

CHAPTER 2

Preliminaries

In this chapter, the overview basic concepts, notations and mathematical definitions of linear systems and NNs with time delays are given. Also, the underlying concepts of stability and synchronization, important definitions, lemma, propositions and results, which will be used in later chapters, are provided.

2.1 Notations

We give some important notations will be used throughout this thesis:

$\mathcal{R}^+$ denote the set of all non-negative real numbers;

$\mathcal{R}^n$ denote the n -dimensional Euclidean space;

Matrix M is positive definite ($M > 0$) if $x^T M x > 0$ for all $x \in \mathcal{R}^n, x \neq 0$;

Matrix M is semi-positive definite ($M \geq 0$) if $x^T M x \geq 0$ for all $x \in \mathcal{R}^n$;

Matrix M is negative definite ($M < 0$) if $x^T M x < 0$ for all $x \in \mathcal{R}^n, x \neq 0$;

Matrix M is semi-negative definite ($M \leq 0$) if $x^T M x \leq 0$ for all $x \in \mathcal{R}^n$;

$M > 0$ ($M \geq 0$) denote the square symmetric positive (semi-) definite matrix;

$M < 0$ ($M \leq 0$) denote the square symmetric negative (semi-) definite matrix;

$M > N$ ($M \geq N$) denote the $M - N$ matrix is square symmetric positive (semi-) definite matrix;

$M < N$ ($M \leq N$) denote the $M - N$ matrix is square symmetric negative (semi-) definite matrix;

$M^{n \times m}$ denote the space of all $(n \times m)$ matrices;

A^T denotes the transpose of the vector/matrix A ;

A^{-1} denote the inverse of a non-singular matrix A ;

A is symmetric if $A = A^T$;

I denotes the identity matrix;

$\lambda(A)$ denotes the set of all eigenvalues of A ;

$\lambda_{\max}(A) = \max \{\operatorname{Re} \lambda : \lambda \in \lambda(A)\}$,

$\lambda_{\min}(A) = \min \{\operatorname{Re} \lambda : \lambda \in \lambda(A)\}$;

$\langle x, y \rangle$ or $x^T y$ — the scalar product of two vector x, y ;

$\|x\|$ denotes the Euclidean vector norm of x ;

$\|\cdot\|_d = \sup_{-h \leq \theta \leq 0} \{\|x(t+\theta)\|, \|\dot{x}(t+\theta)\|\}$;

$L_2([0, t], \mathcal{R}^m)$ denotes the set of all the $\mathcal{R}^m$ -valued square integrable functions on $[0, t]$;

$C([0, t], \mathcal{R}^n)$ denotes the set of all $\mathcal{R}^n$ -valued continuous functions on $[0, t]$;

$\|x\|_c$ denotes the continuous norm $\max_{a \leq \xi \leq b} \|\phi(\xi)\|$ for $\phi \in C([0, t], \mathcal{R}^n)$;

$x_t = \{x(t+s) : s \in [-h, 0]\}$, $\|x_t\| = \sup_{s \in [-h, 0]} \|x(t+s)\|$;

$\otimes$ denote the Kronecker product.

2.2 Model Formulation and Some Preliminaries

2.2.1 Neural networks

In this section, an isolated NNs (the i th) with constant delay will be considered, which can be described by the following differential equation:

$$\dot{x}_i(t) = -Cx_i(t) + Af(x_i(t)) + Bf(x_i(t-\tau)) + I(t), \quad (2.1)$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \dots, x_{in}(t))^T \in \mathcal{R}^n$ is the state vector of the i th delayed neural network. In each neural network $C = \text{diag}(c_1, c_2, \dots, c_n) > 0$ is a diagonal matrix with c_j representing the rate with which the j th neuron will reset its potential to the resting state in isolation, $A = (a_{ij})_{n \times n}$ and $B = (b_{ij})_{n \times n}$ denote the connection weight and the discretely delayed connection weight matrices, respectively, $f(x_i(t)) = (f_1(x_{i1}(t)), f_2(x_{i2}(t)), \dots, f_n(x_{in}(t)))^T$ denote the neuron activation function vector, constant $\tau > 0$ stands for the discrete time delay, and $I(t) = (I_1(t), I_2(t), \dots, I_n(t))^T \in \mathcal{R}^n$ is an external input vector.

The initial conditions of the above i th delayed NNs (2.1) are always assumed to be:

$$x_{ij}(\theta) = \phi_{ij}(\theta) \in \mathcal{C}([-\tau, 0], \mathcal{R}), \quad \forall j = 1, 2, \dots, n, \quad (2.2)$$

in which $\phi_{ij}(\theta)$, ($j = 1, 2, \dots, n$) are continuous functions.

Next, the configuration of an array of coupled NNs is formulated. We consider an array of linearly delay coupled system consisting of N identical delayed NNs with each network being an n -dimensional dynamical system as (2.1). The dynamical behavior of the array of coupled systems can be described by the following differential equations:

$$\begin{cases} \dot{x}_i(t) = -Cx_i(t) + Af(x_i(t)) + Bf(x_i(t-\tau)) + I(t) \\ \quad + \sum_{j=1}^N g_{ij}\Gamma x_j(t), & i = 1, 2, \dots, N, \end{cases} \quad (2.3)$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \dots, x_{in}(t))^T \in \mathcal{R}^n$ is the state vector of the i th delayed neural network. $\Gamma = \text{diag}(\gamma_1, \gamma_2, \dots, \gamma_n) \in \mathcal{R}^{n \times n}$ is the constant inner-coupling matrix of nodes, which describe the individual coupling between networks. $G = (g_{ij})_{N \times N}$ is the outer-coupling configuration matrix of the system.

For the coupling configuration matrix G and the activation functions in the NNs (2.3), we have the following conditions:

1. The coupling configuration matrix $G = (g_{ij})_{N \times N}$ is symmetric $G = G^T$, and satisfies:

$$\sum_{j=1}^N g_{ij} = \sum_{j=1}^N g_{ji} = 0, \quad i = 1, 2, \dots, N.$$

2. $f_r(\cdot) (r = 1, 2, \dots, n)$ are Lipschitz continuous, i.e., there exist constants $l_r > 0, (r = 1, 2, \dots, n)$ such that

$$|f_r(x_1) - f_r(x_2)| \leq l_r |x_1 - x_2|, \quad r = 1, 2, \dots, n,$$

where $x_1, x_2 \in \mathcal{R}$.

For convenience, we denote $L = \text{diag}(l_1, l_2, \dots, l_n)$.

Example 2.2.1. Consider the system (2.1) is presented in [10], where $x(t) = [x_1(t), x_2(t)]^T$ is the state vector of the network. $f(x_i(t)) = 0.5(|x_i + 1| - |x_i - 1|)$ is the activation functions. $I(t) = [0, 0]^T$ is an external input vector. $\tau = 0.95$ is constant delay and the other parameters are as follows:

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 + \frac{\pi}{4} & 20 \\ 0.1 & 1 + \frac{\pi}{4} \end{bmatrix}, \quad B = \begin{bmatrix} -\frac{1.3\pi\sqrt{2}}{4} & 0.1 \\ 0.1 & -\frac{1.3\pi\sqrt{2}}{4} \end{bmatrix},$$

which is described by the following differential equations:

$$\begin{cases} \dot{x}_1(t) = -x_1(t) + 1.79f(x_1(t)) + 20f(x_2(t)) - 1.44f(x_1(t - 0.95)) + 0.1f(x_2(t - 0.95)) \\ \dot{x}_2(t) = -x_2(t) + 0.1f(x_1(t)) + 1.79f(x_2(t)) + 0.1f(x_1(t - 0.95)) - 1.44f(x_2(t - 0.95)). \end{cases}$$

Obviously, the activation function f is globally Lipschitz continuous with $L = \text{diag}(1, 1)$. The chaotic behavior of neural network (2.1) shown in Fig. 1.1, with the initial condition $x(t) = (0.1, 0.1)^T, t \in [-1, 0]$.

Next, consider a dynamical system (2.3) consisting of three linearly coupled identical models. Let the outer-coupling and the inner-coupling configuration matrix are as follows, respectively.

$$G = \begin{bmatrix} -8 & 2 & 6 \\ 2 & -4 & 2 \\ 6 & 2 & -8 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

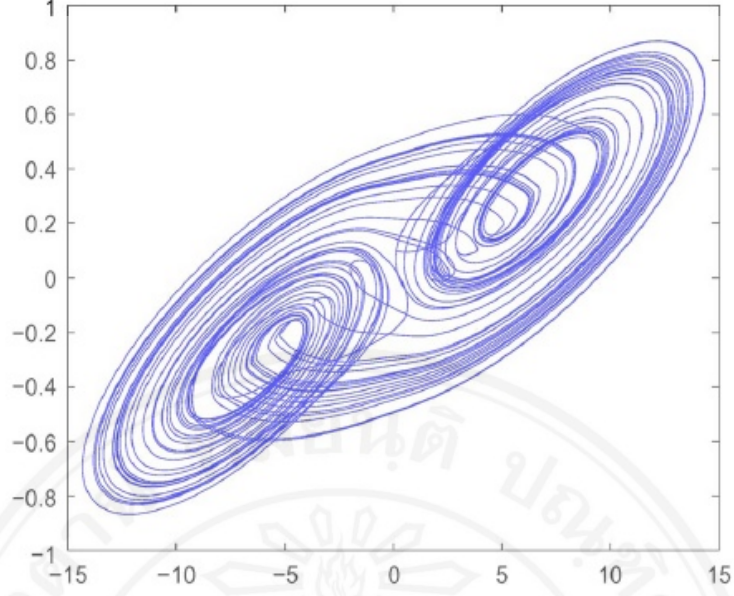


Figure 2.1: Chaotic trajectory of NNs (2.1) with initial condition $x(t) = (0.1, 0.1)^T$, $t \in [-1, 0]$.

Then, the coupled NNs can be described by the following differential equations:

$$\left\{ \begin{array}{l} \dot{x}_{11}(t) = 1.79f(x_{11}(t)) + 20f(x_{12}(t)) - 1.44f(x_{11}(t - 0.95)) + 0.1f(x_{12}(t - 0.95)) \\ \quad - 9x_{11}(t) + 2x_{21}(t) + 6x_{31}(t), \\ \dot{x}_{12}(t) = 0.1f(x_{11}(t)) + 1.79f(x_{12}(t)) + 0.1f(x_{11}(t - 0.95)) - 1.44f(x_{12}(t - 0.95)) \\ \quad - 9x_{12}(t) + 2x_{22}(t) + 6x_{32}(t), \\ \dot{x}_{21}(t) = 1.79f(x_{21}(t)) + 20f(x_{22}(t)) - 1.44f(x_{21}(t - 0.95)) + 0.1f(x_{22}(t - 0.95)) \\ \quad - 5x_{21}(t) + 2x_{11}(t) + 2x_{31}(t), \\ \dot{x}_{22}(t) = 0.1f(x_{21}(t)) + 1.79f(x_{22}(t)) + 0.1f(x_{21}(t - 0.95)) - 1.44f(x_{22}(t - 0.95)) \\ \quad - 5x_{22}(t) + 2x_{12}(t) + 2x_{32}(t), \\ \dot{x}_{31}(t) = 1.79f(x_{31}(t)) + 20f(x_{32}(t)) - 1.44f(x_{31}(t - 0.95)) + 0.1f(x_{32}(t - 0.95)) \\ \quad - 9x_{31}(t) + 6x_{11}(t) + 2x_{21}(t), \\ \dot{x}_{32}(t) = 0.1f(x_{31}(t)) + 1.79f(x_{32}(t)) + 0.1f(x_{31}(t - 0.95)) - 1.44f(x_{32}(t - 0.95)) \\ \quad - 9x_{32}(t) + 6x_{12}(t) + 2x_{22}(t). \end{array} \right.$$

2.2.2 Controller

- **Sampled-data feedback control**

We consider the synchronization for general complex dynamical networks

with control denoted by the form:

$$\dot{x}_i(t) = f(x_i(t)) + c \sum_{j=1}^N g_{ij} A x_j(t - \tau(t)) + u_i(t), \quad i = 1, 2, \dots, N, \quad (2.4)$$

where $x_i(t) \in \mathcal{R}^n$ and $u_i(t) \in \mathcal{R}^n$ are the state variable and the control input of the node i , respectively. $f : \mathcal{R}^n \rightarrow \mathcal{R}^n$ is a continuous vector-valued function. The scalar $\tau(t)$ denotes the time-varying delay satisfying

$$0 \leq \tau(t) \leq \mu, \quad \dot{\tau}(t) \leq \nu,$$

where μ and ν are known positive constants. $c > 0$ is the coupling strength. $A = (a_{ij})_{n \times n} \in \mathcal{R}^{n \times n}$ is a constant inner-coupling matrix of the nodes. $G = (g_{ij})_{N \times N}$ is the outer-coupling matrix of the network, where g_{ij} is defined as follows: if there is a connection between node i and node j ($j \neq i$), then $g_{ij} = g_{ji} = 1$; otherwise, $g_{ij} = g_{ji} = 0$, and the diagonal elements of matrix G are defined by

$$g_{ii} = - \sum_{j=1, j \neq i}^N g_{ij} = - \sum_{j=1, j \neq i}^N g_{ji}, \quad i = 1, 2, \dots, N.$$

Let the synchronization error of system (2.4) be $e_i(t) = x_i(t) - s(t)$ where $s(t) \in \mathcal{R}^n$ is the state trajectory of the unforced isolate node $\dot{s}(t) = f(s(t))$. Then, synchronization error dynamics of complex network is given by:

$$\dot{e}_i(t) = \bar{f}(e_i(t)) + c \sum_{j=1}^N g_{ij} A e_j(t - \tau(t)) + u_i(t), \quad i = 1, 2, \dots, N, \quad (2.5)$$

where $\bar{f}(e_i(t)) = f(x_i(t)) - f(s(t))$.

The following sampled-data feedback controllers are adopted:

$$u_i(t) = K_i e_i(t_k), \quad t_k \leq t < t_{k+1},$$

where K_i is feedback controllers to be determined. $e(t_k)$ is discrete measurement of $e(t)$ at the sampling instant t_k , and t_k satisfies the following conditions:

$$0 = t_0 < t_1 < t_2 \cdots < t_k < \cdots < \lim_{k \rightarrow +\infty} t_k = +\infty.$$

It is assumed that $t_{k+1} - t_k \leq \rho$ for $k \geq 0$ where $\rho > 0$ is a positive scalar and the largest sampling interval, i.e., the sampling interval is bounded. Define a sawtooth function $d(t) : [0, \infty) \rightarrow \mathcal{R}$ as follows:

$$d(t) = t - t_k, \quad t_k \leq t < t_{k+1}.$$

It is easily found that

$$0 < d(t) < \rho,$$

and the sampled-data state feedback controllers takes the following form

$$u_i(t) = K_i e_i(t - d(t)).$$

Consequently, the error system of the complex network is obtained as follows:

$$\dot{e}_i(t) = \bar{f}(e_i(t)) + c \sum_{j=1}^N g_{ij} A e_j(t - \tau(t)) + K_i e_i(t - d(t)), \quad i = 1, 2, \dots, N. \quad (2.6)$$

• Intermittent feedback control

We present the synchronization by the intermittent feedback method. Consider a class of chaotic master (driver) systems:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bf(x(t)), & t > 0, \\ x(t_0) = x_0. \end{cases} \quad (2.7)$$

In using intermittent feedback controls to synchronize system (2.7), the corresponding slave (response) system is designed as

$$\begin{cases} \dot{y}(t) = Ay(t) + Bf(y(t)) + k(t)(x(t) - y(t)), & t > 0, \\ y(t_0) = y_0. \end{cases} \quad (2.8)$$

where $x, y \in \mathcal{R}^n$ are the state vectors of systems (2.7) and (2.8), respectively. $A, B \in \mathcal{R}^{n \times n}$. $f : \mathcal{R}^n \rightarrow \mathcal{R}^n$ are nonlinear functions. We assume that $f : \mathcal{R}^n \rightarrow \mathcal{R}^n$ is a Lipschitz continuous function: there exists a positive constant L such that, for all $x, y \in \mathcal{R}^n$,

$$\| f(x) - f(y) \| \leq L \| x - y \|,$$

and $k(t)$ is the intermittent control gain defined by:

$$k(t) = \begin{cases} K, & t_0 + n\omega \leq t \leq t_0 + n\omega + \delta \\ 0, & t_0 + n\omega + \delta < t \leq t_0 + (n+1)\omega \end{cases},$$

where $K \in \mathcal{R}^{n \times n}$ is a constant control gain. $\omega > 0$ is the control period. And $\delta > 0$ is the control width. This controller, our goal is to design suitable δ, ω and K such that system (2.8) synchronizes system (2.7).

Let $e(t) = y(t) - x(t)$ be the synchronization error between the states of the drive system (2.7) and the response system (2.8). Then, for $t \in (0, \infty)$, we have the following error system:

$$\dot{e}(t) = \begin{cases} Ae(t) + B(f(y(t)) - f(x(t))) - Ke(t), & t_0 + n\omega \leq t \leq t_0 + n\omega + \delta \\ Ae(t) + B(f(y(t)) - f(x(t))), & t_0 + n\omega + \delta < t \leq t_0 + (n+1)\omega. \end{cases}$$

2.3 Definitions and Lemmas

Consider a dynamical system described by

$$\dot{x}(t) = f(t, x(t)) \quad (2.9)$$

where $x \in \mathcal{R}^n$ and f is a vector having components $f_i(t, x_1, \dots, x_n)$, $i = 1, 2, \dots, n$. We shall assume that the f_i are continuous and satisfy standard conditions, such as having continuous first partial derivatives so that the solution of (2.9) exists and is unique for the given initial conditions. If f_i do not depend explicitly on t , (2.9) is called autonomous (otherwise, nonautonomous). If $f(t, c) = 0$ for all t , where c is some constant vector, then it follows at once from (2.9) that if $x(t_0) = c$ then $x(t) = c$ for all $t \geq t_0$. Thus solutions starting at c remain there, and c is said to be an *equilibrium* or *critical point*. Clearly, by introducing new variables $\acute{x}_i = x_i - c_i$ we can arrange for the equilibrium point to be transferred to the origin; we shall assume that this has been done for any equilibrium point under consideration (there may well be several for a given system (2.9)) so that we then have $f(t, 0) = 0$, $t \geq t_0$.

2.3.1 Autonomous systems

Consider the autonomous system

$$\dot{x} = f(x) \quad (2.10)$$

where $f : D \rightarrow \mathcal{R}^n$ is locally Lipschitz map from a domain $D \subset \mathcal{R}^n$ into $\mathcal{R}^n$. We shall always assume that $f(x)$ satisfies $f(0) = 0$, and study stability of the origin $x = 0$.

Definition 2.3.1. [20]: The equilibrium point $x = 0$ of (2.10) is

- (i) **stable** if for each $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$ such that

$$\|x(0)\| < \delta \Rightarrow \|x\| < \epsilon, \quad \forall t \geq 0,$$

- (ii) **unstable** if it is not stable, that is, there exists $\epsilon > 0$ such that for every $\delta > 0$ there exist an $x(0)$ with $\|x(0)\| < \delta$ so that $\|x(t_1)\| \geq \epsilon$ for some $t_1 > 0$. If this holds for every $x(0)$ in $\|x(0)\| < \delta$ the equilibrium is completely unstable.

- (iii) **asymptotically stable** if it is stable and δ can be chosen such that

$$\|x(0)\| < \delta \Rightarrow \lim_{t \rightarrow \infty} x(t) = 0.$$

Definition 2.3.2. [20]: A function $V(\cdot) : \mathcal{R}^n \rightarrow \mathcal{R}$ is said to be Lyapunov function if it satisfies the following:

1. $V(x)$ and all its partial derivatives $\frac{\partial V}{\partial x_i}$ are continuous.
2. $V(x)$ is positive definite, i.e. $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$ in some neighbourhood $\|x\| \leq k$ of the origin.
3. The derivative of V with respect to (2.10), namely

$$\begin{aligned}\dot{V} &= \frac{\partial V}{\partial x_1} \dot{x}_1 + \frac{\partial V}{\partial x_2} \dot{x}_2 + \dots + \frac{\partial V}{\partial x_n} \dot{x}_n \\ &= \frac{\partial V}{\partial x_1} f_1 + \frac{\partial V}{\partial x_2} f_2 + \dots + \frac{\partial V}{\partial x_n} f_n\end{aligned}\quad (2.11)$$

is negative semidefinite i.e. $\dot{V}(0) = 0$, and for all x satisfy $\|x\| \leq k$, $\dot{V}(x) \leq 0$.

Theorem 2.3.1. [20]: Let $x = 0$ be an equilibrium point for (2.10) and $D \subset \mathcal{R}^n$ be a domain containing $x = 0$. Let $V(x) : D \rightarrow \mathcal{R}$ be a continuously differentiable function, such that

$$\begin{aligned}V(0) &= 0 \quad \text{and} \quad V(x) > 0 \quad \text{in} \quad D - \{0\}, \\ \dot{V}(x) &\leq 0 \quad \text{in} \quad D.\end{aligned}$$

Then, $x = 0$ is stable. Moreover, if

$$\dot{V}(x) < 0 \quad \text{in} \quad D - \{0\},$$

then $x = 0$ is asymptotically stable.

Theorem 2.3.2. [20]: Let $x = 0$ be an equilibrium point for (2.10). Let $V(x) : \mathcal{R}^n \rightarrow \mathcal{R}$ be a continuously differentiable function, such that

$$V(0) = 0 \quad \text{and} \quad V(x) > 0, \quad \forall x \neq 0,$$

$$\|x\| \rightarrow \infty \Rightarrow V(x) \rightarrow \infty,$$

$$\dot{V}(x) < 0, \quad \forall x \neq 0,$$

then $x = 0$ is globally asymptotically stable.

Theorem 2.3.3. [20]: Let $x = 0$ be an equilibrium point for (2.10) and $f : D \rightarrow \mathcal{R}^n$ is continuously differentiable and D is a neighborhood of the origin. Let

$$A = \frac{\partial f}{\partial x}(x) \big|_{x=0}.$$

Then,

1. The origin is asymptotically stable if $\text{Re}(\lambda_i) < 0$ for all eigenvalues of A .
2. The origin is unstable if $\text{Re}(\lambda_i) > 0$ for one or more of the eigenvalues of A .

2.3.2 Nonautonomous systems

Consider the nonautonomous system

$$\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0, \quad x \in \mathcal{R}^n, \quad t \in \mathcal{R}^+, \quad (2.12)$$

where $f : \mathcal{R}^+ \times D \rightarrow \mathcal{R}^n$ is piecewise continuous in t and locally Lipschitz in x on $\mathcal{R}^+ \times \mathcal{R}^n$ and $D \subset \mathcal{R}^n$ is domain that contains the origin $x = 0$. The origin is an equilibrium point for (2.12) if

$$f(t, 0) = 0, \quad \forall t \geq t_0.$$

Definition 2.3.3. [20]: The equilibrium point $x = 0$ of the system (2.12) is

- (i) **stable** if, for each $\epsilon > 0$, there is $\delta = \delta(\epsilon, t_0) > 0$ such that

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \quad \forall t \geq t_0 \geq 0, \quad (2.13)$$

- (ii) **uniformly stable** if, for each $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$, independent of t_0 , such that (2.20) is satisfied,

- (iii) **unstable** if not stable,

- (iv) **asymptotically stable** if it is stable and there is $c = c(t_0) > 0$ such that $x(t) \rightarrow 0$ as $t \rightarrow \infty$, for all $\|x(t_0)\| < c$,

- (v) **uniformly asymptotically stable** if it is uniformly stable and there is $c > 0$, independent of t_0 , such that for all $\|x(t_0)\| < c$, $x(t) \rightarrow 0$ as $t \rightarrow \infty$, uniformly in t_0 , for each $\epsilon > 0$, there is $T = T(\epsilon) > 0$ such that

$$\|x(t)\| < \epsilon, \quad \forall t \geq t_0 + T(\epsilon), \quad \forall \|x(t_0)\| < c,$$

- (vi) **globally uniformly asymptotically stable** if it is uniformly stable and, for each pair of positive numbers ϵ and c , there is $T = T(\epsilon, c) > 0$ such that

$$\|x(t)\| < \epsilon, \quad \forall t \geq t_0 + T(\epsilon, c), \quad \forall \|x(t_0)\| < c.$$

Definition 2.3.4. [39]: The equilibrium point $x = 0$ of the system (2.12) is *exponentially stable* if there exist three positive real constants ϵ, K and λ such that

$$\|x(t)\| \leq K \|x_0\| e^{-\lambda(t-t_0)}, \quad \forall \|x_0\| < \epsilon, \quad t \geq t_0,$$

The largest constant λ which may be utilized in above inequality is called the rate of convergence.

Definition 2.3.5. [20]: A continuous function $\alpha : [0, a) \rightarrow [0, \infty]$ is said to belong to class K if it is strictly increasing and $\alpha(0) = 0$. It is said to belong class K_∞ if $a = \infty$ and $\alpha(r) \rightarrow \infty$ as $r \rightarrow \infty$.

Example 2.3.4. [20]: We give some example for class K and class K_∞ :

1. $\alpha(r) = \tan^{-1} r$ is strictly increasing since $\alpha'(r) = \frac{1}{1+r^2} > 0$. It belong to class K , but not to class K_∞ since $\lim_{r \rightarrow \infty} \alpha(r) = \frac{\pi}{2} < \infty$.
2. $\alpha(r) = r^c$, for any positive real number c , is strictly increasing since $\alpha'(r) = cr^{c-1} > 0$. Moreover, $\lim_{r \rightarrow \infty} \alpha(r) = \infty$, thus, it belong to class K_∞ .

Definition 2.3.6. [20]: A continuous function $\beta : [0, a) \times [0, \infty] \rightarrow [0, \infty]$ is said to belong to class KL if, for each fixed s , the mapping $\beta(r, s)$ belong to class K with respect to r and, for each fixed r , the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \rightarrow 0$ as $s \rightarrow \infty$.

Example 2.3.5. [20]: We give some example for class KL :

1. $\beta(r, s) = \frac{r}{ksr+1}$, for any positive real number k , is strictly increasing in r since

$$\frac{\partial \beta}{\partial r} = \frac{1}{(ks+1)^2} > 0$$

and strictly decreasing in s since

$$\frac{\partial \beta}{\partial s} = \frac{-kr^2}{(ks+1)^2} < 0.$$

Moreover, $\beta(r, s) \rightarrow 0$ as $s \rightarrow \infty$. Hence, it belong to class KL .

Lemma 2.3.6. [20]: The equilibrium point $x = 0$ of (2.12) is

- (1) uniformly stable if and only if there exist a class K function $\alpha(\cdot)$ and a positive constant c , independent of t_0 , such that

$$\|x(t)\| \leq \alpha(\|x(t_0)\|), \quad \forall t \geq t_0 \geq 0, \quad \forall \|x(t_0)\| < c, \quad (2.14)$$

- (2) uniformly asymptotically stable if and only if there exist a class KL function $\beta(\cdot, \cdot)$ and a positive constant c , independent of t_0 , such that

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \quad \forall t \geq t_0 \geq 0, \quad \forall \|x(t_0)\| < c, \quad (2.15)$$

- (3) globally uniformly asymptotically stable if and only if inequality (2.15) is satisfied for any initial state $x(t_0)$.

Definition 2.3.7. [20]: The equilibrium point $x = 0$ of (2.12) is exponentially stable if inequality (2.15) is satisfied with

$$\beta(r, s) = kre^{-\gamma s}, \quad k > 0, \quad \gamma > 0$$

and is globally exponentially stable if this condition is satisfied for any initial state.

Definition 2.3.8. [32]: The function $W(x)$ is said to be positive (negative) definite if $W(x) > 0$ ($-W(x) > 0$) and $W(x) = 0$ if and only if $x = 0$. The function $W(x)$ is said to be positive (negative) semi-definite if $W(x) \geq 0$ ($-W(x) \geq 0$).

Definition 2.3.9. [32]: The function $W(x)$ is said to be radially unbounded, positive definite if $W(x)$ is positive definite and $W(x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$.

Let B_ϵ be a ball of size ϵ around the origin,

$$B_\epsilon = \{x \in \mathcal{R}^n : \|x\| < \epsilon\}.$$

Definition 2.3.10. [38]: A function $V(\cdot) : \mathcal{R}^+ \times \mathcal{R}^n \rightarrow \mathcal{R}$ is said to be *Lyapunov function* if it satisfies the following:

- (i) $V(t, x)$ and all its partial derivatives $\frac{\partial V}{\partial t}, \frac{\partial V}{\partial x_i}$ are continuous for all $i = 1, 2, 3, \dots, n$.
- (ii) $V(t, x)$ is positive definite function, i.e., $V(0) = 0$ and $V(t, x) > 0, x \neq 0, \forall x \in B_\epsilon$.
- (iii) The derivative of $V(t, x)$ with respect to system (2.12), namely

$$\begin{aligned} \dot{V}(t, x) &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_1} \dot{x}_1 + \frac{\partial V}{\partial x_2} \dot{x}_2 + \dots + \frac{\partial V}{\partial x_n} \dot{x}_n \\ &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_1} f_1 + \frac{\partial V}{\partial x_2} f_2 + \dots + \frac{\partial V}{\partial x_n} f_n. \end{aligned} \quad (2.16)$$

$\dot{V}(t, x)$ is negative semi-definite i.e., $\dot{V}(t, 0) = 0$ and $\forall x \in B_\epsilon, \dot{V}(t, x) \leq 0$.

Theorem 2.3.7. [20]: Let $x = 0$ be an equilibrium point for (2.12) and $D \subset \mathcal{R}^n$ be a domain containing $x = 0$. Let $V : \mathcal{R}^+ \times D \rightarrow \mathcal{R}$ be a continuously differentiable function, such that

$$W_1(x) \leq V(t, x) \leq W_2(x), \quad (2.17)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -W_3(x) \quad (2.18)$$

$\forall t \geq t_0 \geq 0, \forall x \in D$ where $W_1(x), W_2(x)$ and $W_3(x)$ are continuous positive definite functions on D . Then, $x = 0$ is uniformly asymptotically stable.

Corollary 2.3.8. [20]: Suppose that all the assumptions of Theorem 2.3.7 are satisfied globally (for all $x \in \mathcal{R}^n$) and $W_1(x)$ is radially unbounded. Then, $x = 0$ is globally uniformly asymptotically stable.

Corollary 2.3.9. [20]: Suppose all the assumptions of Theorem 2.3.7 are satisfied with

$$W_1(x) \geq k_1 \|x\|^c, \quad W_2(x) \leq k_2 \|x\|^c, \quad W_3(x) \geq k_3 \|x\|^c$$

for some positive constants k_1, k_2, k_3 and c . Then, $x = 0$ is exponentially stable. Moreover, if the assumptions hold globally, then, $x = 0$ is globally exponentially stable.

Theorem 2.3.10. [20]: Let $x = 0$ be an equilibrium point for the nonlinear system

$$\dot{x}(t) = f(t, x)$$

where $f : [0, \infty) \times D \rightarrow \mathcal{R}^n$ is continuously differentiable, $D = \{x \in \mathcal{R}^n \mid \|x\|_2 < r\}$, and Jacobian matrix $\left[\frac{\partial f}{\partial x}\right]$ is bounded and Lipschitz on D , uniformly in t . Let

$$A(t) = \frac{\partial f}{\partial x}(t, x)|_{x=0}.$$

Then, the origin is an exponentially stable equilibrium point for nonlinear system if and only if it is an exponentially stable equilibrium point for linear system

$$\dot{x}(t) = A(t)x.$$

Theorem 2.3.11. [20]: Let $x = 0$ be an equilibrium point for the nonlinear system

$$\dot{x}(t) = f(t, x)$$

where $f : [0, \infty) \times D \rightarrow \mathcal{R}^n$ is continuously differentiable, $D = \{x \in \mathcal{R}^n \mid \|x\| < r\}$, and Jacobian matrix $\left[\frac{\partial f}{\partial x}\right]$ is bounded and Lipschitz on D , uniformly in t . Let $\beta(\cdot, \cdot)$ be a class KL function and r_0 be a positive constant such that $\beta(r_0, 0) < r$. Let $D_0 = \{x \in \mathcal{R}^n \mid \|x\| < r_0\}$. Assume that the trajectory of the system satisfied

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \quad \forall x(t_0) \in D_0, \quad \forall t \geq t_0 \geq 0.$$

Then, there is a continuously differentiable function $V : [0, \infty) \times D_0 \rightarrow \mathcal{R}$ that satisfies the inequalities

$$\begin{aligned} \alpha_1(\|x\|) \leq V(t, x) &\leq \alpha_2(\|x\|), \\ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) &\leq -\alpha_3(\|x\|), \\ \left\| \frac{\partial V}{\partial x} \right\| &\leq \alpha_4(\|x\|), \end{aligned}$$

where $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, $\alpha_3(\cdot)$ and $\alpha_4(\cdot)$ are class K function defined on $[0, r_0]$. If the system is autonomous, V can be chosen independent of t .

2.3.3 Nonautonomous systems with time delay

We consider the nonautonomous system with time-delay of the form [18]

$$\begin{aligned}\dot{x}(t) &= f(t, x(t-h)), & \forall t \geq 0, \\ x(t_0 + \theta) &= \phi(\theta), & \forall \theta \in [-h, 0],\end{aligned}\tag{2.19}$$

where $x(t) \in \mathcal{R}^n$ is the state variable, $h \in \mathcal{R}^+$ is the delay and $f : \mathcal{R}^+ \times C(C([-h, 0], \mathcal{R}^n)) \rightarrow \mathcal{R}^n$. $\phi(t)$ is a continuous vector-valued initial condition. We assume $f(t, 0) = 0$ so that system (2.19) admits the trivial solution. We also assume that system (2.19) has an existence and uniqueness solution.

Definition 2.3.11. For the system described by (2.19), the trivial solution $x(t) = 0$ is said to be

- (i) **stable** if for any $t_0 \in \mathcal{R}$ and any $\epsilon > 0$, there exists a $\delta = \delta(t_0, \epsilon) > 0$ such that

$$\|x_{t_0}\|_c < \delta \Rightarrow \|x(t)\| < \epsilon, \quad \forall t \geq t_0,$$

- (ii) **uniformly stable** if it is stable and $\delta(t_0, \epsilon)$ can be chosen independently of t_0 ,

- (iii) **asymptotically stable** if it is stable, and for any $t_0 \in \mathcal{R}$ and any $\epsilon > 0$ there exists a $\delta_a = \delta_a(t_0, \epsilon) > 0$ such that

$$\|x_{t_0}\|_c < \delta_a \Rightarrow \lim_{t \rightarrow \infty} \|x(t)\| = 0,$$

- (iv) **uniformly asymptotically stable** if it is uniformly stable and there exists a $\delta_a > 0$ such that for any $\eta > 0$, there exists a $T = T(\delta_a, \eta)$, such that

$$\|x_{t_0}\|_c < \delta_a \Rightarrow \|x(t)\| < \eta, \quad \forall t \geq t_0 + T.$$

Suppose that $u, v, w : \mathcal{R}^+ \rightarrow \mathcal{R}^+$ are continuous nondecreasing functions, where additionally $u(s)$ and $v(s)$ are positive for $s > 0$, and $u(0) = v(0) = 0$. If there exist a continuous differentiable functional $V : \mathcal{R}^+ \times C \rightarrow \mathcal{R}$ such that

$$u(\|\phi(0)\|) \leq V(t, x(t)) \leq v(\|\phi\|),$$

the equilibrium point $x^* = 0$ of system (2.19) is

- (i) **uniformly stable** if

$$\dot{V}(t, x(t)) \leq -w(\|\phi(0)\|),$$

(ii) uniformly asymptotically stable if

$$\dot{V}(t, x(t)) \leq -w(\|\phi(0)\|),$$

where $w(s) > 0$ for $s > 0$,

(iii) globally uniformly asymptotically stable if

$$\dot{V}(t, x(t)) \leq -w(\|\phi(0)\|),$$

and $u(s)$ is radially unbounded.

Definition 2.3.12. [21]: A functional $V : \mathcal{R}^+ \times C \rightarrow \mathcal{R}^+$ is called a Lyapunov-Krasovskii functional for the system (2.19) if it has the following properties. There exist $\lambda_1, \lambda_2, \lambda_3 > 0$ such that

$$(i) \quad \lambda_1 \|x(t)\|^2 \leq V(t, x_t) \leq \lambda_2 \|x_t\|^2,$$

$$(ii) \quad \dot{V}(t, x_t) \leq -\lambda_3 \|x(t)\|^2.$$

Lemma 2.3.12. [18]: Consider the non autonomous time-delay system (2.19). If there exist a Lyapunov function $V(t, x_t)$ and $\lambda_1, \lambda_2 > 0$ such that for every solution $x(t)$ of the system, the following conditions hold,

$$(i) \quad \lambda_1 \|x(t)\|^2 \leq V(t, x_t) \leq \lambda_2 \|x_t\|^2,$$

$$(ii) \quad \dot{V}(t, x_t) \leq 0,$$

then the solution of the system is bounded, i.e., there exists $N > 0$ such that $\|x(t, \phi)\| \leq N \|\phi\|, \forall t \geq 0$.

Lemma 2.3.13. [21]: Consider the autonomous time-delay system (2.19). If there exist a Lyapunov-Krasovskii function $V(x_t)$ and $\lambda_1, \lambda_2, \lambda_3 > 0$ such that for every solution $x(t)$ of the system, the following conditions hold,

$$(i) \quad \lambda_1 \|x(t)\|^2 \leq V(x_t) \leq \lambda_2 \|x_t\|^2,$$

$$(ii) \quad \dot{V}(x_t) \leq -\lambda_3 \|x(t)\|^2,$$

then the solution of the system (2.19) is exponentially stable.

Lemma 2.3.14. [6] : Let $\otimes$ denote the notation of Kronecker product. Then, the following relationships hold:

- (1) $(\alpha A) \otimes B = A \otimes (\alpha B)$,
- (2) $(A + B) \otimes C = A \otimes C + B \otimes C$,
- (3) $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$.

Lemma 2.3.15. [64]: Let $e = (1, 1, \dots, 1)^T$, $E_N = ee^T$, and $U = NI_N - E_N$, $P \in \mathcal{R}^{n \times n}$, $x = (x_1^T, \dots, x_N^T)^T$, and $y = (y_1^T, \dots, y_N^T)^T$ with $x_k, y_k \in \mathcal{R}^n$, ($k = 1, 2, \dots, N$), then

$$x^T(U \otimes P)y = \sum_{1 \leq i < j \leq N} (x_i - x_j)^T P(y_i - y_j).$$

Proposition 2.3.16. [18] (Cauchy inequality): For any symmetric positive definite matrix $N \in M^{n \times n}$ and $x, y \in \mathcal{R}^n$ we have

$$\pm 2x^T y \leq x^T N x + y^T N^{-1} y.$$

Lemma 2.3.17. For P is an $n \times n$ positive definite matrix, and let B_1, B_2 be an $n \times n$ any real matrices, $U = NI_N - E_N$, $x = (x_1^T, \dots, x_N^T)^T$, and $y = (y_1^T, \dots, y_N^T)^T$ are defined as Lemma 2.3.15. Then, for $x = (x_1^T, \dots, x_N^T)^T$ and $y = (y_1^T, \dots, y_N^T)^T$ with appropriate dimensions, the following holds:

$$\pm 2x^T(U \otimes B_1 B_2)y \leq x^T(U \otimes B_1 P^{-1} B_1^T)x + y^T(U \otimes B_2^T P B_2)y$$

Proof: By Lemma 2.3.16, we have

$$\pm 2x^T B_1 B_2 y \leq x^T B_1 P^{-1} B_1^T x + y^T B_2^T P B_2 y,$$

where $P \in \mathcal{R}^{n \times n}$ is positive definite matrix, $x, y \in \mathcal{R}^n$, and B_1, B_2 are any real matrices, we can obtain that:

$$\begin{aligned} \pm 2x^T(U \otimes B_1 B_2)y &= \pm 2 \sum_{1 \leq i < j \leq N} (x_i - x_j)^T B_1 B_2 (y_i - y_j), \\ &\leq x^T(U \otimes B_1 P^{-1} B_1^T)x + y^T(U \otimes B_2^T P B_2)y. \end{aligned}$$

□

Proposition 2.3.18. [18](Jensen's inequality): For any symmetric positive definite matrix $M > 0$, scalar $\gamma > 0$ and vector function $\omega : [0, \gamma] \rightarrow \mathcal{R}^n$ such that the integrations concerned are well defined, the following inequality holds

$$\left(\int_0^\gamma \omega(s) ds \right)^T M \left(\int_0^\gamma \omega(s) ds \right) \leq \gamma \left(\int_0^\gamma \omega^T(s) M \omega(s) ds \right).$$

Lemma 2.3.19. [70]: For any constant symmetric matrix $M \in \mathcal{R}^{n \times n}$, $M = M^T > 0$, $0 \leq h_m \leq h(t) \leq h_M$, $t \geq 0$, and any differentiable vector function $x(t) \in \mathcal{R}^n$, we have

$$\begin{aligned} (a) \quad & \left(\int_{t-h_m}^t \dot{x}(s) ds \right)^T M \left(\int_{t-h_m}^t \dot{x}(s) ds \right) \leq h_m \int_{t-h_m}^t \dot{x}^T(s) M \dot{x}(s) ds, \\ (b) \quad & \left(\int_{t-h(t)}^{t-h_m} \dot{x}(s) ds \right)^T M \left(\int_{t-h(t)}^{t-h_m} \dot{x}(s) ds \right) \leq (h(t) - h_m) \int_{t-h(t)}^{t-h_m} \dot{x}^T(s) M \dot{x}(s) ds \\ & \leq (h_M - h_m) \int_{t-h(t)}^{t-h_m} \dot{x}^T(s) M \dot{x}(s) ds. \end{aligned}$$

Lemma 2.3.20. [70]: For any constant symmetric matrix $M > 0$, scalar $h > 0$, and vector function $\dot{x}(t) : [-h, 0] \rightarrow \mathcal{R}^n$ such that the following integral is well defined, then

$$-h \int_{t-h}^t \dot{x}^T(s) R \dot{x}(s) ds \leq \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}^T \begin{bmatrix} -M & M \\ M & -M \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}.$$

Lemma 2.3.21. [34]: Let M be a positive semi-definite matrix, $\alpha(\cdot) : (-\infty, a] \rightarrow [0, +\infty)$ be a scalar function and $F(\cdot) : (-\infty, a] \rightarrow \mathcal{R}^n$ be a vector function. If the integrations concerned are well defined, the following inequality holds:

$$\left(\int_{-\infty}^a \alpha(s) F(s) ds \right)^T M \left(\int_{-\infty}^a \alpha(s) F(s) ds \right) \leq \int_{-\infty}^a \alpha(s) ds \int_{-\infty}^a \alpha(s) F^T(s) M F(s) ds.$$

Lemma 2.3.22. [18] (Schur complement lemma): Given constant symmetric matrices X, Y, Z with appropriate dimensions satisfying $X = X^T, Y = Y^T > 0$. Then $X + Z^T Y^{-1} Z < 0$ if and only if

$$\begin{bmatrix} X & Z^T \\ Z & -Y \end{bmatrix} < 0 \quad \text{or} \quad \begin{bmatrix} -Y & Z \\ Z^T & X \end{bmatrix} < 0.$$

Lemma 2.3.23. [61]: Given matrices $Q = Q^T$, H , E and $M = M^T > 0$ with appropriate dimensions. Then

$$Q + HFE + E^T F^T H^T < 0,$$

for all F satisfying $F^T F \leq M$, if and only if there exists an $\epsilon > 0$ such that

$$Q + \epsilon HH^T + \epsilon^{-1} E^T M E < 0.$$

Lemma 2.3.24. [18]: Given a positive definite matrix $Q \in \mathcal{R}^{n \times n}$ and $x \in \mathcal{R}^n$, then

$$\lambda_m(Q) x^T x \leq x^T Q x \leq \lambda_M(Q) x^T x$$

where $\lambda_M(Q) = \max\{\text{Re}\lambda : \lambda \in \lambda(Q)\}$ and $\lambda_m(Q) = \min\{\text{Re}\lambda : \lambda \in \lambda(Q)\}$.

2.3.4 Synchronization

Consider a dynamical system described by

$$\dot{x} = f(t, x(t)), \quad (2.20)$$

$$\dot{y} = g(t, x(t), y(t)), \quad (2.21)$$

where $x, y \in \mathcal{R}^n$, $f : \mathcal{R}^+ \times \mathcal{R}^n \rightarrow \mathcal{R}^n$ and $g : \mathcal{R}^+ \times \mathcal{R}^n \times \mathcal{R}^n \rightarrow \mathcal{R}^n$ are assumed to be analytic functions.

Let $x(t, x_0)$ and $y(t, x_0, y_0)$ be solutions to (2.20) and (2.21) respectively. The solutions $x(t, x_0)$ and $y(t, x_0, y_0)$ are said to be *synchronized* if

$$\lim_{t \rightarrow \infty} \|x(t, x_0) - y(t, x_0, y_0)\| = 0.$$

Definition 2.3.13. [44]: System (2.3) is said to be asymptotically synchronized if the following holds:

$$\lim_{t \rightarrow \infty} \|x_i(t) - x_j(t)\| = 0, \quad i, j = 1, 2, \dots, N.$$

Definition 2.3.14. [10]: System (2.3) is said to be exponentially synchronized if there exist two constants $\alpha > 0$ and $M > 0$ such that for all $\phi_{i0}(s)$ ($i = 1, 2, \dots, N$) and for sufficiently large $T > 0$:

$$\|x_i(t) - x_j(t)\| \leq M e^{-\alpha t},$$

for all $t > T$, $i, j = 1, 2, \dots, N$.

Definition 2.3.15. [63]: The complex dynamical networks (2.4) is said to be synchronized if

$$x_1(t) = x_2(t) = \dots = s(t) \quad \text{as } t \rightarrow \infty, \quad (2.22)$$

where $s(t)$ is a solution of an isolated node, satisfying

$$\dot{s}(t) = f(s(t), s(t - h(t))).$$

2.3.5 Numerical analysis

Fourth-Order Runge-Kutta Method

In order to solve an initial-value problem

$$\frac{dx}{dt} = f(t, x), \quad x(t_0) = x_0,$$

where $x = [x_1, x_2, \dots, x_n]^T$ and $f = [f_1, f_2, \dots, f_n]^T$.

The best known Runge-Kutta method of the first stage and fourth order is given by

$$X_{i+1} = X_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where

$$\begin{aligned}k_1 &= hf(t_i, X_i), \\k_2 &= hf(t_i + \frac{h}{2}, X_i + \frac{k_1}{2}), \\k_3 &= hf(t_i + \frac{h}{2}, X_i + \frac{k_2}{2}), \\k_4 &= hf(t_i + h, X_i + k_3),\end{aligned}$$

where X_i is an approximation of $x(t_i)$ when $X_i = [X_{i1}, X_{i2}, \dots, X_{in}]^T$, $t_i = t_0 + ih$, h is step size and $k_i = [k_{i1}, k_{i2}, \dots, k_{in}]^T$, $\forall i = 1, \dots, 4$.