Approach

Consider the rotordynamic system with friction at an interaction plane. The state space model for lateral vibration of rotor-stator structure is written as

$$\begin{aligned} \dot{\mathbf{W}} &= \mathbf{A}\mathbf{W} + \mathbf{B}_f \mathbf{T}_\mu \mathbf{f} \\ \mathbf{z} &= \mathbf{C}\mathbf{W} + \mathbf{z}_d, \quad \mathbf{f} = \beta(\mathbf{z}) \end{aligned} \tag{5.31}$$

where $\mathbf{T}_\mu$ is a constant matrix that models the effects of friction:

$$\mathbf{T}_\mu = \begin{bmatrix} 1 & -\mu \\ \mu & 1 \end{bmatrix}$$

Physically, backward whirl will be possible only if the friction force is sufficient to drive whirl in a reverse direction as discussed in Chapter 2.

The contact geometry for the friction-driven backward whirl case is shown in figure 5.7. The friction force μf is acting on the rotor in a reverse direction of rotor rotation. The interaction constraint (5.20) is still valid for backward whirl case as long as the geometry condition $|\mathbf{p}| < |\mathbf{z}| \cos \alpha$ holds. Hence, the condition for global stability of a backward whirl is

$$\mathbf{W}^T(\mathbf{PA}+\mathbf{A}^T\mathbf{P})\mathbf{W}+2\mathbf{f}^T\left[\mathbf{T}_\mu^T\mathbf{B}_f^T\mathbf{P}-\nu\left(\mathbf{CA}+\omega\Pi\mathbf{C}\right)-2\sigma\mathbf{C}\right]\mathbf{W}-2\sigma\kappa^{-1}\mathbf{f}^T\mathbf{f}<0 \tag{5.32}$$

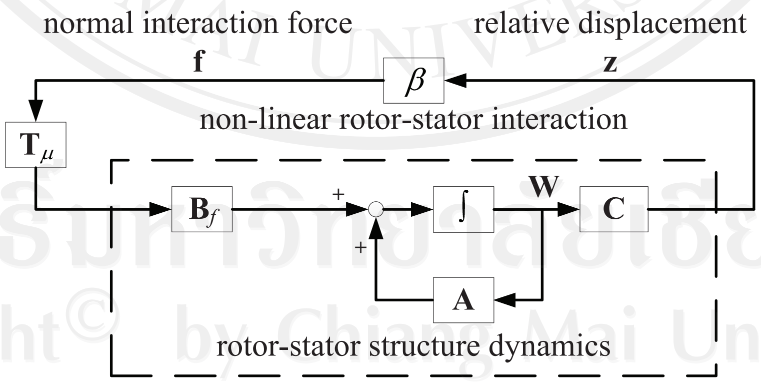

Figure 5.6: Schematic diagram of rotordynamic system with friction at interaction plane

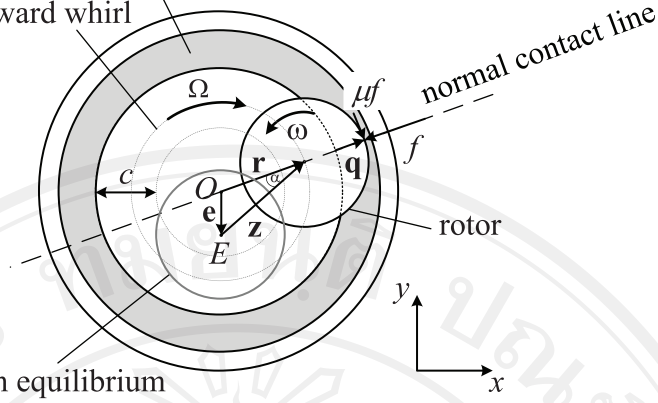

Figure 5.7: Schematic diagram of rotordynamic system with friction at interaction plane

This can be written in equivalent form as

$$\begin{bmatrix} \mathbf{W} \\ \mathbf{f} \end{bmatrix}^T \bar{\mathbb{V}} \begin{bmatrix} \mathbf{W} \\ \mathbf{f} \end{bmatrix} < 0$$

where

$$\bar{\mathbb{V}} = \begin{bmatrix} \mathbf{PA} + \mathbf{A}^T\mathbf{P} & \mathbf{PB}_f\mathbf{T}_\mu - \nu\left(\mathbf{A}^T\mathbf{C}^T + \omega\mathbf{C}^T\Pi^T\right) - \sigma\mathbf{C}^T \\ \mathbf{T}_\mu^T\mathbf{B}_f^T\mathbf{P} - \nu\left(\mathbf{CA} + \omega\Pi\mathbf{C}\right) - \sigma\mathbf{C} & -2\sigma k^{-1}\mathbf{I} \end{bmatrix}$$

which provides the following stability condition:

A sufficient condition for global asymptotic stability with friction: If there exist $\mathbf{P} = \mathbf{P}^T > 0$, $\nu \geq 0$ and $\sigma \geq 0$ such that

$$\begin{bmatrix} \mathbf{PA} + \mathbf{A}^T\mathbf{P} & \mathbf{PB}_f\mathbf{T}_\mu - \nu\left(\mathbf{A}^T\mathbf{C}^T + \omega\mathbf{C}^T\Pi^T\right) - \sigma\mathbf{C}^T \\ \mathbf{T}_\mu^T\mathbf{B}_f^T\mathbf{P} - \nu\left(\mathbf{CA} + \omega\Pi\mathbf{C}\right) - \sigma\mathbf{C} & -2\sigma k^{-1}\mathbf{I} \end{bmatrix} < 0 \tag{5.33}$$

then the system (5.31) is *globally asymptotically stable* at the equilibrium point $\mathbf{w} = \mathbf{w}_d$.

5.5 Numerical Simulation

5.5.1 Forward whirl

The multi-mode rotordynamic model developed in Chapter 4 will be used to illustrate the Lyapunov based approach and compare it with the Nyquist based approach. To determine a stability boundary in terms of k, the objective is to maximize the penetration stiffness coefficient k over free variables $\mathbf{P}$ and ν subject to stability condition (5.30).

Optimization problem: Maximize k over ν and $\mathbf{P} = \mathbf{P}^T > 0$ subject to

$$\begin{bmatrix} \mathbf{PA} + \mathbf{A}^T\mathbf{P} & \mathbf{PB}_f - \nu\left(\mathbf{A}^T\mathbf{C}^T + \omega\mathbf{C}^T\Pi^T\right) - \sigma\mathbf{C}^T \\ \mathbf{B}_f^T\mathbf{P} - \nu\left(\mathbf{CA} + \omega\Pi\mathbf{C}\right) - \sigma\mathbf{C} & -2\sigma k^{-1}\mathbf{I} \end{bmatrix} < 0$$

This is a generalized eigenvalue problem which can be directly solved by using efficient numerical routines based on convex optimization [40]. Figure 5.8 shows the stability boundaries from the Nyquist based approach and the Lyapunov based approach.

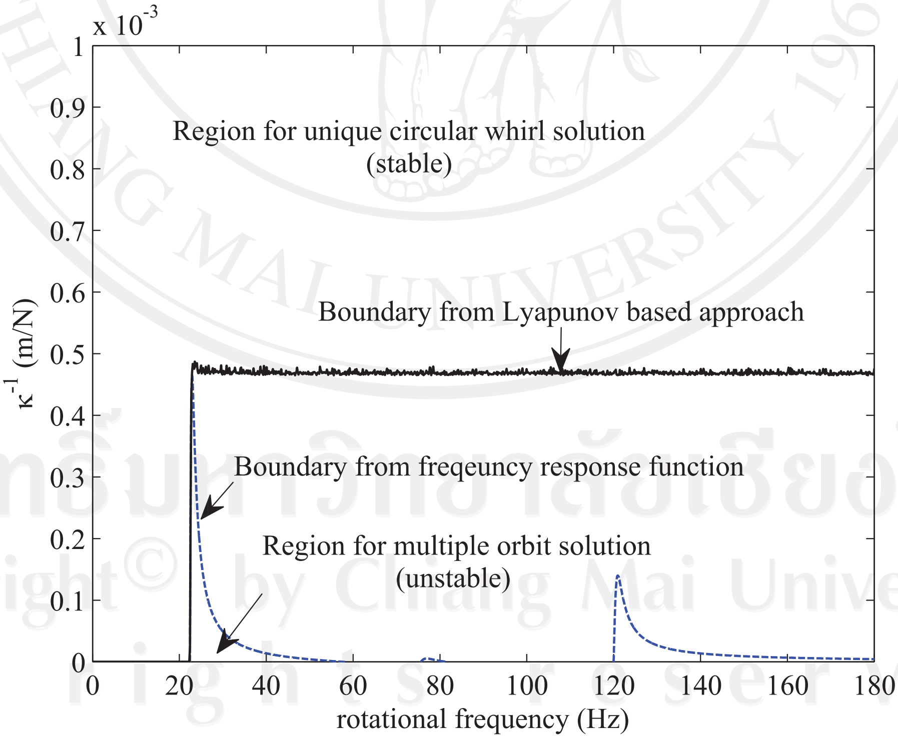

Figure 5.8: Comparison of boundaries for possibility of an alternative vibration response involving rotor-stator interaction.

For the boundary from the Lyapunov based approach, if the actual value of κ^{-1} is below the boundary, stability of the system cannot be established. Also shown is the region for multiple orbit solutions, as calculated from the Nyquist based approach. As expected, this region is contained within the region where stability cannot be established via the Lyapunov based approach. Note, however, that the Lyapunov boundary indicates not only where multiple orbit solutions are eliminated but also any other periodic, aperiodic or unbounded response. This result indicates that the Lyapunov based approach is very useful for checking stability over a finite speed range. With a design perspective, it may be used in the system design process to avoid unstable whirl conditions, i.e. by decreasing penetration dependent stiffness coefficient (by changing materials in contact plane) or changing characteristics of the system (mass, stiffness or damping coefficient).

5.5.2 Backward whirl

In this subsection, friction between the rotor and stator at a contact plane is included in the model as shown in (5.31). For the Lyapunov based approach, the instability boundary for a stable forward whirl can be obtained by solving the following optimization problem.

Optimization problem: Maximize k over ν and $\mathbf{P} = \mathbf{P}^T > 0$ subject to

$$\begin{bmatrix} \mathbf{PA} + \mathbf{A}^T\mathbf{P} & \mathbf{PB}_f\mathbf{T}_\mu - \nu\left(\mathbf{A}^T\mathbf{C}^T + \omega\mathbf{C}^T\Pi^T\right) - \sigma\mathbf{C}^T \\ \mathbf{T}_\mu^T\mathbf{B}_f^T\mathbf{P} - \nu\left(\mathbf{CA} + \omega\Pi\mathbf{C}\right) - \sigma\mathbf{C} & -2\sigma k^{-1}\mathbf{I} \end{bmatrix} < 0$$

Although, the condition (2.21) is developed in order to apply to a single degree of freedom rotordynamic system, it can also be applied to multi-mode rotordynamic system by using an equivalent damping ratio at a considered contact plane as shown in section 5.2. Figure 5.9 shows the boundaries from the condition (2.21) for the rotor test rig and the Lyapunov based approach. The stability boundary from the Lyapunov based approach is very similar to the boundary for possibility of backward whirl solution obtained from condition (2.21). To anticipate the region for possibility of backward whirl, The Lyapunov based approach is more appropriate to deal with the multi-mode rotordynamic system rather than the condition (2.21) because it can deal with multiple contact plane and the stability of the forward whirl can be checked.

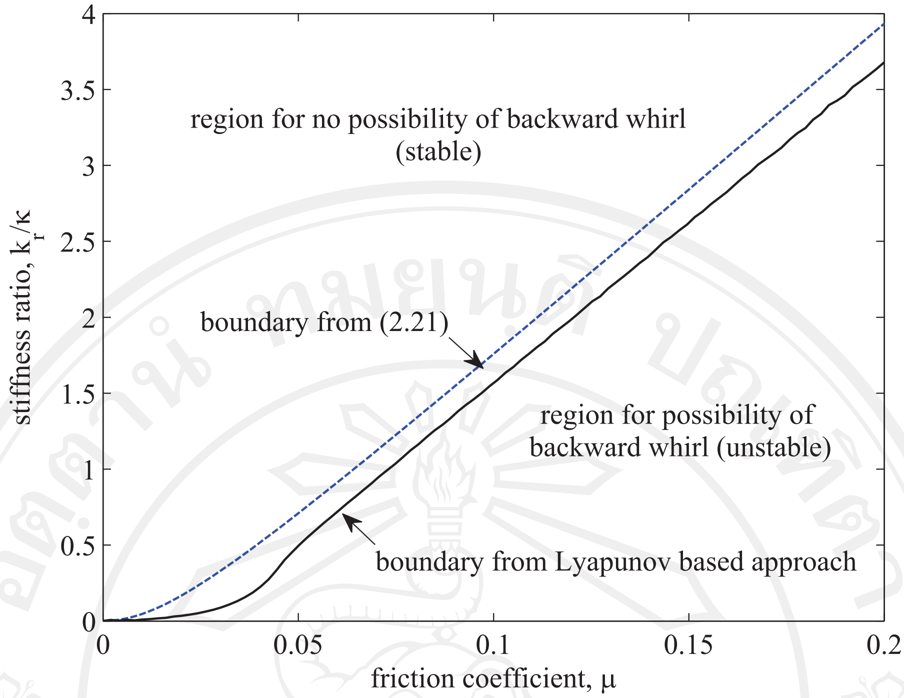

Figure 5.9: Comparison of boundaries for possibility of a backward whirl solution.

5.6 Summary

Two vibrational stability prediction approaches, which are Nyquist stability approach and Lyapunov based approach, have been presented. The Nyquist stability approach is based on the frequency response function of the rotor system. This allows a plot to be made showing the parametric region (in terms of speed and orbit size) where a jump to a high amplitude vibration response with contact is possible. This plot is called a “whirl mode map”. The whirl mode map can be used to anticipate the maximum amplitude of rotor vibration that avoids the possibility of amplitude jump to a vibration response involving contact. To anticipate the possibility of backward whirl, the condition (2.21) can be applied to the multi-mode rotordynamic system model by using an equivalent damping ratio.

Another approach is a Lyapunov based approach which can be used to derive stability conditions in the form of linear matrix inequalities (LMIs). The stability conditions for three different cases were considered

1) The stability condition for a fixed equilibrium without friction.

2) The stability condition for a steady contact-free orbit without friction.

3) The stability condition for a steady contact-free orbit with friction.

A sufficient condition for globally stability of the rotordynamic system for each case involves a combination of the Lyapunov stability condition and a constraint based on the geometry of contact. The Lyapunov stability based approach will be further employed in the controller design procedures that will be presented in the next chapter.