

CHAPTER 3

Global Synchronization of Coupled Delayed Neural Networks with Hybrid Couplings

In this chapter, we make the great efforts to investigate the global synchronization of hybrid coupled NNs with interval time-varying and unbounded distributed delays in the case of constant and interval time-varying delay couplings. The designed controllers ensure the global synchronization via feedback and sampled-data feedback controls. By constructing an augmented Lyapunov-Krasovskii functional and Kronecker product properties, novel sufficient conditions are obtained in terms of LMIs. The criteria are derived in terms of LMIs solved efficiently by standard convex optimization algorithms. The feasibility of the conditions can be easily checked by resorting to Matlab LMI Toolbox. Finally, the effectiveness of the proposed criteria can be demonstrated by utilizing numerical examples.

3.1 Problem Formulation

Consider the coupled dynamical system consisting of N linearly and diffusively coupled recurrent NNs with interval time-varying and unbounded distributed delays can be described by:

$$\left\{ \begin{array}{l} \dot{x}_i(t) = -Cx_i(t) + Af(x_i(t)) + Bg(x_i(t - \tau(t))) \\ \quad + D \int_{-\infty}^t k(t-s)h(x_i(s))ds + EU_i(t) + \sum_{j=1}^N g_{ij}^1 \Gamma_1 x_j(t) \\ \quad + \sum_{j=1}^N g_{ij}^2 \Gamma_2 x_j(t - \tau(t)), \\ i = 1, 2, \dots, N, \end{array} \right. \quad (3.1)$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \dots, x_{in}(t))^T \in R^n$ is the state vector of the i th network at time t . $C = \text{diag}(c_1, c_2, \dots, c_n) > 0$ denotes the rate with which the cell i resets its potential to the resting state when being isolated from other cells and inputs, A, B, D and E represent the connection weight matrix, $G^q = (g_{ij}^q)_{N \times N}$, ($q = 1, 2$) represent the outer-coupling matrix of the network, $\Gamma_1, \Gamma_2 \in R^{n \times n}$ represent a constant inner-coupling matrix, $f(x_i(t)) = (f_1(x_{i1}(t)), \dots, f_n(x_{in}(t)))^T$, $g(x_i(t - \tau(t))) = (g_1(x_{i1}(t - \tau(t))), \dots, g_n(x_{in}(t - \tau(t))))^T$.

$\tau(t)))^T$, and $h(x_i(s)) = (h_1(x_{i1}(s)), \dots, h_n(x_{in}(s)))^T$ represent the activation function vectors, and $U_i(t) \in R^n$ represent the control input vector.

The interval time-varying delay $\tau(t)$ are assumed to satisfy the following conditions:

$$0 \leq \tau_1 \leq \tau(t) \leq \tau_2, \quad \dot{\tau}(t) \leq \mu < \infty, \quad \eta = \tau_2 - \tau_1,$$

and the delay kernel: $k(\cdot) : [0, +\infty) \rightarrow [0, +\infty)$.

We design the feedback controller satisfy the following conditions:

$$U_i(t) = Kx_i(t), \quad t \geq 0, \quad (3.2)$$

where $K \in R^{n \times n}$ is the feedback controller gain matrix to be determined.

Remark 3.1.1. In the coupled NNs (3.1), we have studied the unbounded distributed delay $\int_{-\infty}^t k(t-s)h(x_i(s))ds$. It would have a great important practical applications on the stability of the total coupled system, due to the time-delay s can vary from $-\infty$ to t in a distributed way [34, 58, 71].

Let

$$\begin{aligned} x^T(t) &= [x_1^T(t), \dots, x_N^T(t)], \\ f^T(x(t)) &= [f^T(x_1(t)), \dots, f^T(x_N(t))], \\ g^T(x(t-\tau(t))) &= [g^T(x_1(t-\tau(t))), \dots, g^T(x_N(t-\tau(t)))], \\ h^T(x(s)) &= [h^T(x_1(s)), \dots, h^T(x_N(s))], \\ U^T(t) &= [U_1^T(t), \dots, U_N^T(t)]^T. \end{aligned}$$

By employing the Kronecker product, we can reformulate the system (3.1) as the following form:

$$\begin{cases} \dot{x}(t) = -(I_N \otimes C)x(t) + (I_N \otimes A)f(x(t)) + (I_N \otimes B)g(x(t-\tau(t))) \\ \quad + (I_N \otimes D) \int_{-\infty}^t k(t-s)h(x(s))ds + (I_N \otimes E)U(t) \\ \quad + (G^1 \otimes \Gamma_1)x(t) + (G^2 \otimes \Gamma_2)x(t-\tau(t)), \end{cases} \quad (3.3)$$

and its identical Kronecker product from is

$$U(t) = (I_N \otimes K)x(t).$$

We introduce the following assumptions:

Assumption 3.1 [34], the outer-coupling configuration matrices of the NNs satisfy:

$$g_{ij}^q = g_{ji}^q \geq 0, \quad g_{ii}^q = - \sum_{j=1, j \neq i}^N g_{ij}^q,$$

where $i, j = 1, 2, \dots, N$, $i \neq j$ and $q = 1, 2$.

Assumption 3.2 [34], for some constants $l_r^-, l_r^+, \sigma_r^-, \sigma_r^+, v_r^-, v_r^+$, the neuron activation functions are bounded, and satisfy:

$$\begin{aligned} l_r^- &\leq \frac{f_r(s_1) - f_r(s_2)}{s_1 - s_2} \leq l_r^+, \\ \sigma_r^- &\leq \frac{g_r(s_1) - g_r(s_2)}{s_1 - s_2} \leq \sigma_r^+, \\ v_r^- &\leq \frac{h_r(s_1) - h_r(s_2)}{s_1 - s_2} \leq v_r^+, \end{aligned}$$

where $r \in \{1, 2, \dots, n\}$.

We denote

$$\begin{aligned} F_1 &= \text{diag}(l_1^+ l_1^-, \dots, l_n^+ l_n^-), \quad F_2 = \text{diag}\left(\frac{l_1^+ + l_1^-}{2}, \dots, \frac{l_n^+ + l_n^-}{2}\right), \\ J_1 &= \text{diag}(\sigma_1^+ \sigma_1^-, \dots, \sigma_n^+ \sigma_n^-), \quad J_2 = \text{diag}\left(\frac{\sigma_1^+ + \sigma_1^-}{2}, \dots, \frac{\sigma_n^+ + \sigma_n^-}{2}\right), \\ L_1 &= \text{diag}(v_1^+ v_1^-, \dots, v_n^+ v_n^-), \quad L_2 = \text{diag}\left(\frac{v_1^+ + v_1^-}{2}, \dots, \frac{v_n^+ + v_n^-}{2}\right). \end{aligned}$$

Assumption 3.3 [34], the delay kernel $k(\cdot) : [0, +\infty) \rightarrow [0, +\infty)$ is continuous, integrable and satisfies:

$$\int_0^{+\infty} k(s) \, ds < +\infty,$$

and

$$\int_0^{+\infty} s k(s) \, ds < +\infty.$$

3.2 Synchronization Criteria with Feedback Control

In this section, we will present the global synchronization of coupled NNs (3.3) in terms of LMIs by using the Lyapunov-Krasovskii functionals and some Kronecker product techniques. The feedback controller is designed to achieve the global synchronization of coupled NNs (3.3). Finally, an example is given to show the effectiveness of the results obtained in theorem.

Theorem 3.2.1. *Assume that assumptions 3.1-3.3 hold. The system (3.1) is globally asymptotically synchronized under the controller (3.2), if there exist positive definite matrices $P, Q_1, Q_2, Q_3, Q_4, R, S_1, S_2, S_3, Y$, three diagonal matrices $\Lambda = \text{diag}(\iota_1, \dots, \iota_n) > 0$, $\Omega = \text{diag}(\kappa_1, \dots, \kappa_n) > 0$, $\Delta = \text{diag}(\eta_1, \dots, \eta_n) > 0$ and any real matrices T_1, T_2 with*

appropriate dimensions, such that the following linear matrix inequalities (LMIs) hold for all $1 = i < j = N$:

$$\begin{aligned}\Sigma_{1ij} &= \mathcal{W}_{ij} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & -I & 0 \end{bmatrix}^T \\ &\quad \times S_3 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & -I & 0 \end{bmatrix} < 0,\end{aligned}\quad (3.4)$$

$$\begin{aligned}\Sigma_{2ij} &= \mathcal{W}_{ij} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -I & I & 0 & 0 \end{bmatrix}^T \\ &\quad \times S_3 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -I & I & 0 & 0 \end{bmatrix} < 0,\end{aligned}\quad (3.5)$$

$$\Sigma_3 = \begin{bmatrix} -0.1(S_1 + S_2) & 3Y^T & PE & T_1E \\ * & -3P & 0 & 0 \\ * & * & -P & 0 \\ * & * & * & -P \end{bmatrix} < 0,\quad (3.6)$$

$$\Sigma_4 = \begin{bmatrix} -T_2 & T_2E \\ * & -P \end{bmatrix} < 0,\quad (3.7)$$

$$\mathcal{W}_{ij} = \begin{bmatrix} W_{1,1} & W_{1,2} & W_{1,3} & W_{1,4} & \Omega J_2 & \Delta L_2 & W_{1,7} & S_1 & S_2 & W_{1,10} \\ * & W_{2,2} & T_2B & T_2A & 0 & 0 & W_{2,7} & 0 & 0 & T_2D \\ * & * & W_{3,3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\Lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & W_{5,5} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \bar{k}R - \Delta & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & W_{7,7} & 0 & 0 & 0 \\ * & * & * & * & * & * & * & W_{8,8} & 0 & 0 \\ * & * & * & * & * & * & * & * & W_{9,9} & 0 \\ * & * & * & * & * & * & * & * & * & -\frac{1}{\bar{k}}R \end{bmatrix},$$

where $\bar{k} = \int_0^{+\infty} k(s) ds$ and

$$\begin{aligned}W_{1,1} &= -PC - C^TP - Ng_{ij}^1 P \Gamma_1 - Ng_{ij}^1 \Gamma_1^T P - T_1C - C^T T_1^T \\ &\quad - Ng_{ij}^1 T_1 \Gamma_1 - Ng_{ij}^1 \Gamma_1^T T_1^T + \sum_{i=1}^3 Q_i - 0.9S_1 - 0.9S_2 \\ &\quad - \Lambda F_1 - \Omega J_1 - \Delta L_1,\end{aligned}$$

$$W_{1,2} = -T_1 - C^T T_2^T - Ng_{ij}^1 \Gamma_1^T T_2^T,$$

$$W_{1,3} = (P + T_1)B,$$

$$W_{1,4} = (P + T_1)A + \Lambda F_2,$$

$$W_{1,7} = -Ng_{ij}^2 P \Gamma_2 - Ng_{ij}^2 T_1 \Gamma_2,$$

$$\begin{aligned}
W_{1,10} &= (P + T_1)D, \\
W_{2,2} &= \tau_1^2 S_1 + \tau_2^2 S_2 + \eta^2 S_3 - T_2, \\
W_{2,7} &= -N g_{ij}^2 T_2 \Gamma_2, \\
W_{3,3} &= -(1 - \mu)Q_4, \\
W_{5,5} &= Q_4 - \Omega, \\
W_{7,7} &= -(1 - \mu)Q_1 - 2S_3, \\
W_{8,8} &= -Q_2 - S_1 - S_3, \\
W_{9,9} &= -Q_3 - S_2 - S_3.
\end{aligned}$$

Moreover the memoryless feedback control is $K = -P^{-1}Y$.

Proof : Consider the following Lyapunov-Krasovskii functional candidate for the system (3.3):

$$V(t, x_t) = V_1(t, x_t) + V_2(t, x_t) + V_3(t, x_t) + V_4(t, x_t) + V_5(t, x_t), \quad (3.8)$$

where

$$\begin{aligned}
V_1(t, x_t) &= x^T(t)(U \otimes P)x(t), \\
V_2(t, x_t) &= \int_{t-\tau(t)}^t x^T(s)(U \otimes Q_1)x(s) ds \\
&\quad + \int_{t-\tau_1}^t x^T(s)(U \otimes Q_2)x(s) ds \\
&\quad + \int_{t-\tau_2}^t x^T(s)(U \otimes Q_3)x(s) ds \\
&\quad + \int_{t-\tau(t)}^t g^T(x(s))(U \otimes Q_4)g(x(s)) ds, \\
V_3(t, x_t) &= \tau_1 \int_{-\tau_1}^0 \int_{t+s}^t \dot{x}^T(\theta)(U \otimes S_1)\dot{x}(\theta) d\theta ds \\
&\quad + \tau_2 \int_{-\tau_2}^0 \int_{t+s}^t \dot{x}^T(\theta)(U \otimes S_2)\dot{x}(\theta) d\theta ds, \\
V_4(t, x_t) &= \eta \int_{-\tau_2}^{-\tau_1} \int_{t+s}^t \dot{x}^T(\theta)(U \otimes S_3)\dot{x}(\theta) d\theta ds, \\
V_5(t, x_t) &= \int_0^{+\infty} k(s) \int_{t-s}^t h^T(x(\theta))(U \otimes R)h(x(\theta)) d\theta ds.
\end{aligned}$$

Calculating the time derivative of $V(t, x_t)$ along the trajectories of system (3.3), define a

matrix, if X is a matrix with zero column sums, then $UX = NX$, we obtains

$$\begin{aligned}
\dot{V}_1(t, x_t) &= 2x^T(t)(U \otimes P)\dot{x}(t), \\
&= 2x^T(t)(U \otimes P) \left[- (I_N \otimes C)x(t) + (I_N \otimes A)f(x(t)) \right. \\
&\quad + (I_N \otimes B)g(x(t - \tau(t))) + (I_N \otimes D) \int_{-\infty}^t k(t-s)h(x(s))ds \\
&\quad \left. + (I_N \otimes EK)x(t) + (G^1 \otimes \Gamma_1)x(t) + (G^2 \otimes \Gamma_2)x(t - \tau(t)) \right], \\
&= 2 \sum_{1 \leq i < j \leq N}^N (x_i(t) - x_j(t))^T \left[(-PC - Ng_{ij}^1 P\Gamma_1 + PEK)(x_i(t) - x_j(t)) \right. \\
&\quad + PA(f(x_i(t)) - f(x_j(t))) + PB(g(x_i(t - \tau(t))) - g(x_j(t - \tau(t)))) \\
&\quad + PD \int_{-\infty}^t k(t-s)(h(x_i(s)) - h(x_j(s)))ds \\
&\quad \left. - Ng_{ij}^2 P\Gamma_2(x_i(t - \tau(t)) - x_j(t - \tau(t))) \right]. \tag{3.9}
\end{aligned}$$

Using Lemma 2.3.17, we get

$$\begin{aligned}
&- \sum_{1 \leq i < j \leq N}^N 2(x_i(t) - x_j(t))^T PEK(x_i(t) - x_j(t)) \\
&= - \sum_{1 \leq i < j \leq N}^N 2(x_i(t) - x_j(t))^T PEP^{-1}Y(x_i(t) - x_j(t)), \\
&\leq \sum_{1 \leq i < j \leq N}^N (x_i(t) - x_j(t))^T \left[PEP^{-1}E^T P^T + Y^T P^{-1}Y \right] (x_i(t) - x_j(t)). \tag{3.10}
\end{aligned}$$

$$\begin{aligned}
\dot{V}_2(t, x_t) &\leq x^T(t)(U \otimes Q_1)x(t) - (1 - \mu)x^T(t - \tau(t))(U \otimes Q_1)x(t - \tau(t)) \\
&\quad + x^T(t)(U \otimes Q_2)x(t) - x^T(t - \tau_1)(U \otimes Q_2)x(t - \tau_1) \\
&\quad + x^T(t)(U \otimes Q_3)x(t) - x^T(t - \tau_2)(U \otimes Q_3)x(t - \tau_2) \\
&\quad + g^T(x(t))(U \otimes Q_4)g(x(t)) \\
&\quad - (1 - \mu)g^T(x(t - \tau(t)))(U \otimes Q_4)g(x(t - \tau(t))), \\
&= \sum_{1 \leq i < j \leq N}^N \left[(x_i(t) - x_j(t))^T (Q_1 + Q_2 + Q_3)(x_i(t) - x_j(t)) \right. \\
&\quad - (1 - \mu)(x_i(t - \tau(t)) - x_j(t - \tau(t)))^T Q_1(x_i(t - \tau(t)) - x_j(t - \tau(t))) \\
&\quad - (x_i(t - \tau_1) - x_j(t - \tau_1))^T Q_2(x_i(t - \tau_1) - x_j(t - \tau_1)) \\
&\quad - (x_i(t - \tau_2) - x_j(t - \tau_2))^T Q_3(x_i(t - \tau_2) - x_j(t - \tau_2)) \\
&\quad + (g(x_i(t)) - g(x_j(t)))^T Q_4(g(x_i(t)) - g(x_j(t - \tau_2))) \\
&\quad - (1 - \mu)(g(x_i(t - \tau(t))) - g(x_j(t - \tau(t))))^T Q_4 \\
&\quad \left. \times (g(x_i(t - \tau(t))) - g(x_j(t - \tau(t)))) \right]. \tag{3.11}
\end{aligned}$$

Calculating the time derivative of $V_3(t, x_t)$ and $V_4(t, x_t)$, respectively, with Lemma 2.3.20, we obtains

$$\begin{aligned}
\dot{V}_3(t, x_t) &= \tau_1^2 \dot{x}^T(t)(U \otimes S_1)\dot{x}(t) - \tau_1 \int_{t-\tau_1}^t \dot{x}^T(s)(U \otimes S_1)\dot{x}(s) ds \\
&\quad + \tau_2^2 \dot{x}^T(t)(U \otimes S_2)\dot{x}(t) - \tau_2 \int_{t-\tau_2}^t \dot{x}^T(s)(U \otimes S_2)\dot{x}(s) ds, \\
&\leq \tau_1^2 \dot{x}^T(t)(U \otimes S_1)\dot{x}(t) + \tau_2^2 \dot{x}^T(t)(U \otimes S_2)\dot{x}(t) \\
&\quad - \left[x(t) - x(t - \tau_1) \right]^T (U \otimes S_1) \left[x(t) - x(t - \tau_1) \right] \\
&\quad - \left[x(t) - x(t - \tau_2) \right]^T (U \otimes S_2) \left[x(t) - x(t - \tau_2) \right], \\
&= \sum_{1 \leq i < j \leq N} \left\{ (\dot{x}_i(t) - \dot{x}_j(t))^T (\tau_1^2 S_1 + \tau_2^2 S_2) (\dot{x}_i(t) - \dot{x}_j(t)) \right. \\
&\quad - \left[(x_i(t) - x_j(t)) - (x_i(t - \tau_1) - x_j(t - \tau_1)) \right]^T S_1 \\
&\quad \times \left[(x_i(t) - x_j(t)) - (x_i(t - \tau_1) - x_j(t - \tau_1)) \right] \\
&\quad - \left[(x_i(t) - x_j(t)) - (x_i(t - \tau_2) - x_j(t - \tau_2)) \right]^T S_2 \\
&\quad \times \left. \left[(x_i(t) - x_j(t)) - (x_i(t - \tau_2) - x_j(t - \tau_2)) \right] \right\}. \tag{3.12}
\end{aligned}$$

$$\dot{V}_4(t, x_t) = \eta^2 \dot{x}^T(t)(U \otimes S_3)\dot{x}(t) - \eta \int_{t-\tau_2}^{t-\tau_1} \dot{x}^T(s)(U \otimes S_3)\dot{x}(s) ds, \tag{3.13}$$

let $\eta = (\tau_2 - \tau(t)) + (\tau(t) - \tau_1)$, we can acquire

$$\begin{aligned}
-\eta \int_{t-\tau_2}^{t-\tau_1} \dot{x}^T(s)(U \otimes S_3)\dot{x}(s) ds &= -(\tau_2 - \tau(t)) \int_{t-\tau_2}^{t-\tau(t)} \dot{x}^T(s)(U \otimes S_3)\dot{x}(s) ds \\
&\quad - (\tau(t) - \tau_1) \int_{t-\tau_2}^{t-\tau(t)} \dot{x}^T(s)(U \otimes S_3)\dot{x}(s) ds \\
&\quad - (\tau_2 - \tau(t)) \int_{t-\tau(t)}^{t-\tau_1} \dot{x}^T(s)(U \otimes S_3)\dot{x}(s) ds \\
&\quad - (\tau(t) - \tau_1) \int_{t-\tau(t)}^{t-\tau_1} \dot{x}^T(s)(U \otimes S_3)\dot{x}(s) ds.
\end{aligned}$$

Using Lemma 2.3.20, we get

$$\begin{aligned}
&-(\tau_2 - \tau(t)) \int_{t-\tau_2}^{t-\tau(t)} \dot{x}^T(s)(U \otimes S_3)\dot{x}(s) ds \\
&\leq - \left[x(t - \tau(t)) - x(t - \tau_2) \right]^T (U \otimes S_3) \left[x(t - \tau(t)) - x(t - \tau_2) \right], \tag{3.14}
\end{aligned}$$

$$\begin{aligned}
&-(\tau(t) - \tau_1) \int_{t-\tau(t)}^{t-\tau_1} \dot{x}^T(s)(U \otimes S_3)\dot{x}(s) ds \\
&\leq - \left[x(t - \tau_1) - x(t - \tau(t)) \right]^T (U \otimes S_3) \left[x(t - \tau_1) - x(t - \tau(t)) \right]. \tag{3.15}
\end{aligned}$$

Let $\beta = \frac{\tau_2 - \tau(t)}{\tau_2 - \tau_1} \leq 1$. Then

$$\begin{aligned}
& -(\tau_2 - \tau(t)) \int_{t-\tau(t)}^{t-\tau_1} \dot{x}^T(s)(U \otimes S_3) \dot{x}(s) ds \\
& = -\beta(\tau_2 - \tau_1) \int_{t-\tau(t)}^{t-\tau_1} \dot{x}^T(s)(U \otimes S_3) \dot{x}(s) ds, \\
& \leq -\beta(\tau(t) - \tau_1) \int_{t-\tau(t)}^{t-\tau_1} \dot{x}^T(s)(U \otimes S_3) \dot{x}(s) ds, \\
& \leq -\beta \left[x(t - \tau_1) - x(t - \tau(t)) \right]^T (U \otimes R_3) \left[x(t - \tau_1) - x(t - \tau(t)) \right], \quad (3.16) \\
& -(\tau(t) - \tau_1) \int_{t-\tau_2}^{t-\tau(t)} \dot{x}^T(s)(U \otimes S_3) \dot{x}(s) ds \\
& = -(1 - \beta)(\tau_2 - \tau_1) \int_{t-\tau_2}^{t-\tau(t)} \dot{x}^T(s)(U \otimes S_3) \dot{x}(s) ds, \\
& \leq -(1 - \beta)(\tau_2 - \tau(t)) \int_{t-\tau_2}^{t-\tau(t)} \dot{x}^T(s)(U \otimes S_3) \dot{x}(s) ds, \\
& \leq -(1 - \beta) \left[x(t - \tau(t)) - x(t - \tau_2) \right]^T (U \otimes S_3) \\
& \quad \times \left[x(t - \tau(t)) - x(t - \tau_2) \right]. \quad (3.17)
\end{aligned}$$

Therefore from (3.13) to (3.17), we can estimate $\dot{V}_4(x_t)$ as

$$\begin{aligned}
\dot{V}_4(t, x_t) & \leq \sum_{1 \leq i < j \leq N}^N \left\{ \eta^2 (\dot{x}_i(t) - \dot{x}_j(t))^T S_3 (\dot{x}_i(t) - \dot{x}_j(t)) \right. \\
& \quad - \left[(x_i(t - \tau(t)) - x_j(t - \tau(t))) - (x_i(t - \tau_2) - x_j(t - \tau_2)) \right]^T S_3 \\
& \quad \times \left[(x_i(t - \tau(t)) - x_j(t - \tau(t))) - (x_i(t - \tau_2) - x_j(t - \tau_2)) \right] \\
& \quad - \left[(x_i(t - \tau_1) - x_j(t - \tau_1)) - (x_i(t - \tau(t)) - x_j(t - \tau(t))) \right]^T S_3 \\
& \quad \times \left[(x_i(t - \tau_1) - x_j(t - \tau_1)) - (x_i(t - \tau(t)) - x_j(t - \tau(t))) \right] \\
& \quad - \beta \left[(x_i(t - \tau_1) - x_j(t - \tau_1)) - (x_i(t - \tau(t)) - x_j(t - \tau(t))) \right]^T S_3 \\
& \quad \times \left[(x_i(t - \tau_1) - x_j(t - \tau_1)) - (x_i(t - \tau(t)) - x_j(t - \tau(t))) \right] \\
& \quad \left. - (1 - \beta) \left[(x_i(t - \tau(t)) - x_j(t - \tau(t))) - (x_i(t - \tau_2) - x_j(t - \tau_2)) \right]^T S_3 \right. \\
& \quad \left. \times \left[(x_i(t - \tau(t)) - x_j(t - \tau(t))) - (x_i(t - \tau_2) - x_j(t - \tau_2)) \right] \right\}. \quad (3.18)
\end{aligned}$$

Similarly, calculating the time derivative of $V_5(t, x_t)$, with Lemma 2.3.21, we obtain

$$\begin{aligned}
\dot{V}_5(t, x_t) & = \int_0^{+\infty} k(s) ds \, h^T(x(t))(U \otimes R) h(x(t)) \\
& \quad - \int_0^{+\infty} k(s) h^T(x(t - s))(U \otimes R) h(x(t - s)) ds,
\end{aligned}$$

$$\begin{aligned}
\dot{V}_5(t, x_t) &= \bar{k} h^T(x(t))(U \otimes R)h(x(t)) \\
&\quad - \int_{-\infty}^t k(t-s)h^T(x(s))(U \otimes R)h(x(s))ds, \\
&\leq \bar{k} h^T(x(t))(U \otimes R)h(x(t)) - \frac{1}{\int_{-\infty}^t k(t-s)ds} \\
&\quad \times \left(\int_{-\infty}^t k(t-s)h(x(s))ds \right)^T (U \otimes R) \left(\int_{-\infty}^t k(t-s)h(x_i(s))ds \right), \\
&= \sum_{1 \leq i < j \leq N}^N \left\{ \bar{k} [h(x_i(t)) - h(x_j(t))]^T R [h(x_i(t)) - h(x_j(t))] \right. \\
&\quad - \frac{1}{\bar{k}} \left(\int_{-\infty}^t k(t-s)[h(x_i(s)) - h(x_j(s))]ds \right)^T R \\
&\quad \times \left. \left(\int_{-\infty}^t k(t-s)[h(x_i(s)) - h(x_j(s))]ds \right) \right\}. \tag{3.19}
\end{aligned}$$

By the assumption 3.2, for $r \in \{1, 2, \dots, n\}$, we have

$$\begin{aligned}
&\left(\frac{f_r(x_{ir}(t)) - f_r(x_{jr}(t))}{x_{ir}(t) - x_{jr}(t)} - l_r^+ \right) \left(\frac{f_r(x_{ir}(t)) - f_r(x_{jr}(t))}{x_{ir}(t) - x_{jr}(t)} - l_r^- \right) \leq 0, \\
&\left(\frac{g_r(x_{ir}(t)) - g_r(x_{jr}(t))}{x_{ir}(t) - x_{jr}(t)} - \sigma_r^+ \right) \left(\frac{g_r(x_{ir}(t)) - g_r(x_{jr}(t))}{x_{ir}(t) - x_{jr}(t)} - \sigma_r^- \right) \leq 0, \\
&\left(\frac{h_r(x_{ir}(t)) - h_r(x_{jr}(t))}{x_{ir}(t) - x_{jr}(t)} - v_r^+ \right) \left(\frac{h_r(x_{ir}(t)) - h_r(x_{jr}(t))}{x_{ir}(t) - x_{jr}(t)} - v_r^- \right) \leq 0.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\left((f_r(x_{ir}(t)) - f_r(x_{jr}(t))) - l_r^+(x_{ir}(t) - x_{jr}(t)) \right) \\
&\quad \times \left((f_r(x_{ir}(t)) - f_r(x_{jr}(t))) - l_r^-(x_{ir}(t) - x_{jr}(t)) \right) \leq 0, \\
&\left((g_r(x_{ir}(t)) - g_r(x_{jr}(t))) - \sigma_r^+(x_{ir}(t) - x_{jr}(t)) \right) \\
&\quad \times \left((g_r(x_{ir}(t)) - g_r(x_{jr}(t))) - \sigma_r^-(x_{ir}(t) - x_{jr}(t)) \right) \leq 0, \\
&\left((h_r(x_{ir}(t)) - h_r(x_{jr}(t))) - v_r^+(x_{ir}(t) - x_{jr}(t)) \right) \\
&\quad \times \left((h_r(x_{ir}(t)) - h_r(x_{jr}(t))) - v_r^-(x_{ir}(t) - x_{jr}(t)) \right) \leq 0.
\end{aligned}$$

which are equivalent to

$$\begin{aligned}
&\begin{bmatrix} x_i(t) - x_j(t) \\ f(x_i(t)) - f(x_j(t)) \end{bmatrix}^T \begin{bmatrix} l_r^+ l_r^- e_r e_r^T & -\frac{l_r^+ + l_r^-}{2} e_r e_r^T \\ -\frac{l_r^+ + l_r^-}{2} e_r e_r^T & e_r e_r^T \end{bmatrix} \begin{bmatrix} x_i(t) - x_j(t) \\ f(x_i(t)) - f(x_j(t)) \end{bmatrix} \leq 0, \\
&\begin{bmatrix} x_i(t) - x_j(t) \\ g(x_i(t)) - g(x_j(t)) \end{bmatrix}^T \begin{bmatrix} \sigma_r^+ \sigma_r^- e_r e_r^T & -\frac{\sigma_r^+ + \sigma_r^-}{2} e_r e_r^T \\ -\frac{\sigma_r^+ + \sigma_r^-}{2} e_r e_r^T & e_r e_r^T \end{bmatrix} \begin{bmatrix} x_i(t) - x_j(t) \\ g(x_i(t)) - g(x_j(t)) \end{bmatrix} \leq 0, \\
&\begin{bmatrix} x_i(t) - x_j(t) \\ h(x_i(t)) - h(x_j(t)) \end{bmatrix}^T \begin{bmatrix} v_r^+ v_r^- e_r e_r^T & -\frac{v_r^+ + v_r^-}{2} e_r e_r^T \\ -\frac{v_r^+ + v_r^-}{2} e_r e_r^T & e_r e_r^T \end{bmatrix} \begin{bmatrix} x_i(t) - x_j(t) \\ h(x_i(t)) - h(x_j(t)) \end{bmatrix} \leq 0.
\end{aligned}$$

where e_r denotes the unit column vector having 1 element on its r th row and zeros elsewhere. For any diagonal matrix $\Lambda = \text{diag}(\iota_1, \dots, \iota_n) > 0$, $\Omega = \text{diag}(\kappa_1, \dots, \kappa_n) > 0$, and $\Delta = \text{diag}(\eta_1, \dots, \eta_n) > 0$ with appropriate dimensions, it follows that:

$$\begin{aligned}
0 &\leq -\sum_{r=1}^n \iota_r \begin{bmatrix} x_i(t) - x_j(t) \\ f(x_i(t)) - f(x_j(t)) \end{bmatrix}^T \begin{bmatrix} \iota_r^+ \iota_r^- e_r e_r^T & -\frac{\iota_r^+ + \iota_r^-}{2} e_r e_r^T \\ -\frac{\iota_r^+ + \iota_r^-}{2} e_r e_r^T & e_r e_r^T \end{bmatrix} \\
&\quad \times \begin{bmatrix} x_i(t) - x_j(t) \\ f(x_i(t)) - f(x_j(t)) \end{bmatrix} \\
&\quad - \sum_{r=1}^n \kappa_r \begin{bmatrix} x_i(t) - x_j(t) \\ g(x_i(t)) - g(x_j(t)) \end{bmatrix}^T \begin{bmatrix} \sigma_r^+ \sigma_r^- e_r e_r^T & -\frac{\sigma_r^+ + \sigma_r^-}{2} e_r e_r^T \\ -\frac{\sigma_r^+ + \sigma_r^-}{2} e_r e_r^T & e_r e_r^T \end{bmatrix} \\
&\quad \times \begin{bmatrix} x_i(t) - x_j(t) \\ g(x_i(t)) - g(x_j(t)) \end{bmatrix} \\
&\quad - \sum_{r=1}^n \eta_r \begin{bmatrix} x_i(t) - x_j(t) \\ h(x_i(t)) - h(x_j(t)) \end{bmatrix}^T \begin{bmatrix} \nu_r^+ \nu_r^- e_r e_r^T & -\frac{\nu_r^+ + \nu_r^-}{2} e_r e_r^T \\ -\frac{\nu_r^+ + \nu_r^-}{2} e_r e_r^T & e_r e_r^T \end{bmatrix} \\
&\quad \times \begin{bmatrix} x_i(t) - x_j(t) \\ h(x_i(t)) - h(x_j(t)) \end{bmatrix}, \\
&= \begin{bmatrix} x_i(t) - x_j(t) \\ f(x_i(t)) - f(x_j(t)) \end{bmatrix}^T \begin{bmatrix} -\Lambda F_1 & \Lambda F_2 \\ \Lambda F_2 & -\Lambda \end{bmatrix} \begin{bmatrix} x_i(t) - x_j(t) \\ f(x_i(t)) - f(x_j(t)) \end{bmatrix} \\
&\quad + \begin{bmatrix} x_i(t) - x_j(t) \\ g(x_i(t)) - g(x_j(t)) \end{bmatrix}^T \begin{bmatrix} -\Omega J_1 & \Omega J_2 \\ \Omega J_2 & -\Omega \end{bmatrix} \begin{bmatrix} x_i(t) - x_j(t) \\ g(x_i(t)) - g(x_j(t)) \end{bmatrix} \\
&\quad + \begin{bmatrix} x_i(t) - x_j(t) \\ h(x_i(t)) - h(x_j(t)) \end{bmatrix}^T \begin{bmatrix} -\Delta L_1 & \Delta L_2 \\ \Delta L_2 & -\Delta \end{bmatrix} \begin{bmatrix} x_i(t) - x_j(t) \\ h(x_i(t)) - h(x_j(t)) \end{bmatrix}. \quad (3.20)
\end{aligned}$$

On the other hand, by (3.3) and properties of Kronecker product in Lemma 2.3.14 and Lemma 2.3.15, we have the following zero equations holds for any matrices $T_1, T_2 \in \mathcal{R}^{n \times n}$:

$$\begin{aligned}
0 &= 2 \left[x^T(t)(U \otimes T_1) + \dot{x}^T(t)(U \otimes T_2) \right] \\
&\quad \times \left[-\dot{x}(t) - (I_N \otimes C)x(t) + (I_N \otimes A)f(x(t)) + (I_N \otimes B)g(x(t - \tau(t))) \right. \\
&\quad \left. + (I_N \otimes D) \int_{-\infty}^t k(t-s)h(x(s))ds + (I_N \otimes EK)x(t) + (G^1 \otimes l_1)x(t) \right. \\
&\quad \left. + (G^2 \otimes l_2)x(t - \tau(t)) \right],
\end{aligned}$$

$$\begin{aligned}
&= \sum_{1 \leq i < j \leq N}^N 2(x_i(t) - x_j(t))^T \left[-T_1(\dot{x}_i(t) - \dot{x}_j(t)) + (-T_1 C - N g_{ij}^1 T_1 \Gamma_1 \right. \\
&\quad + T_1 E K)(x_i(t) - x_j(t)) + T_1 A(f(x_i(t)) - f(x_j(t))) + T_1 B(g(x_i(t - \tau(t))) \\
&\quad - g(x_j(t - \tau(t)))) - N g_{ij}^2 T_1 \Gamma_2(x_i(t - \tau(t)) - x_j(t - \tau(t))) \\
&\quad \left. + T_1 D \int_{-\infty}^t k(t-s)(h(x_i(s)) - h(x_j(s))) ds \right] \\
&\quad + \sum_{1 \leq i < j \leq N}^N 2(\dot{x}_i(t) - \dot{x}_j(t))^T \left[-T_2(\dot{x}_i(t) - \dot{x}_j(t)) + (-T_2 C - N g_{ij}^1 T_2 \Gamma_1 \right. \\
&\quad + T_2 E K)(x_i(t) - x_j(t)) + T_2 A(f(x_i(t)) - f(x_j(t))) + T_2 B(g(x_i(t - \tau(t))) \\
&\quad - g(x_j(t - \tau(t)))) - N g_{ij}^2 T_2 \Gamma_2(x_i(t - \tau(t)) - x_j(t - \tau(t))) \\
&\quad \left. + T_2 D \int_{-\infty}^t k(t-s)(h(x_i(s)) - h(x_j(s))) ds \right]. \tag{3.21}
\end{aligned}$$

Using Lemma 2.3.17, we get

$$\begin{aligned}
&- \sum_{1 \leq i < j \leq N}^N 2(x_i(t) - x_j(t))^T T_1 E K (x_i(t) - x_j(t)) \\
&= - \sum_{1 \leq i < j \leq N}^N 2(x_i(t) - x_j(t))^T T_1 E P^{-1} Y (x_i(t) - x_j(t)), \\
&\leq \sum_{1 \leq i < j \leq N}^N (x_i(t) - x_j(t))^T \left[T_1 E P^{-1} E^T T_1^T + Y^T P^{-1} Y \right] \\
&\quad \times (x_i(t) - x_j(t)), \tag{3.22}
\end{aligned}$$

and

$$\begin{aligned}
&- \sum_{1 \leq i < j \leq N}^N 2(\dot{x}_i(t) - \dot{x}_j(t))^T T_2 E K (x_i(t) - x_j(t)) \\
&= - \sum_{1 \leq i < j \leq N}^N 2(\dot{x}_i(t) - \dot{x}_j(t))^T T_2 E P^{-1} Y (x_i(t) - x_j(t)), \\
&\leq \sum_{1 \leq i < j \leq N}^N \left[(\dot{x}_i(t) - \dot{x}_j(t))^T T_2 E P^{-1} E^T T_2^T (\dot{x}_i(t) - \dot{x}_j(t)) \right. \\
&\quad \left. + (x_i(t) - x_j(t))^T Y^T P^{-1} Y (x_i(t) - x_j(t)) \right]. \tag{3.23}
\end{aligned}$$

By setting $\zeta_{ij} = \left[(x_i(t) - x_j(t))^T, (\dot{x}_i(t) - \dot{x}_j(t))^T, (g(x_i(t - \tau(t))) - g(x_j(t - \tau(t))))^T, (f(x_i(t)) - f(x_j(t)))^T, (g(x_i(t)) - g(x_j(t)))^T, (h(x_i(t)) - h(x_j(t)))^T, (x_i(t - \tau(t)) - x_j(t - \tau(t)))^T, (x_i(t - \tau_1) - x_j(t - \tau_1))^T, (x_i(t - \tau_2) - x_j(t - \tau_2))^T, \left(\int_{-\infty}^t k(t-s)(h(x_i(s)) - h(x_j(s))) ds \right)^T \right]^T$.

From (3.9) - (3.23), we can obtain that

$$\begin{aligned}\dot{V}(t, x_t) \leq & \sum_{1 \leq i < j \leq N}^N \left\{ \zeta_{ij}^T(t) [(1 - \beta_i) \Sigma_{1ij} + \beta_i \Sigma_{2ij}] \zeta_{ij}(t) \right. \\ & + (x_i(t) - x_j(t))^T \Pi_1 (x_i(t) - x_j(t)) \\ & \left. + (\dot{x}_i(t) - \dot{x}_j(t))^T \Pi_2 (\dot{x}_i(t) - \dot{x}_j(t)) \right\},\end{aligned}\quad (3.24)$$

where Σ_{1ij} and Σ_{2ij} are defined in (3.4), (3.5), respectively, and

$$\begin{aligned}\Pi_1 &= -0.1S_1 - 0.1S_2 + 3Y^T P^{-1}Y + PEP^{-1}E^T P + T_1 EP^{-1}E^T T_1^T, \\ \Pi_2 &= -T_2 + T_2 EP^{-1}E^T T_2^T.\end{aligned}$$

Since $0 \leq \beta \leq 1$, $(1 - \beta)\Sigma_{1ij} + \beta\Sigma_{2ij}$ is a convex combination of Σ_{1ij} and Σ_{2ij} . So, $(1 - \beta)\Sigma_{1ij} + \beta\Sigma_{2ij} < 0$ is equivalent to $\Sigma_{1ij} < 0$ and $\Sigma_{2ij} < 0$. By Lemma 2.3.22, the inequalities Π_1 and Π_2 are equivalent to $\Sigma_3 < 0$ and $\Sigma_4 < 0$. Thus, it follows from (3.4)-(3.7) and (3.24) such that

$$\lim_{t \rightarrow \infty} \|x_i(t) - x_j(t)\| = 0.$$

According to Definition 2.3.13, we can conclude that system (3.3) is globally asymptotically synchronized. The proof is completed. $\square$

Next, we present a simulation example so as to illustrate the usefulness of our Theorem 3.2.1.

Example 3.2.2. Consider a dynamical system (3.1) consisting of five linearly coupled identical models under the controller (3.2) with a hybrid coupling. The state equations of the entire array are

$$\begin{cases} \dot{x}_i(t) = -Cx_i(t) + Af(x_i(t)) + Bg(x_i(t - \tau(t))) \\ \quad + D \int_{-\infty}^t k(t-s)h(x_i(s))ds + EU_i(t) + \sum_{j=1}^5 g_{ij}^1 \Gamma_1 x_j(t) \\ \quad + \sum_{j=1}^5 g_{ij}^2 \Gamma_2 x_j(t - \tau(t)), \end{cases} \quad (3.25)$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t))^T$, $i = 1, 2, 3, 4, 5$ are the state vector of the i th network, and the other parameters are as follows:

$$\begin{aligned}C &= \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & -0.2 \\ 0.2 & -0.9 \end{bmatrix}, \quad B = \begin{bmatrix} 0.2 & 0.5 \\ 0.2 & -0.1 \end{bmatrix}, \\ D &= \begin{bmatrix} -0.1 & 0 \\ 0 & -0.1 \end{bmatrix}, \quad E = \begin{bmatrix} -0.1 & 0 \\ 0.3 & 0.1 \end{bmatrix}, \quad \Gamma_1 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \\ \Gamma_2 &= \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad \tau_1 = 0.2, \quad \tau_2 = 1.0, \quad \mu = 0.3, \quad k(s) = e^{-s}.\end{aligned}$$

The outer-coupling matrix $G^q = (g_{ij}^q)_{N \times N}$, ($q = 1, 2$) are defined as:

$$g^1 = \begin{bmatrix} -2 & 1 & 0 & 0 & 1 \\ 1 & -3 & 1 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 1 & 1 & -3 & 1 \\ 1 & 0 & 0 & 1 & -2 \end{bmatrix}, \quad g^2 = \begin{bmatrix} -4 & 1 & 1 & 1 & 1 \\ 1 & -4 & 1 & 1 & 1 \\ 1 & 1 & -4 & 1 & 1 \\ 1 & 1 & 1 & -4 & 1 \\ 1 & 1 & 1 & 1 & -4 \end{bmatrix}.$$

Take the activation function as follows:

$$\begin{aligned} f_1(x_{i1}(t)) &= 0.5(|x_{i1}(t) + 1| - |x_{i1}(t) - 1|), \\ f_2(x_{i2}(t)) &= 0.25(|x_{i2}(t) + 1| - |x_{i2}(t) - 1|), \\ g_1(x_{i1}(s)) &= \tanh(-x_{i1}(s)), \\ g_2(x_{i2}(s)) &= \tanh(-0.8x_{i2}(s)), \\ h_1(x_{i1}(s)) &= \tanh(-x_{i1}(s)), \\ h_2(x_{i2}(s)) &= \tanh(-0.8x_{i2}(s)). \end{aligned}$$

It is easy to see:

$$\begin{aligned} F_1 &= \begin{bmatrix} -1 & 0 \\ 0 & -0.25 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \\ J_1 &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad J_2 = \begin{bmatrix} -0.5 & 0 \\ 0 & -0.4 \end{bmatrix}, \\ L_1 &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad L_2 = \begin{bmatrix} -0.5 & 0 \\ 0 & -0.4 \end{bmatrix}. \end{aligned}$$

By applying the MATLAB LMI Toolbox in Theorem 3.2.1, we obtain the solution set as follows:

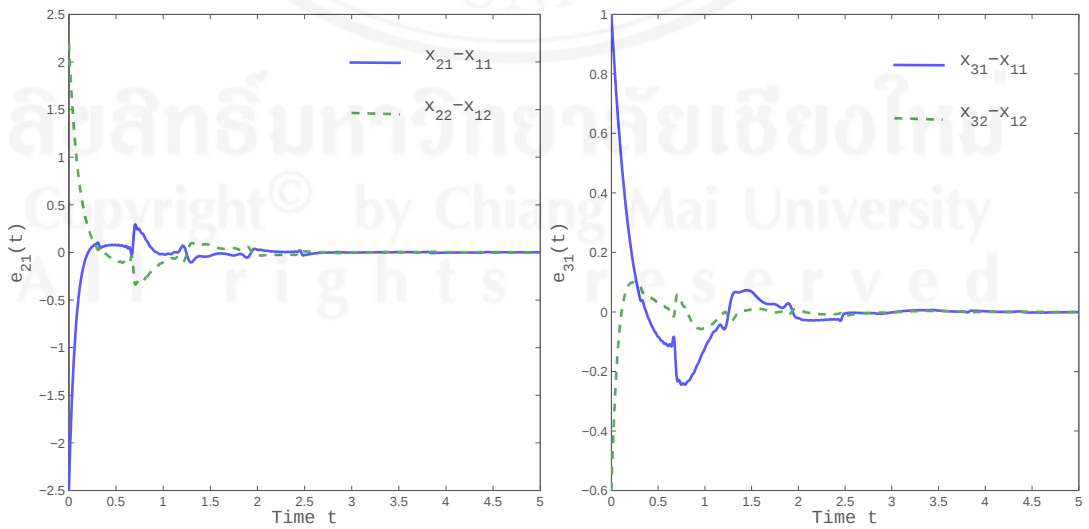
$$\begin{aligned} P &= \begin{bmatrix} 0.1242 & 0.0106 \\ 0.0106 & 0.0751 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 0.1789 & 0.0137 \\ 0.0137 & 0.1035 \end{bmatrix}, \\ Q_2 &= \begin{bmatrix} 0.0631 & 0.0032 \\ 0.0032 & 0.0230 \end{bmatrix}, \quad Q_3 = \begin{bmatrix} 0.0786 & 0.0022 \\ 0.0022 & 0.0350 \end{bmatrix}, \\ Q_4 &= \begin{bmatrix} 0.0855 & 0.0064 \\ 0.0064 & 0.0809 \end{bmatrix}, \quad R = \begin{bmatrix} 0.0687 & 0.0010 \\ 0.0010 & 0.0545 \end{bmatrix}, \\ S_1 &= \begin{bmatrix} 0.02475 & -0.0095 \\ -0.0095 & 0.0451 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0.0001 & 2.853 \times 10^{-5} \\ 2.853 \times 10^{-5} & 5.831 \times 10^{-5} \end{bmatrix}, \end{aligned}$$

$$\begin{aligned}
S_3 &= \begin{bmatrix} 0.0003 & 8.179 \times 10^{-5} \\ 8.179 \times 10^{-5} & 0.0001 \end{bmatrix}, & T_1 &= \begin{bmatrix} -0.0433 & -0.0063 \\ -0.0025 & -0.0106 \end{bmatrix}, \\
T_2 &= \begin{bmatrix} 0.0095 & 0.0012 \\ 0.0014 & 0.0055 \end{bmatrix}, & Y &= \begin{bmatrix} 0.0047 & 0.0012 \\ 0.0011 & 0.0015 \end{bmatrix}, \\
\Lambda &= \begin{bmatrix} 0.1524 & 0 \\ 0 & 0.1576 \end{bmatrix}, & \Omega &= \begin{bmatrix} 0.2221 & 0 \\ 0 & 0.2048 \end{bmatrix}, \\
\Delta &= \begin{bmatrix} 0.1983 & 0 \\ 0 & 0.1657 \end{bmatrix}.
\end{aligned}$$

Then, the controller gain is

$$K = \begin{bmatrix} -0.0369 & -0.0077 \\ -0.0103 & -0.0192 \end{bmatrix}. \quad (3.26)$$

Therefore, it follows from Theorem 3.2.1 that the error system (3.25) under controller gain (3.26) with given parameters is globally synchronized, which is further verified by the simulation result of the synchronization error given in Figure 3.1. The figures show that the synchronization errors trajectories $e_{i1}(t) = x_i(t) - x_1(t)$, ($i = 2, 3, 4, 5$) of the system (3.25) with initial values of the state are $x_1(0) = [1, -1]^T$, $x_2(0) = [-1.5, 1.2]^T$, $x_3(0) = [2, -1.6]^T$, $x_4(0) = [0.7, 1.1]^T$, $x_5(0) = [-1.2, -1.8]^T$. The numerical simulations are carried out using the explicit Runge-Kutta-like method (dde45), interpolation and extrapolation by spline of the third order.



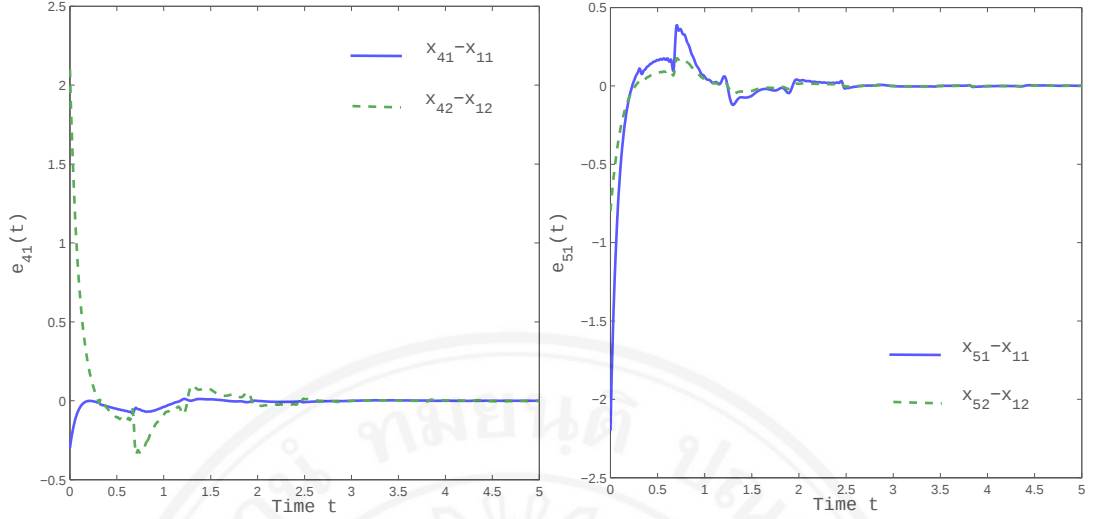


Figure 3.1: Synchronization errors trajectories, $e_{i1}(t) = x_i(t) - x_1(t)$, $i = 2, 3, 4, 5$.

3.3 Synchronization Criteria with Sampled-Data Feedback Control

In this section, we study the global synchronization of coupled NNs (3.1) by design the sampled-data feedback controller satisfy the following conditions:

$$U_i(t) = Kx_i(t_k), \quad t_k \leq t < t_{k+1}, \quad (3.27)$$

where $K \in R^{n \times n}$ is sampled-data feedback controller gain matrix to be determined, the sampling instant t_k satisfies the following conditions:

$$0 = t_0 < t_1 < \dots < t_k < \dots < \lim_{k \rightarrow +\infty} t_k = +\infty.$$

It is assumed that $t_{k+1} - t_k \leq h$ for $k \geq 0$ where $h > 0$ is a positive scalar and the largest sampling interval, i.e., the sampling interval is bounded.

Moreover, it is difficult to analyze the synchronization of NNs because of the discrete term $x(t_k)$. Define a sawtooth function $d(t) : [0, \infty) \rightarrow \mathcal{R}$ as follows:

$$d(t) = t - t_k, \quad t_k \leq t < t_{k+1}. \quad (3.28)$$

It is easily seen that

$$0 \leq d(t) < h. \quad (3.29)$$

The control input of node i can be rewritten as $Kx_i(t_k) = Kx_i(t - d(t))$. Then, with Kronecker product can be represented as

$$U(t) = (I_N \otimes K) x(t).$$

We obtain the following result.

Theorem 3.3.1. *Assume that assumptions 3.1-3.3 hold. The system (3.1) is globally asymptotically synchronized under the controller (3.27), if there exist positive definite matrices $P, Q_1, Q_2, Q_3, Q_4, R, S_1, S_2, S_3, Z_1, Z_2, Y$, three diagonal matrices $\Lambda = \text{diag}(\iota_1, \dots, \iota_n) > 0, \Omega = \text{diag}(\kappa_1, \dots, \kappa_n) > 0, \Delta = \text{diag}(\eta_1, \dots, \eta_n) > 0$ and any real matrices T_1, T_2 with appropriate dimensions, such that the following linear matrix inequalities (LMIs) hold for all $1 = i < j = N$:*

$$\begin{aligned} \Phi_{1ij} &= \mathcal{M}_{ij} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & -I & 0 & 0 & 0 \end{bmatrix}^T \\ &\quad \times S_3 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & -I & 0 & 0 & 0 \end{bmatrix} < 0, \end{aligned} \quad (3.30)$$

$$\begin{aligned} \Phi_{2ij} &= \mathcal{M}_{ij} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -I & I & 0 & 0 & 0 & 0 \end{bmatrix}^T \\ &\quad \times S_3 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -I & I & 0 & 0 & 0 & 0 \end{bmatrix} < 0, \end{aligned} \quad (3.31)$$

$$\Phi_3 = \begin{bmatrix} -0.1(S_1 + S_2) & PE & T_1 E \\ * & -P & 0 \\ * & * & -P \end{bmatrix} < 0, \quad (3.32)$$

$$\Phi_4 = \begin{bmatrix} -T_2 & T_2 E \\ * & -P \end{bmatrix} < 0, \quad (3.33)$$

$$\Phi_5 = \begin{bmatrix} -\frac{1}{h}Z_2 & 3Y^T \\ * & -3P \end{bmatrix} < 0, \quad (3.34)$$

$$\mathcal{M}_{ij} = \begin{bmatrix} M_{1,1} & M_{1,2} & M_{1,3} & \cdots & M_{1,12} \\ * & M_{2,2} & M_{2,3} & \cdots & M_{2,12} \\ * & * & M_{3,3} & \cdots & M_{3,12} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & M_{12,12} \end{bmatrix},$$

where $\bar{k} = \int_0^{+\infty} k(s) ds$ and

$$\begin{aligned} M_{1,1} &= -PC - C^T P - Ng_{ij}^1 P \Gamma_1 - Ng_{ij}^1 \Gamma_1^T P - T_1 C - C^T T_1^T \\ &\quad - Ng_{ij}^1 T_1 \Gamma_1 - Ng_{ij}^1 \Gamma_1^T T_1^T + \sum_{i=1}^3 Q_i - 0.9S_1 - 0.9S_2 \\ &\quad - \Lambda F_1 - \Omega J_1 - \Delta L_1 - \frac{1}{h}Z_2 + Z_1, \\ M_{1,2} &= -T_1 - C^T T_2^T - Ng_{ij}^1 \Gamma_1^T T_2^T, \\ M_{1,3} &= (P + T_1)B, \\ M_{1,4} &= (P + T_1)A + \Lambda F_2, \end{aligned}$$

$$\begin{aligned}
M_{1,5} &= \Omega J_2, \\
M_{1,6} &= \Delta L_2, \\
M_{1,7} &= -Ng_{ij}^2 P\Gamma_2 - Ng_{ij}^2 T_1\Gamma_2, \\
M_{1,8} &= S_1, \\
M_{1,9} &= S_2, \\
M_{1,10} &= (P + T_1)D, \\
M_{1,11} &= \frac{1}{h}Z_2, \\
M_{2,2} &= \tau_1^2 S_1 + \tau_2^2 S_2 + \eta^2 S_3 - T_2 + hZ_2, \\
M_{2,3} &= T_2 B, \\
M_{2,4} &= T_2 A, \\
M_{2,7} &= -Ng_{ij}^2 T_2\Gamma_2, \\
M_{2,10} &= T_2 D, \\
M_{3,3} &= -(1 - \mu)Q_4, \\
M_{4,4} &= -\Lambda, \\
M_{5,5} &= Q_4 - \Omega, \\
M_{6,6} &= \bar{k}R - \Delta, \\
M_{7,7} &= -(1 - \mu)Q_1 - 2S_3, \\
M_{8,8} &= -Q_2 - S_1 - S_3, \\
M_{9,9} &= -Q_3 - S_2 - S_3, \\
M_{10,10} &= -\frac{1}{k}R, \\
M_{11,11} &= -\frac{1}{h}Z_2, \\
M_{11,12} &= \frac{1}{h}Z_2, \\
M_{12,12} &= -Z_1 - \frac{1}{h}Z_2, \\
0 &= M_{1,12} = M_{2,5} = M_{2,6} = M_{2,8} = M_{2,9} = M_{2,11} = M_{2,12} = M_{3,4} \\
&= M_{3,5} = M_{3,6} = M_{3,7} = M_{3,8} = M_{3,9} = M_{3,10} = M_{3,11} = M_{3,12} \\
&= M_{4,5} = M_{4,6} = M_{4,7} = M_{4,8} = M_{4,9} = M_{4,10} = M_{4,11} = M_{4,12} \\
&= M_{5,6} = M_{5,7} = M_{5,8} = M_{5,9} = M_{5,10} = M_{5,11} = M_{5,12} = M_{6,7} \\
&= M_{6,8} = M_{6,9} = M_{6,10} = M_{6,11} = M_{6,12} = M_{7,8} = M_{7,9} = M_{7,10} \\
&= M_{7,11} = M_{7,12} = M_{8,9} = M_{8,10} = M_{8,11} = M_{8,12} = M_{9,10} = M_{9,11} \\
&= M_{9,12} = M_{10,11} = M_{10,12}.
\end{aligned}$$

Moreover the memoryless feedback control is $K = -P^{-1}Y$.

Proof : Consider the following piecewise Lyapunov-Krasovskii functional for the system (3.1):

$$V(t, x_t) = \sum_{l=1}^6 V_l(t, x_t), \quad t \in [t_k, t_{k+1}), \quad (3.35)$$

where

$$\begin{aligned} V_1(t, x_t) &= x^T(t)(U \otimes P)x(t), \\ V_2(t, x_t) &= \int_{t-\tau(t)}^t x^T(s)(U \otimes Q_1)x(s) ds \\ &\quad + \int_{t-\tau_1}^t x^T(s)(U \otimes Q_2)x(s) ds \\ &\quad + \int_{t-\tau_2}^t x^T(s)(U \otimes Q_3)x(s) ds \\ &\quad + \int_{t-\tau(t)}^t g^T(x(s))(U \otimes Q_4)g(x(s)) ds, \\ V_3(t, x_t) &= \tau_1 \int_{-\tau_1}^0 \int_{t+s}^t \dot{x}^T(\theta)(U \otimes S_1)\dot{x}(\theta) d\theta ds \\ &\quad + \tau_2 \int_{-\tau_2}^0 \int_{t+s}^t \dot{x}^T(\theta)(U \otimes S_2)\dot{x}(\theta) d\theta ds, \\ V_4(t, x_t) &= \eta \int_{-\tau_2}^{-\tau_1} \int_{t+s}^t \dot{x}^T(\theta)(U \otimes S_3)\dot{x}(\theta) d\theta ds, \\ V_5(t, x_t) &= \int_0^{+\infty} k(s) \int_{t-s}^t h^T(x(\theta))(U \otimes R)h(x(\theta)) d\theta ds, \\ V_6(t, x_t) &= \int_{t-h}^t x^T(s)(U \otimes Z_1)x(s) ds \\ &\quad + \int_{t-h}^t \int_s^t \dot{x}^T(\theta)(U \otimes Z_2)\dot{x}(\theta) d\theta ds. \end{aligned}$$

By taking the derivative of $V_1(t, x_t), \dots, V_5(t, x_t)$ along the trajectories of system (3.1).

We may proof this by using a similar argument as in the proof of Theorem 3.2.1. Thus,

we have the following

$$\begin{aligned} \dot{V}(t, x_t) &\leq \sum_{1 \leq i < j \leq N}^N \left\{ \zeta_{ij}^T(t) [(1 - \beta_i) \Sigma_{1ij} + \beta_i \Sigma_{2ij}] \zeta_{ij}(t) \right. \\ &\quad + (x_i(t) - x_j(t))^T \Pi_1 (x_i(t) - x_j(t)) \\ &\quad \left. + (\dot{x}_i(t) - \dot{x}_j(t))^T \Pi_2 (\dot{x}_i(t) - \dot{x}_j(t)) \right\}, \end{aligned} \quad (3.36)$$

where Σ_{1ij} and Σ_{2ij} are defined in (3.4), (3.5), respectively, and

$$\begin{aligned} \Pi_1 &= -0.1S_1 - 0.1S_2 + 3Y^T P^{-1}Y + PEP^{-1}E^T P + T_1 EP^{-1}E^T T_1^T, \\ \Pi_2 &= -T_2 + T_2 EP^{-1}E^T T_2^T. \end{aligned}$$

In addition $V_6(t, x_t)$, calculating the time derivative of $V_6(t, x_t)$ and using Lemma 2.3.20 yield

$$\begin{aligned}
\dot{V}_6(t, x_t) &= x^T(t)(U \otimes Z_1)x(t) - x^T(t-h)(U \otimes Z_1)x(t-h) \\
&\quad + h\dot{x}^T(t)(U \otimes Z_2)\dot{x}(t) - \int_{t-h}^t \dot{x}^T(s)(U \otimes Z_2)\dot{x}(s)ds, \\
&\leq x^T(t)(U \otimes Z_1)x(t) - x^T(t-h)(U \otimes Z_1)x(t-h) \\
&\quad + h\dot{x}^T(t)(U \otimes Z_2)\dot{x}(t) \\
&\quad - \frac{1}{h} \left[x(t-d(t)) - x(t-h) \right]^T (U \otimes Z_2) \left[x(t-d(t)) - x(t-h) \right] \\
&\quad - \frac{1}{h} \left[x(t) - x(t-d(t)) \right]^T (U \otimes Z_2) \left[x(t) - x(t-d(t)) \right], \\
&= \sum_{1 \leq i < j \leq N}^N \left\{ (x_i(t) - x_j(t))^T Z_1 (x_i(t) - x_j(t)) \right. \\
&\quad - (x_i(t-h) - x_j(t-h))^T Z_1 (x_i(t-h) - x_j(t-h)) \\
&\quad + h(\dot{x}_i(t) - \dot{x}_j(t))^T Z_2 (\dot{x}_i(t) - \dot{x}_j(t)) \\
&\quad - \frac{1}{h} \left[(x_i(t-d(t)) - x_j(t-d(t))) - (x_i(t-h) - x_j(t-h)) \right]^T Z_2 \\
&\quad \times \left[(x_i(t-d(t)) - x_j(t-d(t))) - (x_i(t-h) - x_j(t-h)) \right] \\
&\quad - \frac{1}{h} \left[(x_i(t) - x_j(t)) - (x_i(t-d(t)) - x_j(t-d(t))) \right]^T Z_2 \\
&\quad \times \left. \left[(x_i(t) - x_j(t)) - (x_i(t-d(t)) - x_j(t-d(t))) \right] \right\}. \tag{3.37}
\end{aligned}$$

From (3.36) and (3.37), we obtain

$$\begin{aligned}
\dot{V}(t, x_t) &\leq \sum_{1 \leq i < j \leq N}^N \left\{ \zeta_{ij}^T(t) \left[(1 - \beta_i)\Phi_{1ij} + \beta_i\Phi_{2ij} \right] \zeta_{ij}(t) \right. \\
&\quad + (x_i(t) - x_j(t))^T \Upsilon_1 (x_i(t) - x_j(t)) \\
&\quad + (\dot{x}_i(t) - \dot{x}_j(t))^T \Upsilon_2 (\dot{x}_i(t) - \dot{x}_j(t)) \\
&\quad \left. + (x_i(t-d(t)) - x_j(t-d(t)))^T \Upsilon_3 (x_i(t-d(t)) - x_j(t-d(t))) \right\}, \tag{3.38}
\end{aligned}$$

where Φ_{1ij} and Φ_{2ij} are defined in (3.30), (3.31), respectively, and

$$\begin{aligned}
\Upsilon_1 &= -0.1S_1 - 0.1S_2 + PEP^{-1}E^TP + T_1EP^{-1}E^TT_1^T, \\
\Upsilon_2 &= -T_2 + T_2EP^{-1}E^TT_2^T, \\
\Upsilon_3 &= -\frac{1}{h}Z_2 + 3Y^TP^{-1}Y.
\end{aligned}$$

Applying Schur complement lemma 2.3.22, the inequality Υ_1, Υ_2 and Υ_3 are equivalent to Φ_3, Φ_4 and Φ_5 , respectively, and we may proof the theorem by using a similar argument as in the proof of Theorem 3.2.1. Then, that system (3.3) is globally asymptotically synchronized. $\square$

The next example demonstrate the benefits of the method described above.

Example 3.3.2. Consider a dynamical system (3.1) consisting of five linearly coupled identical models under the controller (3.27) with a hybrid coupling. The state equations of the entire array are

$$\begin{cases} \dot{x}_i(t) = -Cx_i(t) + Af(x_i(t)) + Bg(x_i(t - \tau(t))) \\ \quad + D \int_{-\infty}^t k(t-s)h(x_i(s))ds + EU_i(t) + \sum_{j=1}^5 g_{ij}^1 \Gamma_1 x_j(t) \\ \quad + \sum_{j=1}^5 g_{ij}^2 \Gamma_2 x_j(t - \tau(t)), \end{cases} \quad (3.39)$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t))^T, i = 1, 2, 3, 4, 5$ are the state vector of the i th network, and the other parameters are as follows:

$$\begin{aligned} C &= \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \quad A = \begin{bmatrix} 0.9 & -0.2 \\ 0.2 & -0.5 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 & 0.5 \\ 0.2 & 0.2 \end{bmatrix}, \\ D &= \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad E = \begin{bmatrix} 0.3 & 0 \\ 0.1 & 0.3 \end{bmatrix}, \quad \Gamma_1 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \\ \Gamma_2 &= \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad \tau_1 = 0.2, \quad \tau_2 = 1.0, \quad \mu = 0.3, \quad k(s) = e^{-s}. \end{aligned}$$

The outer-coupling matrix $G^q = (g_{ij}^q)_{N \times N}, (q = 1, 2)$ are defined as:

$$g^1 = \begin{bmatrix} -2 & 1 & 0 & 0 & 1 \\ 1 & -3 & 1 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 1 & 1 & -3 & 1 \\ 1 & 0 & 0 & 1 & -2 \end{bmatrix}, \quad g^2 = \begin{bmatrix} -4 & 1 & 1 & 1 & 1 \\ 1 & -4 & 1 & 1 & 1 \\ 1 & 1 & -4 & 1 & 1 \\ 1 & 1 & 1 & -4 & 1 \\ 1 & 1 & 1 & 1 & -4 \end{bmatrix}.$$

Take the activation function as follows:

$$\begin{aligned} f_1(x_{i1}(t)) &= 0.5(|x_{i1}(t) + 1| - |x_{i1}(t) - 1|), \\ f_2(x_{i2}(t)) &= 0.25(|x_{i2}(t) + 1| - |x_{i2}(t) - 1|), \\ g_1(x_{i1}(t)) &= 0.5(|x_{i1}(t) + 1| - |x_{i1}(t) - 1|), \\ g_2(x_{i2}(t)) &= 0.25(|x_{i2}(t) + 1| - |x_{i2}(t) - 1|), \\ h_1(x_{i1}(s)) &= \tanh(-x_{i1}(s)), \\ h_2(x_{i2}(s)) &= \tanh(-0.8x_{i2}(s)). \end{aligned}$$

It is easy to see:

$$\begin{aligned} F_1 &= \begin{bmatrix} -1 & 0 \\ 0 & -0.25 \end{bmatrix}, & F_2 &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \\ J_1 &= \begin{bmatrix} -1 & 0 \\ 0 & -0.25 \end{bmatrix}, & J_2 &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \\ L_1 &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, & L_2 &= \begin{bmatrix} -0.5 & 0 \\ 0 & -0.4 \end{bmatrix}. \end{aligned}$$

By applying the MATLAB LMI Toolbox in Theorem 3.3.1, we obtain the solution set as follows:

$$\begin{aligned} P &= \begin{bmatrix} 0.8984 & -0.1234 \\ -0.1234 & 0.8241 \end{bmatrix}, & Q_1 &= \begin{bmatrix} 0.5165 & -0.0783 \\ -0.0783 & 0.4281 \end{bmatrix}, \\ Q_2 &= \begin{bmatrix} 0.1032 & -0.0531 \\ -0.0531 & 0.1283 \end{bmatrix}, & Q_3 &= \begin{bmatrix} 0.1838 & -0.0832 \\ -0.0832 & 0.2094 \end{bmatrix}, \\ Q_4 &= \begin{bmatrix} 0.5906 & 0.1324 \\ 0.1324 & 1.0100 \end{bmatrix}, & R &= \begin{bmatrix} 0.2081 & -0.0224 \\ -0.0224 & 0.2879 \end{bmatrix}, \\ S_1 &= \begin{bmatrix} 0.7843 & 0.0941 \\ 0.0941 & 0.9012 \end{bmatrix}, & S_2 &= \begin{bmatrix} 0.0022 & -0.0018 \\ -0.0018 & 0.0038 \end{bmatrix}, \\ S_3 &= \begin{bmatrix} 0.0218 & -0.0079 \\ -0.0079 & 0.0386 \end{bmatrix}, & T_1 &= \begin{bmatrix} -0.2073 & 0.1066 \\ 0.1143 & -0.2539 \end{bmatrix}, \\ T_2 &= \begin{bmatrix} 0.1424 & -0.0361 \\ -0.0167 & 0.1815 \end{bmatrix}, & Z_1 &= \begin{bmatrix} 0.2420 & -0.1069 \\ -0.1069 & 0.3044 \end{bmatrix}, \\ Z_2 &= \begin{bmatrix} 0.0308 & -0.0137 \\ -0.0137 & 0.0409 \end{bmatrix}, & Y &= \begin{bmatrix} 0.0794 & -0.0212 \\ -0.0212 & 0.0877 \end{bmatrix}, \\ \Lambda &= \begin{bmatrix} 0.8174 & 0 \\ 0 & 0.7948 \end{bmatrix}, & \Omega &= \begin{bmatrix} 0.8114 & 0 \\ 0 & 1.5146 \end{bmatrix}, \\ \Delta &= \begin{bmatrix} 0.5892 & 0 \\ 0 & 0.7613 \end{bmatrix}. \end{aligned}$$

Then, the controller gain is

$$K = \begin{bmatrix} -0.0866 & 0.0092 \\ 0.0128 & -0.1050 \end{bmatrix}. \quad (3.40)$$

On the other hand, the error system (3.39) with controller gain (3.40) is simulated, and the synchronization is observed until the maximum sampling interval $h = 0.43$. Therefore,

it follows from Theorem 3.3.1 with given parameters is globally synchronized, which is further verified by the simulation result of the synchronization error given in Figure 3.2. The figures show that the synchronization errors trajectories $e_{i1}(t) = x_i(t) - x_1(t)$, ($i = 2, 3, 4, 5$) of the system (3.39) with initial values of the state are $x_1(0) = [1, -1]^T$, $x_2(0) = [-1.5, 1.2]^T$, $x_3(0) = [2, -1.6]^T$, $x_4(0) = [0.7, 1.1]^T$, $x_5(0) = [-1.2, -1.8]^T$. The numerical simulations are carried out using the explicit Runge-Kutta-like method (dde45), interpolation and extrapolation by spline of the third order.

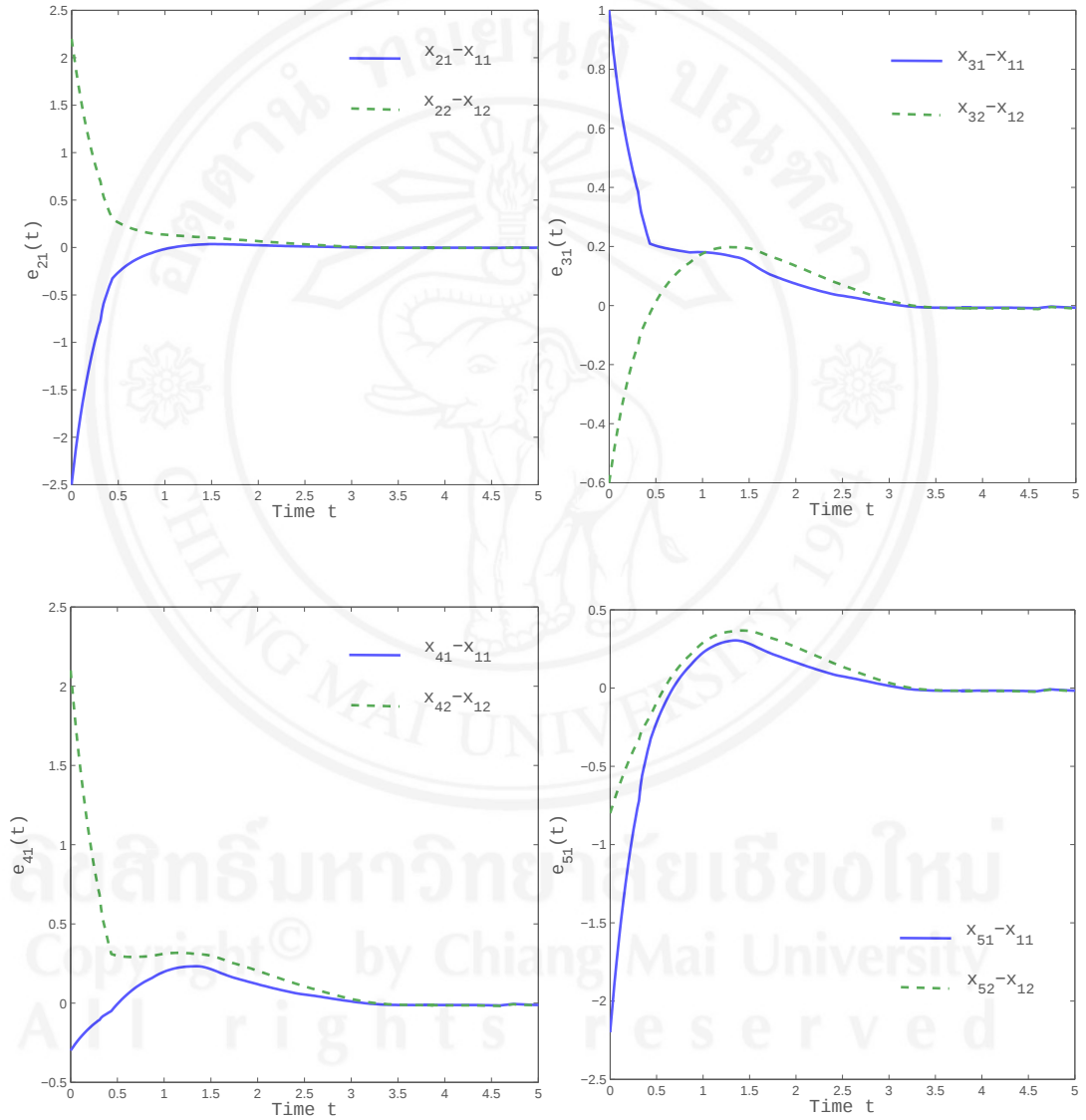


Figure 3.2: Synchronization errors trajectories, $e_{i1}(t) = x_i(t) - x_1(t)$, $i = 2, 3, 4, 5$.