

CHAPTER 4

Exponential Synchronization of Coupled Delayed Neural Networks with Hybrid Couplings

In this chapter, we make the great efforts to investigate the exponential synchronization of coupled NNs with interval time-varying delays, leakage delay and hybrid couplings. The designed controllers ensure the exponential synchronization via feedback and intermittent feedback controls. By constructing an augmented Lyapunov-Krasovskii functional and Kronecker product properties, novel sufficient conditions are obtained in terms of LMIs. The criteria are derived in terms of LMIs solved efficiently by standard convex optimization algorithms. The feasibility of the conditions can be easily checked by resorting to Matlab LMI Toolbox. Finally, the effectiveness of the proposed criteria can be demonstrated by utilizing numerical examples.

4.1 Problem Formulation

Consider the coupled dynamical system consisting of N linearly and diffusively coupled recurrent NNs with time delays can be described by:

$$\left\{ \begin{array}{l} \dot{x}_i(t) = -Cx_i(t - \sigma) + Af(x_i(t)) + Bg(x_i(t - \tau(t))) \\ \quad + D \int_{t-k(t)}^t h(x_i(s)) ds + EU_i(t) + \sum_{j=1}^N g_{ij}^1 \Gamma_1 x_j(t) \\ \quad + \sum_{j=1}^N g_{ij}^2 \Gamma_2 x_j(t - \tau(t)) + \sum_{j=1}^N g_{ij}^3 \Gamma_3 \int_{t-k(t)}^t h(x_j(s)) ds, \\ x_i(t) = \phi_i(t), \quad t \in [\tau_{\max}, 0], \quad \tau_{\max} = \max\{\tau_2, k\}, \quad i = 1, 2, \dots, N, \end{array} \right. \quad (4.1)$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \dots, x_{in}(t))^T \in \mathcal{R}^n$ is the state vector of the i th network at time t . $C = \text{diag}(c_1, c_2, \dots, c_n) > 0$ denotes the rate with which the cell i resets its potential to the resting state when being isolated from other cells and inputs, A, B, D and E represent the connection weight matrix, $G^q = (g_{ij}^q)_{N \times N}$, ($q = 1, 2, 3$) represent the outer-coupling matrix of the network, $\Gamma_1, \Gamma_2, \Gamma_3 \in \mathcal{R}^{n \times n}$ represent a constant inner-coupling matrix, $f(x_i(t)) = (f_1(x_{i1}(t)), \dots, f_n(x_{in}(t)))^T$, $g(x_i(t - \tau(t))) = (g_1(x_{i1}(t - \tau(t))), \dots, g_n(x_{in}(t - \tau(t))))^T$ and $h(x_i(s)) = (h_1(x_{i1}(s)), \dots, h_n(x_{in}(s))))^T$ represent the activation function vectors and $U_i(t) \in \mathcal{R}^n$ represent the control input vector.

The interval time-varying delays, time delays and leakage delay are $\tau(t)$, $k(t)$ and σ , respectively, satisfy the following conditions:

$$\begin{aligned} 0 &\leq \tau_1 \leq \tau(t) \leq \tau_2, \quad \dot{\tau}(t) \leq \mu_1, \quad \eta = \tau_2 - \tau_1, \\ 0 &< k(t) \leq k, \quad \dot{k}(t) \leq \mu_2, \quad 0 < \sigma. \end{aligned}$$

We design the feedback controller satisfy the following conditions:

$$U_i(t) = Kx_i(t), \quad t \geq 0, \quad (4.2)$$

where $K \in R^{n \times n}$ is the feedback controller gain matrix to be determined.

Let

$$\begin{aligned} x^T(t) &= [x_1^T(t), \dots, x_N^T(t)], \\ f^T(x(t)) &= [f^T(x_1(t)), \dots, f^T(x_N(t))], \\ g^T(x(t - \tau(t))) &= [g^T(x_1(t - \tau(t))), \dots, g^T(x_N(t - \tau(t)))], \\ h^T(x(s)) &= [h^T(x_1(s)), \dots, h^T(x_N(s))], \\ U^T(t) &= [U_1^T(t), \dots, U_N^T(t)]^T, \end{aligned}$$

By employing the Kronecker product, we can reformulate the system (4.1) as the following form:

$$\begin{cases} \dot{x}(t) = -(I_N \otimes C)x(t - \sigma) + (I_N \otimes A)f(x(t)) + (I_N \otimes B)g(x(t - \tau(t))) \\ \quad + (I_N \otimes D) \int_{t-k(t)}^t h(x(s))ds + (I_N \otimes E)U(t) + (G^1 \otimes \Gamma_1)x(t) \\ \quad + (G^2 \otimes \Gamma_2)x(t - \tau(t)) + (G^3 \otimes \Gamma_3) \int_{t-k(t)}^t h(x(s))ds, \end{cases} \quad (4.3)$$

and its identical Kronecker product from is

$$U(t) = (I_N \otimes K)x(t), \quad t \geq 0.$$

We introduce the following assumptions:

Assumption 4.1 [34], the outer-coupling configuration matrices of the NNs satisfy:

$$g_{ij}^q = g_{ji}^q \geq 0, \quad g_{ii}^q = - \sum_{j=1, j \neq i}^N g_{ij}^q,$$

where $i, j = 1, 2, \dots, N$, $i \neq j$, and $q = 1, 2, 3$.

Assumption 4.2 [34], for some constants $l_r^-, l_r^+, \varrho_r^-, \varrho_r^+, v_r^-, v_r^+$, the neuron activation functions are bounded, and satisfy:

$$\begin{aligned} l_r^- &\leq \frac{f_r(s_1) - f_r(s_2)}{s_1 - s_2} \leq l_r^+, \\ \varrho_r^- &\leq \frac{g_r(s_1) - g_r(s_2)}{s_1 - s_2} \leq \varrho_r^+, \\ v_r^- &\leq \frac{h_r(s_1) - h_r(s_2)}{s_1 - s_2} \leq v_r^+, \end{aligned}$$

where $r \in \{1, 2, \dots, n\}$.

We denote

$$\begin{aligned} F_1 &= \text{diag}(l_1^+ l_1^-, \dots, l_n^+ l_n^-), \quad F_2 = \text{diag}\left(\frac{l_1^+ + l_1^-}{2}, \dots, \frac{l_n^+ + l_n^-}{2}\right), \\ J_1 &= \text{diag}(\varrho_1^+ \varrho_1^-, \dots, \varrho_n^+ \varrho_n^-), \quad J_2 = \text{diag}\left(\frac{\varrho_1^+ + \varrho_1^-}{2}, \dots, \frac{\varrho_n^+ + \varrho_n^-}{2}\right), \\ L_1 &= \text{diag}(v_1^+ v_1^-, \dots, v_n^+ v_n^-), \quad L_2 = \text{diag}\left(\frac{v_1^+ + v_1^-}{2}, \dots, \frac{v_n^+ + v_n^-}{2}\right). \end{aligned}$$

4.2 Synchronization Criteria with Feedback Control

In this section, we will present the exponential synchronization of coupled NNs (4.3) in terms of LMIs by using the Lyapunov-Krasovskii functionals and Kronecker product techniques. The feedback controller is designed to achieve the exponential synchronization of coupled NNs (4.3). The notations of several matrices are defined as:

$$\left\{ \begin{aligned} \varrho_r &= \max \left\{ |\varrho_r^-|, |\varrho_r^+| \right\}, \quad v_r = \max \left\{ |v_r^-|, |v_r^+| \right\}, \quad r = 1, 2, \dots, n, \\ \varrho_m &= \max \left\{ \varrho_1, \varrho_2, \dots, \varrho_n \right\}, \quad v_m = \max \left\{ v_1, v_2, \dots, v_n \right\}, \\ \|\phi\| &= \|x(0)\|, \quad \|\varphi\| = \sup_{-\tau_{\max} \leq s \leq 0} \|x(s)\|, \\ \lambda_1 &= \lambda_{\min}(U \otimes P), \\ \lambda_2 &= \lambda_{\max}(U \otimes P) + \tau_1^2 \lambda_{\max}(U \otimes R_1) + \tau_2^2 \lambda_{\max}(U \otimes R_2) + \eta^2 \lambda_{\max}(U \otimes R_3) \\ &\quad + \sigma^2 \lambda_{\max}(U \otimes S_1), \\ \lambda_3 &= \sigma \lambda_{\max}(U \otimes Q_1) + \tau_1 \lambda_{\max}(U \otimes Q_2) + \tau_2 \lambda_{\max}(U \otimes Q_3) + \tau_2 \lambda_{\max}(U \otimes Q_4) \\ &\quad + \varrho_m^2 \tau_2 \lambda_{\max}(U \otimes Q_5) + k^2 \lambda_{\max}(U \otimes S_2) - \tau_1^2 \lambda_{\max}(U \otimes R_1) - \tau_2^2 \lambda_{\max}(U \otimes R_2) \\ &\quad - \eta^2 \lambda_{\max}(U \otimes R_3) - \sigma^2 \lambda_{\max}(U \otimes S_1) + v_m^2 k^2 \lambda_{\max}(U \otimes S_3). \end{aligned} \right. \quad (4.4)$$

Then, we have the following theorem.

Theorem 4.2.1. *Given $\alpha > 0$ and suppose that assumption 4.1-4.2 hold, the system (4.1) is exponentially synchronized under the controller (4.2) if there exist symmetric positive definite matrices, $P, Y, Q_1, Q_2, Q_3, Q_4, Q_5, R_l, S_l, (l = 1, 2, 3)$, three diagonal matrices $\Lambda = \text{diag}(\iota_1, \dots, \iota_n) > 0, \Omega = \text{diag}(\kappa_1, \dots, \kappa_n) > 0, \Delta = \text{diag}(\eta_1, \dots, \eta_n) > 0$ and real matrices T with appropriate dimensions, such that the following linear matrix inequalities (LMIs) hold for all $1 = i < j = N$:*

$$\begin{aligned} \Sigma_{1ij} &= \mathcal{M}_{ij} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & -I & 0 & 0 \end{bmatrix}^T \\ &\quad \times e^{-2\alpha\tau_2} R_3 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & -I & 0 & 0 \end{bmatrix} < 0, \end{aligned} \quad (4.5)$$

$$\begin{aligned} \Sigma_{2ij} &= \mathcal{M}_{ij} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I & I & 0 & 0 & 0 \end{bmatrix}^T \\ &\quad \times e^{-2\alpha\tau_2} R_3 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I & I & 0 & 0 & 0 \end{bmatrix} < 0, \end{aligned} \quad (4.6)$$

$$\Sigma_3 = \begin{bmatrix} -0.1e^{-2\alpha\tau_1}R_1 - 0.1e^{-2\alpha\tau_2}R_2 & 2Y^T & PE \\ * & -2P & 0 \\ * & * & -P \end{bmatrix} < 0, \quad (4.7)$$

$$\Sigma_4 = \begin{bmatrix} -T & TE \\ * & -P \end{bmatrix} < 0, \quad (4.8)$$

where

$$\mathcal{M}_{ij} = \begin{bmatrix} M_{1,1} & M_{1,2} & M_{1,3} & \cdots & M_{1,12} \\ * & M_{2,2} & M_{2,3} & \cdots & M_{2,12} \\ * & * & M_{3,3} & \cdots & M_{3,12} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & M_{12,12} \end{bmatrix},$$

$$\begin{aligned} M_{1,1} &= -Ng_{ij}^1 P \Gamma_1 - Ng_{ij}^1 \Gamma_1^T P + 2\alpha P + \sum_{r=1}^4 Q_r - 0.9e^{-2\alpha\tau_1} R_1 \\ &\quad - 0.9e^{-2\alpha\tau_2} R_2 - e^{-2\alpha\sigma} S_1 + kS_2 - \Omega F_1 - \Theta J_1 - \Delta L_1, \end{aligned}$$

$$M_{1,2} = -e^{-2\alpha\sigma} S_1 + C^T P,$$

$$M_{1,3} = -Ng_{ij}^1 T \Gamma_1,$$

$$M_{1,4} = B^T P,$$

$$M_{1,5} = \Omega F_2,$$

$$M_{1,6} = \Theta J_2 + A^T P,$$

$$M_{1,7} = \Delta L_2,$$

$$M_{1,8} = -Ng_{ij}^2 \Gamma_2^T P,$$

$$M_{1,9} = e^{-2\alpha\tau_1} R_1,$$

$$M_{1,10} = e^{-2\alpha\tau_2} R_2,$$

$$M_{1,12} = -Ng_{ij}^3 \Gamma_3^T P + D^T P,$$

$$M_{2,2} = -e^{-2\alpha\sigma} (Q_1 + S_1),$$

$$M_{2,3} = -TC,$$

$$M_{3,3} = \tau_1^2 R_1 + \tau_2^2 R_2 + \eta^2 R_3 + \sigma^2 S_1 - T,$$

$$M_{3,4} = B^T T,$$

$$M_{3,6} = A^T T^T,$$

$$M_{3,8} = -Ng_{ij}^2 \Gamma_2^T T^T,$$

$$M_{3,12} = -Ng_{ij}^3 \Gamma_3^T T^T + D^T T^T,$$

$$\begin{aligned}
M_{4,4} &= -(1 - \mu_1)e^{-2\alpha\tau_2}Q_5, \\
M_{5,5} &= -\Omega - Q_5, \\
M_{6,6} &= -\Theta, \\
M_{7,7} &= kS_3 - \Delta, \\
M_{8,8} &= -(1 - \mu_1)e^{-2\alpha\tau_2}Q_4 - e^{-2\alpha\tau_2}R_3 - e^{-2\alpha\tau_2}R_3, \\
M_{8,9} &= -e^{-2\alpha\tau_2}R_3, \\
M_{8,10} &= -e^{-2\alpha\tau_2}R_3, \\
M_{9,9} &= -e^{-2\alpha\tau_1}(Q_2 + R_1) - e^{-2\alpha\tau_2}R_3, \\
M_{10,10} &= -e^{-2\alpha\tau_2}(R_2 + R_3 + Q_3), \\
M_{11,11} &= -\frac{(1 - \mu_2)}{k}e^{-2\alpha k}S_2, \\
M_{12,12} &= -\frac{(1 - \mu_2)}{k}e^{-2\alpha k}S_3, \\
0 &= M_{1,11} = M_{2,4} = M_{2,5} = M_{2,6} = M_{2,7} = M_{2,8} = M_{2,9} \\
&= M_{2,10} = M_{2,11} = M_{2,12} = M_{3,5} = M_{3,7} = M_{3,9} = M_{3,10} \\
&= M_{3,11} = M_{4,5} = M_{4,6} = M_{4,7} = M_{4,8} = M_{4,9} = M_{4,10} \\
&= M_{4,11} = M_{4,12} = M_{5,6} = M_{5,7} = M_{5,8} = M_{5,9} = M_{5,10} \\
&= M_{5,11} = M_{5,12} = M_{6,7} = M_{6,8} = M_{6,9} = M_{6,10} = M_{6,11} \\
&= M_{6,12} = M_{7,8} = M_{7,9} = M_{7,10} = M_{7,11} = M_{7,12} = M_{8,11} \\
&= M_{8,12} = M_{9,10} = M_{9,11} = M_{9,12} = M_{10,11} = M_{10,12} = M_{11,12}.
\end{aligned}$$

Moreover, the memoryless feedback control is $K = -P^{-1}Y$.

Proof : Consider the following Lyapunov-Krasovskii functional candidate for the system (4.3):

$$V(x(t)) = \sum_{l=1}^6 V_l(x(t)), \quad (4.9)$$

where

$$\begin{aligned}
V_1(x(t)) &= x^T(t)(U \otimes P)x(t), \\
V_2(x(t)) &= \int_{t-\sigma}^t e^{2\alpha(s-t)}x^T(s)(U \otimes Q_1)x(s)ds \\
&\quad + \int_{t-\tau_1}^t e^{2\alpha(s-t)}x^T(s)(U \otimes Q_2)x(s)ds
\end{aligned}$$

$$\begin{aligned}
& + \int_{t-\tau_2}^t e^{2\alpha(s-t)} x^T(s) (U \otimes Q_3) x(s) ds \\
& + \int_{t-\tau(t)}^t e^{2\alpha(s-t)} x^T(s) (U \otimes Q_4) x(s) ds \\
& + \int_{t-\tau(t)}^t e^{2\alpha(s-t)} g(x^T(s)) (U \otimes Q_5) g(x(s)) ds, \\
V_3(x(t)) &= \tau_1 \int_{-\tau_1}^0 \int_{t+s}^t e^{2\alpha(\theta-t)} \dot{x}^T(\theta) (U \otimes R_1) \dot{x}(\theta) d\theta ds \\
& + \tau_2 \int_{-\tau_2}^0 \int_{t+s}^t e^{2\alpha(\theta-t)} \dot{x}^T(\theta) (U \otimes R_2) \dot{x}(\theta) d\theta ds, \\
V_4(x(t)) &= \eta \int_{-\tau_2}^{-\tau_1} \int_{t+s}^t e^{2\alpha(\theta-t)} \dot{x}^T(\theta) (U \otimes R_3) \dot{x}(\theta) d\theta ds, \\
V_5(x(t)) &= \sigma \int_{-\sigma}^0 \int_{t+s}^t e^{2\alpha(\theta-t)} \dot{x}^T(\theta) (U \otimes S_1) \dot{x}(\theta) d\theta ds, \\
V_6(x(t)) &= \int_{-k(t)}^0 \int_{t+s}^t e^{2\alpha(\theta-t)} x^T(\theta) (U \otimes S_2) x(\theta) d\theta ds \\
& + \int_{-k(t)}^0 \int_{t+s}^t e^{2\alpha(\theta-t)} h^T(x(\theta)) (U \otimes S_3) h(x(\theta)) d\theta ds.
\end{aligned}$$

It easy to check that

$$\lambda_1 \|x(t)\|^2 \leq V(x(t)), \quad \forall t \geq 0. \quad (4.10)$$

where λ_1 is defined in (4.4).

Calculating the time derivative of $V(x(t))$ along the trajectories of system (4.3), define a matrix, if X is a matrix with zero column sums, then $UX = NX$, we obtains

$$\begin{aligned}
\dot{V}_1(x(t)) &= 2x^T(t) (U \otimes P) \dot{x}(t), \\
&= 2x^T(t) (I_N \otimes P) \left[- (I_N \otimes C) x(t - \sigma) + (I_N \otimes A) f(x(t)) \right. \\
&\quad + (I_N \otimes B) g(x(t - \tau(t))) + (I_N \otimes D) \int_{t-k(t)}^t h(x(s)) ds \\
&\quad + (I_N \otimes EK) x(t) + (G^1 \otimes \Gamma_1) x(t) + (G^2 \otimes \Gamma_2) x(t - \tau(t)) \\
&\quad \left. + (G^3 \otimes \Gamma_3) \int_{t-k(t)}^t h(x(s)) ds, \right] \\
&= 2 \sum_{1 \leq i < j \leq N}^N (x_i(t) - x_j(t))^T \left[(-Ng_{ij}^1 P \Gamma_1 + PEK) (x_i(t) - x_j(t)) \right. \\
&\quad - PC(x_i(t - \sigma) - x_j(t - \sigma)) + PA(f(x_i(t)) - f(x_j(t))) \\
&\quad + PB(g(x_i(t - \tau(t))) - g(x_j(t - \tau(t)))) \\
&\quad + (PD - Ng_{ij}^3 P \Gamma_3) \int_{t-k(t)}^t (h(x_i(s)) - h(x_j(s))) ds \\
&\quad \left. - Ng_{ij}^2 P \Gamma_2 (x_i(t - \tau(t)) - x_j(t - \tau(t))) \right]. \quad (4.11)
\end{aligned}$$

Using Lemma 2.3.17, we get

$$\begin{aligned}
& - \sum_{1 \leq i < j \leq N}^N 2(x_i(t) - x_j(t))^T P E K (x_i(t) - x_j(t)) \\
& = - \sum_{1 \leq i < j \leq N}^N 2(x_i(t) - x_j(t))^T P E P^{-1} Y (x_i(t) - x_j(t)), \\
& \leq \sum_{1 \leq i < j \leq N}^N (x_i(t) - x_j(t))^T \left[P E P^{-1} E^T P + Y^T P^{-1} Y \right] (x_i(t) - x_j(t)). \quad (4.12)
\end{aligned}$$

$$\begin{aligned}
\dot{V}_2(x(t)) & \leq x^T(t) \left(U \otimes (Q_1 + Q_2 + Q_3 + Q_4) \right) x(t) \\
& \quad - e^{-2\alpha\sigma} x^T(t - \sigma) (U \otimes Q_1) x(t - \sigma) \\
& \quad - e^{-2\alpha\tau_1} x^T(t - \tau_1) (U \otimes Q_2) x(t - \tau_1) \\
& \quad - e^{-2\alpha\tau_2} x^T(t - \tau_2) (U \otimes Q_3) x(t - \tau_2) \\
& \quad + g^T(x(t)) (U \otimes Q_5) g(x(t)) \\
& \quad - (1 - \mu_1) e^{-2\alpha\tau_2} \left[x^T(t - \tau(t)) (U \otimes Q_4) x(t - \tau(t)) \right. \\
& \quad \left. + g^T(x(t - \tau(t))) (U \otimes Q_5) g(x(t - \tau(t))) \right] - 2\alpha V_2, \\
& = \sum_{1 \leq i < j \leq N}^N \left\{ (x_i(t) - x_j(t))^T (Q_1 + Q_2 + Q_3 + Q_4) (x_i(t) - x_j(t)) \right. \\
& \quad - e^{-2\alpha\sigma} (x_i(t - \sigma) - x_j(t - \sigma))^T Q_1 (x_i(t - \sigma) - x_j(t - \sigma)) \\
& \quad - e^{-2\alpha\tau_1} (x_i(t - \tau_1) - x_j(t - \tau_1))^T Q_2 (x_i(t - \tau_1) - x_j(t - \tau_1)) \\
& \quad - e^{-2\alpha\tau_2} (x_i(t - \tau_2) - x_j(t - \tau_2))^T Q_3 (x_i(t - \tau_2) - x_j(t - \tau_2)) \\
& \quad + (g(x_i(t)) - g(x_j(t)))^T Q_5 (g(x_i(t)) - g(x_j(t))) \\
& \quad - (1 - \mu_1) e^{-2\alpha\tau_2} \left[(x_i(t - \tau(t)) - x_j(t - \tau(t)))^T Q_4 \right. \\
& \quad \times (x_i(t - \tau(t)) - x_j(t - \tau(t))) \\
& \quad \left. + (g(x_i(t - \tau(t))) - g(x_j(t - \tau(t))))^T Q_5 \right. \\
& \quad \left. \times (g(x_i(t - \tau(t))) - g(x_j(t - \tau(t)))) \right] \left. \right\} - 2\alpha V_2. \quad (4.13)
\end{aligned}$$

Calculating the time derivative of $V_3(t, x_t)$ and $V_4(t, x_t)$, respectively, with Lemma 2.3.20, we obtains

$$\begin{aligned}
\dot{V}_3(x(t)) & = \tau_1^2 \dot{x}^T(t) (U \otimes R_1) \dot{x}(t) + \tau_2^2 \dot{x}^T(t) (U \otimes R_2) \dot{x}(t) \\
& \quad - \tau_1 e^{-2\alpha\tau_1} \int_{t-\tau_1}^t \dot{x}^T(s) (U \otimes R_1) \dot{x}(s) ds \\
& \quad - \tau_2 e^{-2\alpha\tau_2} \int_{t-\tau_2}^t \dot{x}^T(s) (U \otimes R_2) \dot{x}(s) ds - 2\alpha V_3,
\end{aligned}$$

$$\begin{aligned}
&\leq \tau_1^2 \dot{x}^T(t)(U \otimes R_1) \dot{x}(t) + \tau_2^2 \dot{x}^T(t)(U \otimes R_2) \dot{x}(t) \\
&\quad - e^{-2\alpha\tau_1} \left[x(t) - x(t - \tau_1) \right]^T (U \otimes R_1) \left[x(t) - x(t - \tau_1) \right] \\
&\quad - e^{-2\alpha\tau_2} \left[x(t) - x(t - \tau_2) \right]^T (U \otimes R_2) \left[x(t) - x(t - \tau_2) \right] - 2\alpha V_3, \\
&= \sum_{1 \leq i < j \leq N}^N \left\{ (\dot{x}_i(t) - \dot{x}_j(t))^T (\tau_1^2 R_1 + \tau_2^2 R_2) (\dot{x}_i(t) - \dot{x}_j(t)) \right. \\
&\quad - e^{-2\alpha\tau_1} \left[(x_i(t) - x_j(t)) - (x_i(t - \tau_1) - x_j(t - \tau_1)) \right]^T R_1 \\
&\quad \times \left[(x_i(t) - x_j(t)) - (x_i(t - \tau_1) - x_j(t - \tau_1)) \right] \\
&\quad - e^{-2\alpha\tau_2} \left[(x_i(t) - x_j(t)) - (x_i(t - \tau_2) - x_j(t - \tau_2)) \right]^T R_2 \\
&\quad \times \left. \left[(x_i(t) - x_j(t)) - (x_i(t - \tau_2) - x_j(t - \tau_2)) \right] \right\} - 2\alpha V_3. \tag{4.14}
\end{aligned}$$

$$\begin{aligned}
\dot{V}_4(x(t)) &= \eta^2 \dot{x}^T(t)(U \otimes R_3) \dot{x}(t) - \eta e^{-2\alpha\tau_2} \int_{t-\tau_2}^{t-\tau_1} \dot{x}^T(s)(U \otimes R_3) \dot{x}(s) ds \\
&\quad - 2\alpha V_4, \tag{4.15}
\end{aligned}$$

let $\eta = (\tau_2 - \tau(t)) + (\tau(t) - \tau_1)$, we can acquire

$$\begin{aligned}
-\eta \int_{t-\tau_2}^{t-\tau_1} \dot{x}^T(s)(U \otimes R_3) \dot{x}(s) ds &= -(\tau_2 - \tau(t)) \int_{t-\tau_2}^{t-\tau(t)} \dot{x}^T(s)(U \otimes R_3) \dot{x}(s) ds \\
&\quad - (\tau(t) - \tau_1) \int_{t-\tau_2}^{t-\tau(t)} \dot{x}^T(s)(U \otimes R_3) \dot{x}(s) ds \\
&\quad - (\tau_2 - \tau(t)) \int_{t-\tau(t)}^{t-\tau_1} \dot{x}^T(s)(U \otimes R_3) \dot{x}(s) ds \\
&\quad - (\tau(t) - \tau_1) \int_{t-\tau(t)}^{t-\tau_1} \dot{x}^T(s)(U \otimes R_3) \dot{x}(s) ds.
\end{aligned}$$

Using Lemma 2.3.20, we get

$$\begin{aligned}
&-(\tau_2 - \tau(t)) \int_{t-\tau_2}^{t-\tau(t)} \dot{x}^T(s)(U \otimes R_3) \dot{x}(s) ds \\
&\leq - \left[x(t - \tau(t)) - x(t - \tau_2) \right]^T (U \otimes R_3) \left[x(t - \tau(t)) - x(t - \tau_2) \right], \tag{4.16}
\end{aligned}$$

$$\begin{aligned}
&-(\tau(t) - \tau_1) \int_{t-\tau(t)}^{t-\tau_1} \dot{x}^T(s)(U \otimes R_3) \dot{x}(s) ds \\
&\leq - \left[x(t - \tau_1) - x(t - \tau(t)) \right]^T (U \otimes R_3) \left[x(t - \tau_1) - x(t - \tau(t)) \right]. \tag{4.17}
\end{aligned}$$

Let $\beta = \frac{\tau_2 - \tau(t)}{\tau_2 - \tau_1} \leq 1$, then

$$\begin{aligned}
&-(\tau_2 - \tau(t)) \int_{t-\tau(t)}^{t-\tau_1} \dot{x}^T(s)(U \otimes R_3) \dot{x}(s) ds \\
&= -\beta(\tau_2 - \tau_1) \int_{t-\tau(t)}^{t-\tau_1} \dot{x}^T(s)(U \otimes R_3) \dot{x}(s) ds, \\
&\leq -\beta(\tau(t) - \tau_1) \int_{t-\tau(t)}^{t-\tau_1} \dot{x}^T(s)(U \otimes R_3) \dot{x}(s) ds,
\end{aligned}$$

$$\begin{aligned}
&\leq -\beta \left[x(t - \tau_1) - x(t - \tau(t)) \right]^T (U \otimes R_3) \left[x(t - \tau_1) - x(t - \tau(t)) \right], \quad (4.18) \\
&- (\tau(t) - \tau_1) \int_{t-\tau_2}^{t-\tau(t)} \dot{x}^T(s) (U \otimes R_3) \dot{x}(s) ds \\
&= -(1 - \beta)(\tau_2 - \tau_1) \int_{t-\tau_2}^{t-\tau(t)} \dot{x}^T(s) (U \otimes R_3) \dot{x}(s) ds, \\
&\leq -(1 - \beta)(\tau_2 - \tau(t)) \int_{t-\tau_2}^{t-\tau(t)} \dot{x}^T(s) (U \otimes R_3) \dot{x}(s) ds, \\
&\leq -(1 - \beta) \left[x(t - \tau(t)) - x(t - \tau_2) \right]^T (U \otimes R_3) \\
&\quad \times \left[x(t - \tau(t)) - x(t - \tau_2) \right]. \quad (4.19)
\end{aligned}$$

Therefore from (4.15) to (4.19), we can estimate $\dot{V}_4(t)$ as

$$\begin{aligned}
\dot{V}_4(x(t)) &\leq \sum_{1 \leq i < j \leq N}^N \left\{ (\dot{x}_i(t) - \dot{x}_j(t))^T \eta^2 R_3 (\dot{x}_i(t) - \dot{x}_j(t)) \right. \\
&\quad - e^{-2\alpha\tau_2} \left[(x_i(t - \tau(t)) - x_j(t - \tau(t))) - (x_i(t - \tau_2) - x_j(t - \tau_2)) \right]^T \\
&\quad \quad R_3 \left[(x_i(t - \tau(t)) - x_j(t - \tau(t))) - (x_i(t - \tau_2) - x_j(t - \tau_2)) \right] \\
&\quad - e^{-2\alpha\tau_2} \left[(x_i(t - \tau_1) - x_j(t - \tau_1)) - (x_i(t - \tau(t)) - x_j(t - \tau(t))) \right]^T \\
&\quad \quad R_3 \left[(x_i(t - \tau_1) - x_j(t - \tau_1)) - (x_i(t - \tau(t)) - x_j(t - \tau(t))) \right] \\
&\quad - \beta e^{-2\alpha\tau_2} \left[(x_i(t - \tau_1) - x_j(t - \tau_1)) - (x_i(t - \tau(t)) - x_j(t - \tau(t))) \right]^T \\
&\quad \quad R_3 \left[(x_i(t - \tau_1) - x_j(t - \tau_1)) - (x_i(t - \tau(t)) - x_j(t - \tau(t))) \right] \\
&\quad - (1 - \beta) e^{-2\alpha\tau_2} \left[(x_i(t - \tau(t)) - x_j(t - \tau(t))) - (x_i(t - \tau_2) - x_j(t - \tau_2)) \right]^T \\
&\quad \quad R_3 \left[(x_i(t - \tau(t)) - x_j(t - \tau(t))) - (x_i(t - \tau_2) - x_j(t - \tau_2)) \right] \Big\} \\
&\quad - 2\alpha V_4. \quad (4.20)
\end{aligned}$$

Similarly, calculating the time derivative of $V_5(t, x_t)$ and $V_6(t, x_t)$, respectively, with Lemma 2.3.18 and Lemma 2.3.20, we obtain

$$\begin{aligned}
\dot{V}_5(x(t)) &= \sigma^2 \dot{x}^T(t) (U \otimes S_1) \dot{x}(t) - \sigma e^{-2\alpha\sigma} \int_{t-\sigma}^t \dot{x}^T(s) (U \otimes S_1) \dot{x}(s) ds - 2\alpha V_5, \\
&\leq \sigma^2 \dot{x}^T(t) (U \otimes S_1) \dot{x}(t) \\
&\quad - e^{-2\alpha\sigma} \left[x(t) - x(t - \sigma) \right]^T (U \otimes S_1) \left[x(t) - x(t - \sigma) \right] - 2\alpha V_5, \\
&= \sum_{1 \leq i < j \leq N}^N \left\{ (\dot{x}_i(t) - \dot{x}_j(t))^T \sigma^2 S_1 (\dot{x}_i(t) - \dot{x}_j(t)) \right. \\
&\quad - e^{-2\alpha\sigma} \left[(x_i(t) - x_j(t)) - (x_i(t - \sigma) - x_j(t - \sigma)) \right]^T S_1 \\
&\quad \quad \times \left[(x_i(t) - x_j(t)) - (x_i(t - \sigma) - x_j(t - \sigma)) \right] \Big\} - 2\alpha V_5. \quad (4.21)
\end{aligned}$$

$$\begin{aligned}
\dot{V}_6(x(t)) &\leq kx^T(t)(U \otimes S_2)x(t) + kh^T(x(t))(U \otimes S_3)h(x(t)) - (1 - \mu_2)e^{-2\alpha k} \\
&\quad \times \int_{t-k(t)}^t \left[x^T(s)(U \otimes S_2)x(s) + h^T(x(s))(U \otimes S_3)h(x(s)) \right] ds \\
&\quad - 2\alpha V_6, \\
&\leq kx^T(t)(U \otimes S_2)x(t) + kh^T(x(t))(U \otimes S_3)h(x(t)) \\
&\quad - \frac{(1 - \mu_2)}{k} e^{-2\alpha k} \left(\int_{t-k(t)}^t x(s) ds \right)^T (U \otimes S_2) \left(\int_{t-k(t)}^t x(s) ds \right) \\
&\quad - \frac{(1 - \mu_2)}{k} e^{-2\alpha k} \left(\int_{t-k(t)}^t h(x(s)) ds \right)^T (U \otimes S_3) \left(\int_{t-k(t)}^t h(x(s)) ds \right) \\
&\quad - 2\alpha V_6, \\
&= \sum_{1 \leq i < j \leq N}^N \left\{ (x_i(t) - x_j(t))^T k S_2 (x_i(t) - x_j(t)) \right. \\
&\quad + \left[h(x_i(t)) - h(x_j(t)) \right]^T k S_3 \left[h(x_i(t)) - h(x_j(t)) \right] \\
&\quad - \frac{(1 - \mu_2)}{k} e^{-2\alpha k} \left(\int_{t-k(t)}^t k(t-s)[x_i(s) - x_j(s)] ds \right)^T \\
&\quad \times S_2 \left(\int_{t-k(t)}^t k(t-s)[x_i(s) - x_j(s)] ds \right) \\
&\quad - \frac{(1 - \mu_2)}{k} e^{-2\alpha k} \left(\int_{t-k(t)}^t k(t-s)[h(x_i(s)) - h(x_j(s))] ds \right)^T \\
&\quad \times S_3 \left(\int_{t-k(t)}^t k(t-s)[h(x_i(s)) - h(x_j(s))] ds \right) \left. \right\} - 2\alpha V_6. \tag{4.22}
\end{aligned}$$

By the assumption 4.2, for $r \in \{1, 2, \dots, n\}$, we have

$$\begin{aligned}
&\left(\frac{f_r(x_{ir}(t)) - f_r(x_{jr}(t))}{x_{ir}(t) - x_{jr}(t)} - l_r^+ \right) \left(\frac{f_r(x_{ir}(t)) - f_r(x_{jr}(t))}{x_{ir}(t) - x_{jr}(t)} - l_r^- \right) \leq 0, \\
&\left(\frac{g_r(x_{ir}(t)) - g_r(x_{jr}(t))}{x_{ir}(t) - x_{jr}(t)} - \varrho_r^+ \right) \left(\frac{g_r(x_{ir}(t)) - g_r(x_{jr}(t))}{x_{ir}(t) - x_{jr}(t)} - \varrho_r^- \right) \leq 0, \\
&\left(\frac{h_r(x_{ir}(t)) - h_r(x_{jr}(t))}{x_{ir}(t) - x_{jr}(t)} - v_r^+ \right) \left(\frac{h_r(x_{ir}(t)) - h_r(x_{jr}(t))}{x_{ir}(t) - x_{jr}(t)} - v_r^- \right) \leq 0.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\left((f_r(x_{ir}(t)) - f_r(x_{jr}(t))) - l_r^+(x_{ir}(t) - x_{jr}(t)) \right) \\
&\quad \times \left((f_r(x_{ir}(t)) - f_r(x_{jr}(t))) - l_r^-(x_{ir}(t) - x_{jr}(t)) \right) \leq 0, \\
&\left((g_r(x_{ir}(t)) - g_r(x_{jr}(t))) - \varrho_r^+(x_{ir}(t) - x_{jr}(t)) \right) \\
&\quad \times \left((g_r(x_{ir}(t)) - g_r(x_{jr}(t))) - \varrho_r^-(x_{ir}(t) - x_{jr}(t)) \right) \leq 0, \\
&\left((h_r(x_{ir}(t)) - h_r(x_{jr}(t))) - v_r^+(x_{ir}(t) - x_{jr}(t)) \right) \\
&\quad \times \left((h_r(x_{ir}(t)) - h_r(x_{jr}(t))) - v_r^-(x_{ir}(t) - x_{jr}(t)) \right) \leq 0.
\end{aligned}$$

which are equivalent to

$$\begin{aligned}
& \begin{bmatrix} x_i(t) - x_j(t) \\ f(x_i(t)) - f(x_j(t)) \end{bmatrix}^T \begin{bmatrix} l_r^+ l_r^- e_r e_r^T & -\frac{l_r^+ + l_r^-}{2} e_r e_r^T \\ -\frac{l_r^+ + l_r^-}{2} e_r e_r^T & e_r e_r^T \end{bmatrix} \begin{bmatrix} x_i(t) - x_j(t) \\ f(x_i(t)) - f(x_j(t)) \end{bmatrix} \leq 0, \\
& \begin{bmatrix} x_i(t) - x_j(t) \\ g(x_i(t)) - g(x_j(t)) \end{bmatrix}^T \begin{bmatrix} \varrho_r^+ \varrho_r^- e_r e_r^T & -\frac{\varrho_r^+ + \varrho_r^-}{2} e_r e_r^T \\ -\frac{\varrho_r^+ + \varrho_r^-}{2} e_r e_r^T & e_r e_r^T \end{bmatrix} \begin{bmatrix} x_i(t) - x_j(t) \\ g(x_i(t)) - g(x_j(t)) \end{bmatrix} \leq 0, \\
& \begin{bmatrix} x_i(t) - x_j(t) \\ h(x_i(t)) - h(x_j(t)) \end{bmatrix}^T \begin{bmatrix} v_r^+ v_r^- e_r e_r^T & -\frac{v_r^+ + v_r^-}{2} e_r e_r^T \\ -\frac{v_r^+ + v_r^-}{2} e_r e_r^T & e_r e_r^T \end{bmatrix} \begin{bmatrix} x_i(t) - x_j(t) \\ h(x_i(t)) - h(x_j(t)) \end{bmatrix} \leq 0.
\end{aligned}$$

where e_r denotes the unit column vector having 1 element on its r th row and zeros elsewhere. For any diagonal matrix $\Lambda = \text{diag}(\iota_1, \dots, \iota_n) > 0$, $\Omega = \text{diag}(\kappa_1, \dots, \kappa_n) > 0$, and $\Delta = \text{diag}(\eta_1, \dots, \eta_n) > 0$ with appropriate dimensions, it follows that:

$$\begin{aligned}
0 & \leq - \sum_{r=1}^n \iota_r \begin{bmatrix} x_i(t) - x_j(t) \\ f(x_i(t)) - f(x_j(t)) \end{bmatrix}^T \begin{bmatrix} l_r^+ l_r^- e_r e_r^T & -\frac{l_r^+ + l_r^-}{2} e_r e_r^T \\ -\frac{l_r^+ + l_r^-}{2} e_r e_r^T & e_r e_r^T \end{bmatrix} \\
& \quad \times \begin{bmatrix} x_i(t) - x_j(t) \\ f(x_i(t)) - f(x_j(t)) \end{bmatrix} \\
& - \sum_{r=1}^n \kappa_r \begin{bmatrix} x_i(t) - x_j(t) \\ g(x_i(t)) - g(x_j(t)) \end{bmatrix}^T \begin{bmatrix} \varrho_r^+ \varrho_r^- e_r e_r^T & -\frac{\varrho_r^+ + \varrho_r^-}{2} e_r e_r^T \\ -\frac{\varrho_r^+ + \varrho_r^-}{2} e_r e_r^T & e_r e_r^T \end{bmatrix} \\
& \quad \times \begin{bmatrix} x_i(t) - x_j(t) \\ g(x_i(t)) - g(x_j(t)) \end{bmatrix} \\
& - \sum_{r=1}^n \eta_r \begin{bmatrix} x_i(t) - x_j(t) \\ h(x_i(t)) - h(x_j(t)) \end{bmatrix}^T \begin{bmatrix} v_r^+ v_r^- e_r e_r^T & -\frac{v_r^+ + v_r^-}{2} e_r e_r^T \\ -\frac{v_r^+ + v_r^-}{2} e_r e_r^T & e_r e_r^T \end{bmatrix} \\
& \quad \times \begin{bmatrix} x_i(t) - x_j(t) \\ h(x_i(t)) - h(x_j(t)) \end{bmatrix}, \\
& = \begin{bmatrix} x_i(t) - x_j(t) \\ f(x_i(t)) - f(x_j(t)) \end{bmatrix}^T \begin{bmatrix} -\Lambda F_1 & \Lambda F_2 \\ \Lambda F_2 & -\Lambda \end{bmatrix} \begin{bmatrix} x_i(t) - x_j(t) \\ f(x_i(t)) - f(x_j(t)) \end{bmatrix} \\
& + \begin{bmatrix} x_i(t) - x_j(t) \\ g(x_i(t)) - g(x_j(t)) \end{bmatrix}^T \begin{bmatrix} -\Omega J_1 & \Omega J_2 \\ \Omega J_2 & -\Omega \end{bmatrix} \begin{bmatrix} x_i(t) - x_j(t) \\ g(x_i(t)) - g(x_j(t)) \end{bmatrix} \\
& + \begin{bmatrix} x_i(t) - x_j(t) \\ h(x_i(t)) - h(x_j(t)) \end{bmatrix}^T \begin{bmatrix} -\Delta L_1 & \Delta L_2 \\ \Delta L_2 & -\Delta \end{bmatrix} \begin{bmatrix} x_i(t) - x_j(t) \\ h(x_i(t)) - h(x_j(t)) \end{bmatrix}. \quad (4.23)
\end{aligned}$$

On the other hand, by (4.3) and properties of Kronecker product in Lemma 2.3.14 and

Lemma 2.3.15, we have the following zero equations holds for any matrix $T \in \mathcal{R}^{n \times n}$,

$$\begin{aligned}
0 &= 2\dot{x}^T(t)(U \otimes T) \left[- (I_N \otimes C) x(t - \sigma) + (I_N \otimes A) f(x(t)) \right. \\
&\quad + (I_N \otimes B) g(x(t - \tau(t))) + (I_N \otimes D) \int_{t-k(t)}^t h(x(s)) ds \\
&\quad + (I_N \otimes EK) x(t) + (G^1 \otimes \Gamma_1) x(t) + (G^2 \otimes \Gamma_2) x(t - \tau(t)) \\
&\quad \left. + (G^3 \otimes \Gamma_3) \int_{t-k(t)}^t h(x(s)) ds - \dot{x}(t) \right], \\
&= 2 \sum_{1 \leq i < j \leq N} (\dot{x}_i(t) - \dot{x}_j(t))^T \left[(-Ng_{ij}^1 T \Gamma_1 + TEK)(x_i(t) - x_j(t)) \right. \\
&\quad - TC(x_i(t - \sigma) - x_j(t - \sigma)) + TA(f(x_i(t)) - f(x_j(t))) \\
&\quad + TB(g(x_i(t - \tau(t))) - g(x_j(t - \tau(t)))) \\
&\quad + (TD - Ng_{ij}^3 T \Gamma_3) \int_{t-k(t)}^t (h(x_i(s)) - h(x_j(s))) ds \\
&\quad \left. - Ng_{ij}^2 T \Gamma_2(x_i(t - \tau(t)) - x_j(t - \tau(t))) - T(\dot{x}_i(t) - \dot{x}_j(t)) \right]. \tag{4.24}
\end{aligned}$$

Using Lemma 2.3.17, we get

$$\begin{aligned}
&- \sum_{1 \leq i < j \leq N} 2(\dot{x}_i(t) - \dot{x}_j(t))^T TEK (x_i(t) - x_j(t)) \\
&= - \sum_{1 \leq i < j \leq N} 2(\dot{x}_i(t) - \dot{x}_j(t))^T TEP^{-1}Y (x_i(t) - x_j(t)) \\
&\leq \sum_{1 \leq i < j \leq N} \left[(\dot{x}_i(t) - \dot{x}_j(t))^T TEP^{-1}E^T T^T (\dot{x}_i(t) - \dot{x}_j(t)) \right. \\
&\quad \left. + (x_i(t) - x_j(t))^T Y^T P^{-1}Y (x_i(t) - x_j(t)) \right]. \tag{4.25}
\end{aligned}$$

By setting $\zeta_{ij} = \left[(x_i(t) - x_j(t))^T, (x_i(t - \sigma) - x_j(t - \sigma))^T, (\dot{x}_i(t) - \dot{x}_j(t))^T, (g(x_i(t - \tau(t))) - g(x_j(t - \tau(t))))^T, (g(x_i(t)) - g(x_j(t)))^T, (f(x_i(t)) - f(x_j(t)))^T, (h(x_i(t)) - h(x_j(t)))^T, (x_i(t - \tau(t)) - x_j(t - \tau(t)))^T, (x_i(t - \tau_1) - x_j(t - \tau_1))^T, (x_i(t - \tau_2) - x_j(t - \tau_2))^T, \int_{-\infty}^t k(t - s)(x_i(s) - x_j(s))^T ds, \int_{-\infty}^t k(t - s)(h(x_i(s)) - h(x_j(s)))^T ds \right]$. From (4.11) - (4.25), we can obtain that

$$\begin{aligned}
\dot{V}(x(t)) &\leq \sum_{1 \leq i < j \leq N} \left\{ \zeta_{ij}^T(t) \left[(1 - \beta)\Sigma_{1ij} + \beta\Sigma_{2ij} \right] \zeta_{ij}(t) \right. \\
&\quad + (x_i(t) - x_j(t))^T \Xi_1 (x_i(t) - x_j(t)) \\
&\quad \left. + (\dot{x}_i(t) - \dot{x}_j(t))^T \Xi_2 (\dot{x}_i(t) - \dot{x}_j(t)) \right\} - 2\alpha V(x(t)), \tag{4.26}
\end{aligned}$$

where Σ_{1ij} and Σ_{2ij} are defined in (4.5), (4.6), respectively, and

$$\begin{aligned}
\Xi_1 &= -0.1e^{-2\alpha\tau_1} R_1 - 0.1e^{-2\alpha\tau_2} R_2 + PEP^{-1}E^T P + 2Y^T P^{-1}Y, \\
\Xi_2 &= -T + TEP^{-1}E^T T^T.
\end{aligned}$$

Since $0 \leq \beta \leq 1$, $(1 - \beta)\Sigma_{1ij} + \beta\Sigma_{2ij}$ is a convex combination of Σ_{1ij} and Σ_{2ij} . So, $(1 - \beta)\Sigma_{1ij} + \beta\Sigma_{2ij} < 0$ is equivalent to $\Sigma_{1ij} < 0$ and $\Sigma_{2ij} < 0$. By Lemma 2.3.22, the inequalities Ξ_1 and Ξ_2 are equivalent to $\Sigma_3 < 0$ and $\Sigma_4 < 0$. Thus, it follows from (4.5)-(4.8) and (4.26) such that

$$\dot{V}(x(t)) + 2\alpha V(x(t)) \leq 0, \quad \forall t \geq 0. \quad (4.27)$$

Integrating both sides of (4.27) from 0 to t , we obtain

$$V(x(t)) \leq e^{-2\alpha t} V(0), \quad \forall t \geq 0. \quad (4.28)$$

Furthermore, taking condition (4.10) into account, we have

$$\lambda_1 \|x_i(t) - x_j(t)\|^2 \leq V(x(t)) \leq e^{-2\alpha t} V(0) \leq \lambda_2 e^{-2\alpha t} \|\phi\|^2 + \lambda_3 e^{-2\alpha t} \|\varphi\|^2, \quad \forall t \geq 0.$$

then

$$\|x_i(t) - x_j(t)\| \leq \sqrt{\frac{\lambda_2}{\lambda_1}} e^{-\alpha t} \|\phi\| + \sqrt{\frac{\lambda_3}{\lambda_1}} e^{-\alpha t} \|\varphi\|, \quad \forall t \geq 0. \quad (4.29)$$

This completes the proof. $\square$

Next, we present a simulation example so as to illustrate the usefulness of our Theorem 4.2.1.

Example 4.2.2. Consider a dynamical system (4.1) consisting of five linearly coupled identical models under the controller (4.2) with a hybrid coupling. The state equations of the entire array are

$$\begin{cases} \dot{x}_i(t) = -Cx_i(t - \sigma) + Af(x_i(t)) + Bg(x_i(t - \tau(t))) \\ \quad + D \int_{t-k(t)}^t h(x_i(s)) ds + EU_i(t) + \sum_{j=1}^5 g_{ij}^1 \Gamma_1 x_j(t) \\ \quad + \sum_{j=1}^5 g_{ij}^2 \Gamma_2 x_j(t - \tau(t)) + \sum_{j=1}^5 g_{ij}^3 \Gamma_3 \int_{t-k(t)}^t h(x_j(s)) ds, \end{cases} \quad (4.30)$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t))^T$, $i = 1, 2, 3, 4, 5$ are the state vector of the i th network, and the other parameters are as follows:

$$\begin{aligned} C &= \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, \quad A = \begin{bmatrix} 0.5 & -0.4 \\ 0.4 & 0.5 \end{bmatrix}, \quad B = \begin{bmatrix} 0.1 & 0.1 \\ 0.2 & 0.2 \end{bmatrix}, \\ D &= \begin{bmatrix} 0.4 & 0 \\ 0 & 0.3 \end{bmatrix}, \quad E = \begin{bmatrix} 0.3 & 0 \\ 0.1 & 0.3 \end{bmatrix}, \quad \Gamma_1 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \\ \Gamma_2 &= \begin{bmatrix} 0.05 & 0 \\ 0 & 0.05 \end{bmatrix}, \quad \Gamma_3 = \begin{bmatrix} 0.05 & 0 \\ 0 & 0.05 \end{bmatrix}, \\ \tau_1 &= 0.2, \quad \tau_2 = 1.0, \quad k = 1, \quad \sigma = 0.02, \quad \mu_1 = 0.3, \quad \mu_2 = 0.2. \end{aligned}$$

The outer-coupling matrix $G^q = (g_{ij}^q)_{N \times N}$, $(q = 1, 2, 3)$ are defined as:

$$g^1 = \begin{bmatrix} -2 & 1 & 0 & 0 & 1 \\ 1 & -3 & 1 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 1 & 1 & -3 & 1 \\ 1 & 0 & 0 & 1 & -2 \end{bmatrix}, \quad g^2 = g^3 = \begin{bmatrix} -4 & 1 & 1 & 1 & 1 \\ 1 & -4 & 1 & 1 & 1 \\ 1 & 1 & -4 & 1 & 1 \\ 1 & 1 & 1 & -4 & 1 \\ 1 & 1 & 1 & 1 & -4 \end{bmatrix}.$$

Take the activation function as follows:

$$\begin{aligned} f_1(x_{i1}(t)) &= 0.5(|x_{i1}(t) + 1| - |x_{i1}(t) - 1|), \\ f_2(x_{i2}(t)) &= 0.25(|x_{i2}(t) + 1| - |x_{i2}(t) - 1|), \\ g_1(x_{i1}(t)) &= 0.5(|x_{i1}(t) + 1| - |x_{i1}(t) - 1|), \\ g_2(x_{i2}(t)) &= 0.25(|x_{i2}(t) + 1| - |x_{i2}(t) - 1|), \\ h_1(x_{i1}(s)) &= \tanh(-x_{i1}(s)), \\ h_2(x_{i2}(s)) &= \tanh(-0.8x_{i2}(s)). \end{aligned}$$

It is easy to see:

$$\begin{aligned} F_1 &= \begin{bmatrix} -1 & 0 \\ 0 & -0.25 \end{bmatrix}, & F_2 &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \\ J_1 &= \begin{bmatrix} -1 & 0 \\ 0 & -0.25 \end{bmatrix}, & J_2 &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \\ L_1 &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, & L_2 &= \begin{bmatrix} -0.5 & 0 \\ 0 & -0.4 \end{bmatrix}. \end{aligned}$$

By applying the MATLAB LMI Toolbox, Theorem 4.2.1 can be solved to yield the following feasible solutions:

$$\begin{aligned} P &= \begin{bmatrix} 19.9722 & -0.9649 \\ -0.9649 & 16.2956 \end{bmatrix}, & Q_1 &= \begin{bmatrix} 0.4759 & 0.0594 \\ 0.0594 & 0.8187 \end{bmatrix}, \\ Q_2 &= \begin{bmatrix} 3.6265 & -0.2895 \\ -0.2895 & 4.4498 \end{bmatrix}, & Q_3 &= \begin{bmatrix} 6.3244 & -0.6504 \\ -0.6504 & 7.6949 \end{bmatrix}, \\ Q_4 &= \begin{bmatrix} 6.3244 & -0.6504 \\ -0.6504 & 7.6950 \end{bmatrix}, & Q_5 &= \begin{bmatrix} 8.3350 & 0.8309 \\ 0.8309 & 22.9277 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned}
R_1 &= \begin{bmatrix} 21.6167 & 3.1821 \\ 3.1821 & 19.6786 \end{bmatrix}, & R_2 &= \begin{bmatrix} 0.0330 & -0.0106 \\ -0.0106 & 0.0239 \end{bmatrix}, \\
R_3 &= \begin{bmatrix} 0.1450 & -0.0601 \\ -0.0601 & 0.0649 \end{bmatrix}, & S_1 &= \begin{bmatrix} 80.2454 & -3.2911 \\ -3.2911 & 64.9877 \end{bmatrix}, \\
S_2 &= \begin{bmatrix} 3.6598 & -0.2077 \\ -0.2077 & 4.0363 \end{bmatrix}, & S_3 &= \begin{bmatrix} 4.0931 & -0.0845 \\ -0.0845 & 4.6057 \end{bmatrix}, \\
T &= \begin{bmatrix} 3.0487 & -0.1580 \\ -0.1717 & 2.6529 \end{bmatrix}, & Y &= \begin{bmatrix} 0.3942 & -0.0615 \\ -0.0615 & 0.3314 \end{bmatrix}, \\
\Omega &= \begin{bmatrix} 15.6411 & 0 \\ 0 & 23.6617 \end{bmatrix}, & \Theta &= \begin{bmatrix} 14.3208 & 0 \\ 0 & 36.3797 \end{bmatrix}, \\
\Delta &= \begin{bmatrix} 15.5635 & 0 \\ 0 & 18.6668 \end{bmatrix}.
\end{aligned}$$

Then, the controller gain is

$$K = \begin{bmatrix} -0.0196 & 0.0021 \\ 0.0026 & -0.0202 \end{bmatrix}. \quad (4.31)$$

Therefore, it follows from Theorem 4.2.1 that the system (4.30) under controller gain (4.31) with given parameters is exponentially synchronized, which is further verified by the simulation result of the synchronization error given in Figure 4.1. The figures show that the synchronization errors trajectories $e_{i1}(t) = x_i(t) - x_1(t), (i = 2, 3, 4, 5)$ of the system (4.30) with initial values of the state are $x_1(0) = [1, -1]^T, x_2(0) = [-1.5, 1.2]^T, x_3(0) = [2, -1.6]^T, x_4(0) = [0.7, 1.1]^T, x_5(0) = [-1.2, -1.8]^T$. The numerical simulations are carried out using the explicit Runge-Kutta-like method (dde45), interpolation and extrapolation by spline of the third order.

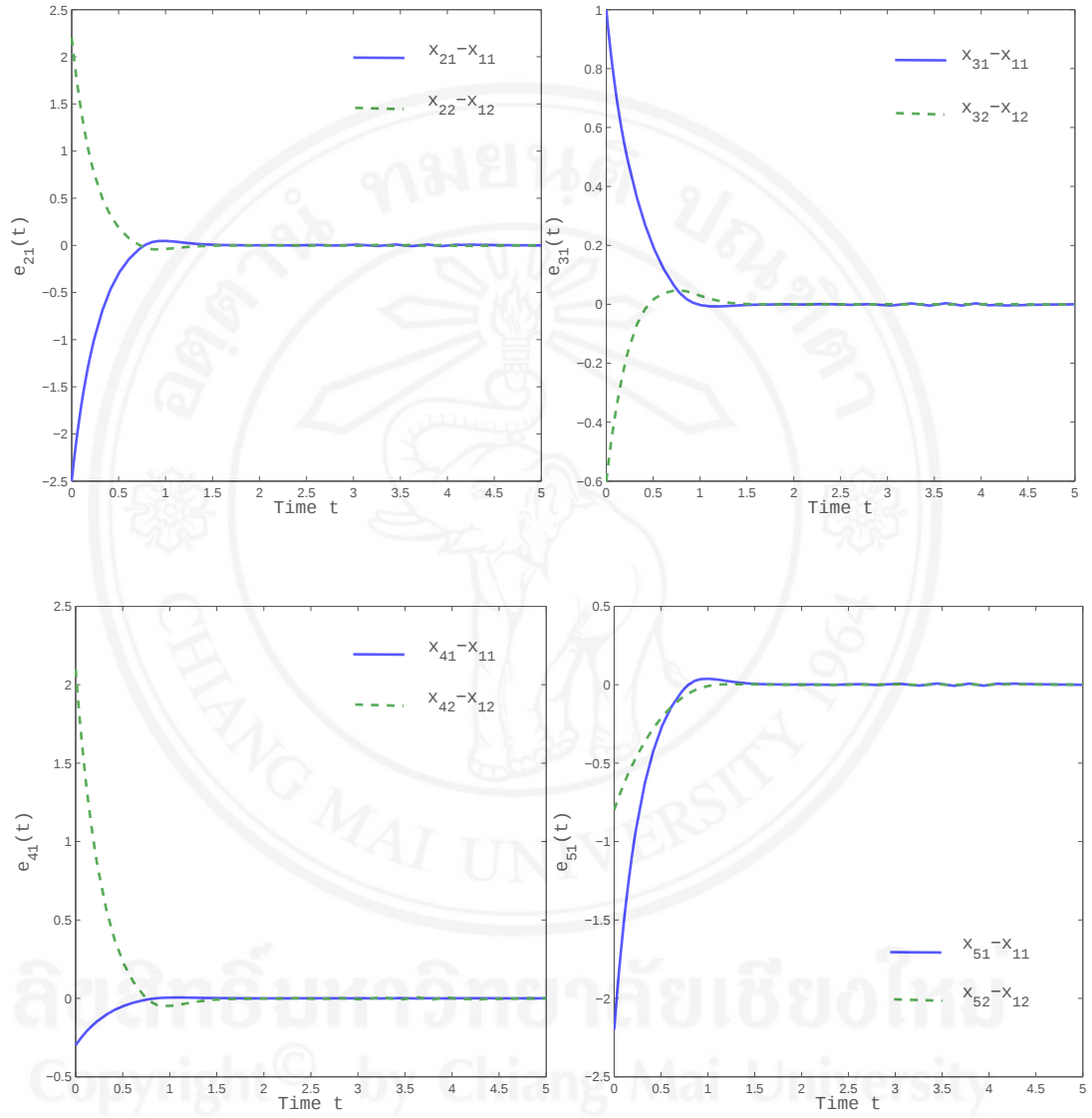


Figure 4.1: Synchronization errors trajectories, $e_{i1}(t) = x_i(t) - x_1(t)$, $i = 2, 3, 4, 5$.

4.3 Synchronization Criteria with Intermittent Feedback Control

In this section, we study the exponential synchronization of coupled NNs (4.1) with hybrid coupling by design the intermittent feedback controller satisfy the following conditions:

$$U_i(t) = \begin{cases} Kx_i(t), & n\omega \leq t \leq n\omega + \delta, \\ 0, & n\omega + \delta < t \leq (n+1)\omega, \end{cases}, \quad (4.32)$$

where K is a constant control gain. $\omega > 0$ is the control period. $\delta > 0$ is the control width (control duration) and n is a non-negative integer.

Then, with Kronecker product (4.32) can be represented as

$$U(t) = \begin{cases} (I_N \otimes K)x(t), & n\omega \leq t \leq n\omega + \delta, \\ 0, & n\omega + \delta < t \leq (n+1)\omega, \end{cases},$$

We obtain the following result.

Theorem 4.3.1. *Given $0 < \alpha < \varepsilon$, and suppose that assumption 4.1-4.2 hold, the system (4.1) is exponentially synchronized under the controller (4.32) if there exist symmetric positive definite matrices, $P, Y, Q_1, Q_2, Q_3, Q_4, Q_5, R_l, S_l, (l = 1, 2, 3)$, three diagonal matrices $\Lambda = \text{diag}(\iota_1, \dots, \iota_n) > 0, \Omega = \text{diag}(\kappa_1, \dots, \kappa_n) > 0, \Delta = \text{diag}(\eta_1, \dots, \eta_n) > 0$ and real matrices T with appropriate dimensions, such that the following linear matrix inequalities (LMIs) hold for all $1 = i < j = N$:*

$$\begin{aligned} \Phi_{1ij} &= Z_{ij} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & -I & 0 & 0 \end{bmatrix}^T \\ &\quad \times e^{-2\alpha\tau_2} R_3 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & -I & 0 & 0 \end{bmatrix} < 0, \end{aligned} \quad (4.33)$$

$$\begin{aligned} \Phi_{2ij} &= Z_{ij} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I & I & 0 & 0 & 0 \end{bmatrix}^T \\ &\quad \times e^{-2\alpha\tau_2} R_3 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I & I & 0 & 0 & 0 \end{bmatrix} < 0, \end{aligned} \quad (4.34)$$

$$\Phi_3 = \begin{bmatrix} -0.1e^{-2\alpha\tau_1} R_1 - 0.1e^{-2\alpha\tau_2} R_2 & 2Y^T & PE \\ * & -2P & 0 \\ * & * & -P \end{bmatrix} < 0, \quad (4.35)$$

$$\Phi_4 = \begin{bmatrix} -T & TE \\ * & -P \end{bmatrix} < 0, \quad (4.36)$$

$$\begin{aligned} \Phi_{5ij} &= \hat{Z}_{ij} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & -I & 0 & 0 \end{bmatrix}^T \\ &\quad \times e^{-2\alpha\tau_2} R_3 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & -I & 0 & 0 \end{bmatrix} < 0, \end{aligned} \quad (4.37)$$

$$\begin{aligned} \Phi_{6ij} &= \hat{Z}_{ij} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I & I & 0 & 0 & 0 \end{bmatrix}^T \\ &\quad \times e^{-2\alpha\tau_2} R_3 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I & I & 0 & 0 & 0 \end{bmatrix} < 0, \end{aligned} \quad (4.38)$$

and

$$-\alpha\delta + (\varepsilon - \alpha)(\omega - \delta) < 0, \quad (4.39)$$

where

$$\begin{aligned} Z_{ij} &= \begin{bmatrix} Z_{1,1} & Z_{1,2} & Z_{1,3} & \cdots & Z_{1,12} \\ * & Z_{2,2} & Z_{2,3} & \cdots & Z_{2,12} \\ * & * & Z_{3,3} & \cdots & Z_{3,12} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & Z_{12,12} \end{bmatrix}, \\ \hat{Z}_{ij} &= \begin{bmatrix} \hat{Z}_{1,1} & Z_{1,2} & Z_{1,3} & \cdots & Z_{1,12} \\ * & Z_{2,2} & Z_{2,3} & \cdots & Z_{2,12} \\ * & * & \hat{Z}_{3,3} & \cdots & Z_{3,12} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & Z_{12,12} \end{bmatrix}, \\ Z_{1,1} &= -Ng_{ij}^1 P \Gamma_1 - Ng_{ij}^1 \Gamma_1^T P + 2\alpha P + \sum_{r=1}^4 Q_r - 0.9e^{-2\alpha\tau_1} R_1 \\ &\quad - 0.9e^{-2\alpha\tau_2} R_2 - e^{-2\alpha\sigma} S_1 + kS_2 - \Omega F_1 - \Theta J_1 - \Delta L_1, \\ Z_{1,2} &= -e^{-2\alpha\sigma} S_1 + C^T P, \\ Z_{1,3} &= -Ng_{ij}^1 T \Gamma_1, \\ Z_{1,4} &= B^T P, \\ Z_{1,5} &= \Omega F_2, \\ Z_{1,6} &= \Theta J_2 + A^T P, \\ Z_{1,7} &= \Delta L_2, \\ Z_{1,8} &= -Ng_{ij}^2 \Gamma_2^T P, \\ Z_{1,9} &= e^{-2\alpha\tau_1} R_1, \\ Z_{1,10} &= e^{-2\alpha\tau_2} R_2, \\ Z_{1,12} &= -Ng_{ij}^3 \Gamma_3^T P + D^T P, \\ Z_{2,2} &= -e^{-2\alpha\sigma} (Q_1 + S_1), \\ Z_{2,3} &= -TC, \\ Z_{3,3} &= \tau_1^2 R_1 + \tau_2^2 R_2 + \eta^2 R_3 + \sigma^2 S_1 - T, \\ Z_{3,4} &= B^T T, \end{aligned}$$

$$\begin{aligned}
Z_{3,6} &= A^T T^T, \\
Z_{3,8} &= -Ng_{ij}^2 \Gamma_2^T T^T, \\
Z_{3,12} &= -Ng_{ij}^3 \Gamma_3^T T^T + D^T T^T, \\
Z_{4,4} &= -(1 - \mu_1) e^{-2\alpha\tau_2} Q_5, \\
Z_{5,5} &= -\Omega - Q_5, \\
Z_{6,6} &= -\Theta, \\
Z_{7,7} &= kS_3 - \Delta, \\
Z_{8,8} &= -(1 - \mu_1) e^{-2\alpha\tau_2} Q_4 - e^{-2\alpha\tau_2} R_3 - e^{-2\alpha\tau_2} R_3, \\
Z_{8,9} &= -e^{-2\alpha\tau_2} R_3, \\
Z_{8,10} &= -e^{-2\alpha\tau_2} R_3, \\
Z_{9,9} &= -e^{-2\alpha\tau_1} (Q_2 + R_1) - e^{-2\alpha\tau_2} R_3, \\
Z_{10,10} &= -e^{-2\alpha\tau_2} (R_2 + R_3 + Q_3), \\
Z_{11,11} &= -\frac{(1 - \mu_2)}{k} e^{-2\alpha k} S_2, \\
Z_{12,12} &= -\frac{(1 - \mu_2)}{k} e^{-2\alpha k} S_3, \\
0 &= Z_{1,11} = Z_{2,4} = Z_{2,5} = Z_{2,6} = Z_{2,7} = Z_{2,8} = Z_{2,9} \\
&= Z_{2,10} = Z_{2,11} = Z_{2,12} = Z_{3,5} = Z_{3,7} = Z_{3,9} = Z_{3,10} \\
&= Z_{3,11} = Z_{4,5} = Z_{4,6} = Z_{4,7} = Z_{4,8} = Z_{4,9} = Z_{4,10} \\
&= Z_{4,11} = Z_{4,12} = Z_{5,6} = Z_{5,7} = Z_{5,8} = Z_{5,9} = Z_{5,10} \\
&= Z_{5,11} = Z_{5,12} = Z_{6,7} = Z_{6,8} = Z_{6,9} = Z_{6,10} = Z_{6,11} \\
&= Z_{6,12} = Z_{7,8} = Z_{7,9} = Z_{7,10} = Z_{7,11} = Z_{7,12} = Z_{8,11} \\
&= Z_{8,12} = Z_{9,10} = Z_{9,11} = Z_{9,12} = Z_{10,11} = Z_{10,12} = Z_{11,12}, \\
\hat{Z}_{1,1} &= -Ng_{ij}^1 P \Gamma_1 - Ng_{ij}^1 \Gamma_1^T P + 2\alpha P + \sum_{r=1}^4 Q_r - e^{-2\alpha\tau_1} R_1 - e^{-2\alpha\tau_2} R_2 \\
&\quad - e^{-2\alpha\sigma} S_1 + kS_2 - \Omega F_1 - \Theta J_1 - \Delta L_1 - 2\varepsilon P, \\
\hat{Z}_{3,3} &= \tau_1^2 R_1 + \tau_2^2 R_2 + \eta^2 R_3 + \sigma^2 S_1 - T - T^T.
\end{aligned}$$

Moreover, the memoryless feedback control is $K = -P^{-1}Y$.

Proof : Choose Lyapunov-Krasovskii functional as in candidate to be

$$V(x(t)) = \sum_{l=1}^6 V_l(x(t)), \quad (4.40)$$

where

$$\begin{aligned}
V_1(x(t)) &= x^T(t)(U \otimes P)x(t), \\
V_2(x(t)) &= \int_{t-\sigma}^t e^{2\alpha(s-t)} x^T(s)(U \otimes Q_1)x(s)ds \\
&\quad + \sum_{l=1}^2 \int_{t-\tau_l}^t e^{2\alpha(s-t)} x^T(s)(U \otimes Q_{l+1})x(s)ds \\
&\quad + \int_{t-\tau(t)}^t e^{2\alpha(s-t)} x^T(s)(U \otimes Q_4)x(s)ds \\
&\quad + \int_{t-\tau(t)}^t e^{2\alpha(s-t)} g(x^T(s))(U \otimes Q_5)g(x(s))ds, \\
V_3(x(t)) &= \tau_1 \int_{-\tau_1}^0 \int_{t+s}^t e^{2\alpha(\theta-t)} \dot{x}^T(\theta)(U \otimes R_1)\dot{x}(\theta) d\theta ds \\
&\quad + \tau_2 \int_{-\tau_2}^0 \int_{t+s}^t e^{2\alpha(\theta-t)} \dot{x}^T(\theta)(U \otimes R_2)\dot{x}(\theta) d\theta ds, \\
V_4(x(t)) &= \eta \int_{-\tau_2}^{-\tau_1} \int_{t+s}^t e^{2\alpha(\theta-t)} \dot{x}^T(\theta)(U \otimes R_3)\dot{x}(\theta) d\theta ds, \\
V_5(x(t)) &= \sigma \int_{-\sigma}^0 \int_{t+s}^t e^{2\alpha(\theta-t)} \dot{x}^T(\theta)(U \otimes S_1)\dot{x}(\theta) d\theta ds, \\
V_6(x(t)) &= \int_{-k(t)}^0 \int_{t+s}^t e^{2\alpha(\theta-t)} x^T(\theta)(U \otimes S_2)x(\theta) d\theta ds \\
&\quad + \int_{-k(t)}^0 \int_{t+s}^t e^{2\alpha(\theta-t)} h^T(x(\theta))(U \otimes S_3)h(x(\theta)) d\theta ds.
\end{aligned}$$

It easy to check that

$$\lambda_1 \|x(t)\|^2 \leq V(x(t)), \quad \forall t \geq 0. \quad (4.41)$$

where λ_1 is defined in (4.4).

Case I: for $U_i(t) = Kx_i(t)$, $n\omega \leq t \leq n\omega + \delta$, by taking the derivative of $V(x(t))$ along the trajectories of system (4.1). We may proof this by using a similar argument as in the proof of Theorem 4.2.1. Thus, we have

$$\dot{V}(x(t)) + 2\alpha V(x(t)) \leq 0, \quad \text{for } n\omega \leq t \leq n\omega + \delta. \quad (4.42)$$

So, by the above differential inequality (4.42), we obtain

$$V(x(t)) \leq V(x(n\omega))e^{-2\alpha(t-n\omega)}, \quad \text{for } n\omega \leq t \leq n\omega + \delta. \quad (4.43)$$

Case II: for $U_i(t) = 0$, $n\omega + \delta \leq t \leq (n+1)\omega$, by taking the derivative of $V(x(t))$ along the trajectories of system (4.1). We are able to do similar estimation as we did in the Theorem 4.2.1, and addition $2\varepsilon V(x(t)) - 2\varepsilon V_1(x(t))$. So, we obtain:

$$\dot{V}(x(t)) - 2(\varepsilon - \alpha)V(x(t)) \leq \sum_{1 \leq i < j \leq N}^N \zeta_{ij}^T(t)[(1 - \beta)\Phi_{5ij} + \beta\Phi_{6ij}]\zeta_{ij}(t), \quad (4.44)$$

where Φ_{5ij} and Φ_{6ij} are defined in (4.37), (4.38), respectively.

By $(1 - \beta)\Phi_{5ij} + \beta\Phi_{6ij} < 0$ holds if and only if $\Phi_{5ij} < 0$ and $\Phi_{6ij} < 0$. Thus, it follows from (4.37)-(4.38) and (4.44), we obtain

$$\dot{V}(x(t)) - 2(\varepsilon - \alpha)V(x(t)) \leq 0, \quad n\omega + \delta < t \leq (n+1)\omega. \quad (4.45)$$

From the above differential inequality (4.45), we have

$$V(x(t)) \leq V(x(n\omega + \delta))e^{2(\varepsilon - \alpha)(t - n\omega - \delta)}, \quad n\omega + \delta < t \leq (n+1)\omega. \quad (4.46)$$

Therefore, by (4.43), (4.46) and conditions (4.32), we obtain

$$\begin{aligned} V(x((n+1)\omega)) &\leq V(x(n\omega + \delta))e^{2(\varepsilon - \alpha)(\omega - \delta)} \\ &\leq V(x(n\omega))e^{-2\alpha\delta}e^{2(\varepsilon - \alpha)(\omega - \delta)} \\ &= V(x(n\omega))e^{-2\alpha\delta + 2(\varepsilon - \alpha)(\omega - \delta)} \\ &\leq V(x((n-1)\omega + \delta))e^{2(\varepsilon - \alpha)(\omega - \delta)}e^{-2\alpha\delta + 2(\varepsilon - \alpha)(\omega - \delta)} \\ &\leq V(x((n-1)\omega))e^{-2\alpha\delta + 2(\varepsilon - \alpha)(\omega - \delta)}e^{-2\alpha\delta + 2(\varepsilon - \alpha)(\omega - \delta)} \\ &= V(x((n-1)\omega))e^{2(-2\alpha\delta + 2(\varepsilon - \alpha)(\omega - \delta))} \\ &\vdots \\ &\leq V(x(0))e^{(-2\alpha\delta + 2(\varepsilon - \alpha)(\omega - \delta))(n+1)}. \end{aligned} \quad (4.47)$$

For any $t > 0$, there is a $n_0 \geq 0$, such that $n_0\omega \leq t \leq (n_0 + 1)\omega$.

For $n_0\omega + \delta \leq t \leq (n_0 + 1)\omega$, using condition (4.39), we have

$$\begin{aligned} V(x(t)) &\leq V(x(n_0\omega + \delta))e^{2(\varepsilon - \alpha)(t - (n_0\omega + \delta))} \\ &\leq V(x(n_0\omega))e^{-2\alpha\delta}e^{2(\varepsilon - \alpha)(t - (n_0\omega + \delta))} \\ &\leq V(x(0))e^{(-2\alpha\delta + 2(\varepsilon - \alpha)(\omega - \delta))n_0}e^{-2\alpha\delta}e^{2(\varepsilon - \alpha)(t - (n_0\omega + \delta))} \\ &\leq V(x(0))e^{(-2\alpha\delta + 2(\varepsilon - \alpha)(\omega - \delta))n_0}e^{-2\alpha\delta}e^{2(\varepsilon - \alpha)((n_0+1)\omega - (n_0\omega + \delta))} \\ &= V(x(0))e^{(-2\alpha\delta + 2(\varepsilon - \alpha)(\omega - \delta))(n_0+1)} \\ &= V(x(0))e^{\frac{(-2\alpha\delta + 2(\varepsilon - \alpha)(\omega - \delta))(n_0+1)\omega}{\omega}} \\ &\leq V(x(0))e^{\frac{(-2\alpha\delta + 2(\varepsilon - \alpha)(\omega - \delta))t}{\omega}}. \end{aligned} \quad (4.48)$$

For $n_0\omega \leq t \leq n_0\omega + \delta$, using condition (4.39), we have

$$\begin{aligned} V(x(t)) &\leq V(x(n_0\omega))e^{-2\alpha(t - n_0\omega)} \\ &\leq V(x(0))e^{(-2\alpha\delta + 2(\varepsilon - \alpha)(\omega - \delta))n_0}e^{-2\alpha(t - n_0\omega)} \\ &\leq V(x(0))e^{(-2\alpha\delta + 2(\varepsilon - \alpha)(\omega - \delta))n_0} \\ &= V(x(0))e^{(-2\alpha\delta + 2(\varepsilon - \alpha)(\omega - \delta))}e^{(-2\alpha\delta + 2(\varepsilon - \alpha)(\omega - \delta))(n_0+1)} \end{aligned}$$

$$\begin{aligned}
&= V(x(0))e^{-(2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))}e^{\frac{(-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))(n_0+1)\omega}{\omega}} \\
&\leq V(x(0))e^{-(2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))}e^{\frac{(-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))t}{\omega}}.
\end{aligned} \tag{4.49}$$

Let $\xi = e^{-(2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))}$. By (4.48) and (4.49), we obtain

$$V(x(t)) \leq \xi V(x(0))e^{\frac{(-2\alpha\delta+2(\varepsilon-\alpha)(\omega-\delta))t}{\omega}}, \quad \forall t \geq 0. \tag{4.50}$$

On the other hand, using the condition (4.41), we have obtained the following:

$$\|x_i(t) - x_j(t)\| \leq \sqrt{\frac{\lambda_2 \xi}{\lambda_1}} e^{\frac{(-\alpha\delta+(\varepsilon-\alpha)(\omega-\delta))t}{\omega}} \|\phi\| + \sqrt{\frac{\lambda_3 \xi}{\lambda_1}} e^{\frac{(-\alpha\delta+(\varepsilon-\alpha)(\omega-\delta))t}{\omega}} \|\varphi\|, \quad \forall t \geq 0.$$

This completes the proof. $\square$

Remark 4.3.2. In this Theorem 4.3.1, it is worth pointing out that there were no exponential synchronization for hybrid coupled NNs with intermittent feedback control. Through a numerical example is given to illustrate the effectiveness of our theoretical results.

The next example demonstrate the benefits of the method described above.

Example 4.3.3. Consider a dynamical system (4.1) consisting of five linearly coupled identical models under the controller (4.32) with a hybrid coupling. The state equations of the entire array are

$$\begin{cases}
\dot{x}_i(t) = -Cx_i(t - \sigma) + Af(x_i(t)) + Bg(x_i(t - \tau(t))) \\
\quad + D \int_{t-k(t)}^t h(x_i(s)) ds + EU_i(t) + \sum_{j=1}^5 g_{ij}^1 \Gamma_1 x_j(t) \\
\quad + \sum_{j=1}^5 g_{ij}^2 \Gamma_2 x_j(t - \tau(t)) + \sum_{j=1}^5 g_{ij}^3 \Gamma_3 \int_{t-k(t)}^t h(x_j(s)) ds,
\end{cases} \tag{4.51}$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t))^T, i = 1, 2, 3, 4, 5$ are the state vector of the i th network, and the other parameters are as follows:

$$\begin{aligned}
C &= \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}, A = \begin{bmatrix} 0.8 & -0.2 \\ 0.2 & -0.5 \end{bmatrix}, B = \begin{bmatrix} 0.5 & 0.5 \\ 0.2 & 0.2 \end{bmatrix}, \\
D &= \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \Gamma_1 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \\
\Gamma_2 &= \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \Gamma_3 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \\
\tau_1 &= 0.2, \quad \tau_2 = 1.0, \quad k = 1, \quad \sigma = 0.02, \quad \mu_1 = 0.3, \quad \mu_2 = 0.2, \\
\alpha &= 0.2, \quad \delta = 0.5, \quad \varepsilon = 0.3, \quad \omega = 1.0.
\end{aligned}$$

The outer-coupling matrix $G^q = (g_{ij}^q)_{N \times N}$, ($q = 1, 2, 3$) are defined as:

$$g^1 = g^2 = \begin{bmatrix} -2 & 1 & 0 & 0 & 1 \\ 1 & -3 & 1 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 1 & 1 & -3 & 1 \\ 1 & 0 & 0 & 1 & -2 \end{bmatrix}, \quad g^3 = \begin{bmatrix} -4 & 1 & 1 & 1 & 1 \\ 1 & -4 & 1 & 1 & 1 \\ 1 & 1 & -4 & 1 & 1 \\ 1 & 1 & 1 & -4 & 1 \\ 1 & 1 & 1 & 1 & -4 \end{bmatrix}.$$

Take the activation function as follows:

$$\begin{aligned} f_1(x_{i1}(t)) &= 0.5(|x_{i1}(t) + 1| - |x_{i1}(t) - 1|), \\ f_2(x_{i2}(t)) &= 0.25(|x_{i2}(t) + 1| - |x_{i2}(t) - 1|), \\ g_1(x_{i1}(t)) &= 0.25(|x_{i1}(t) + 1| - |x_{i1}(t) - 1|), \\ g_2(x_{i2}(t)) &= 0.5(|x_{i2}(t) + 1| - |x_{i2}(t) - 1|), \\ h_1(x_{i1}(s)) &= \tanh(-1.2x_{i1}(s)), \\ h_2(x_{i2}(s)) &= \tanh(-0.8x_{i2}(s)). \end{aligned}$$

It is easy to see

$$\begin{aligned} F_1 &= \begin{bmatrix} -1 & 0 \\ 0 & -0.25 \end{bmatrix}, & F_2 &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \\ J_1 &= \begin{bmatrix} -0.25 & 0 \\ 0 & -1 \end{bmatrix}, & J_2 &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \\ L_1 &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, & L_2 &= \begin{bmatrix} -0.6 & 0 \\ 0 & -0.4 \end{bmatrix}. \end{aligned}$$

By applying the MATLAB LMI Toolbox, Theorem 4.3.1 can be solved to yield the following feasible solutions:

$$\begin{aligned} P &= \begin{bmatrix} 3.3815 & -0.4108 \\ -0.4108 & 3.6823 \end{bmatrix}, & Q_1 &= \begin{bmatrix} 0.0551 & -0.0061 \\ -0.0061 & 0.0603 \end{bmatrix}, \\ Q_2 &= \begin{bmatrix} 0.7608 & -0.2542 \\ -0.2542 & 0.7411 \end{bmatrix}, & Q_3 &= \begin{bmatrix} 1.5097 & -0.4644 \\ -0.4644 & 1.5188 \end{bmatrix}, \\ Q_4 &= \begin{bmatrix} 2.4981 & -0.4650 \\ -0.4650 & 2.5341 \end{bmatrix}, & Q_5 &= \begin{bmatrix} 8.3809 & 0.4855 \\ 0.4855 & 2.1210 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned}
R_1 &= \begin{bmatrix} 3.3124 & 0.5747 \\ 0.5747 & 4.4357 \end{bmatrix}, & R_2 &= \begin{bmatrix} 0.0018 & -0.0014 \\ -0.0014 & 0.0032 \end{bmatrix}, \\
R_3 &= \begin{bmatrix} 0.0040 & -0.0042 \\ -0.0042 & 0.0088 \end{bmatrix}, & S_1 &= \begin{bmatrix} 19.5547 & -0.3603 \\ -0.3603 & 15.0032 \end{bmatrix}, \\
S_2 &= \begin{bmatrix} 0.8871 & -0.1703 \\ -0.1703 & 0.8924 \end{bmatrix}, & S_3 &= \begin{bmatrix} 1.06749 & -0.0700 \\ -0.0701 & 1.4542 \end{bmatrix}, \\
T &= \begin{bmatrix} 0.3787 & -0.0117 \\ -0.0914 & 0.5761 \end{bmatrix}, & Y &= \begin{bmatrix} 0.0438 & -0.0163 \\ -0.0163 & 0.0559 \end{bmatrix}, \\
\Omega &= \begin{bmatrix} 3.3612 & 0 \\ 0 & 4.1875 \end{bmatrix}, & \Theta &= \begin{bmatrix} 11.2175 & 0 \\ 0 & 3.4441 \end{bmatrix}, \\
\Delta &= \begin{bmatrix} 3.2877 & 0 \\ 0 & 4.5159 \end{bmatrix}.
\end{aligned}$$

Then, the controller gain is

$$K = \begin{bmatrix} -0.0126 & 0.0030 \\ 0.0030 & -0.0149 \end{bmatrix}. \quad (4.52)$$

Therefore, it follows from Theorem 4.3.1 that the system (4.51) under controller gain (4.52) with given parameters is exponentially synchronized, which is further verified by the simulation result of the synchronization error given in Figure 4.2. The figures show that the synchronization errors trajectories $e_{i1}(t) = x_i(t) - x_1(t), (i = 2, 3, 4, 5)$ of the system (4.51) with initial values of the state are $x_1(0) = [1, -1]^T, x_2(0) = [-1.5, 1.2]^T, x_3(0) = [2, -1.6]^T, x_4(0) = [0.7, 1.1]^T, x_5(0) = [-1.2, -1.8]^T$. The numerical simulations are carried out using the explicit Runge-Kutta-like method (dde45), interpolation and extrapolation by spline of the third order.

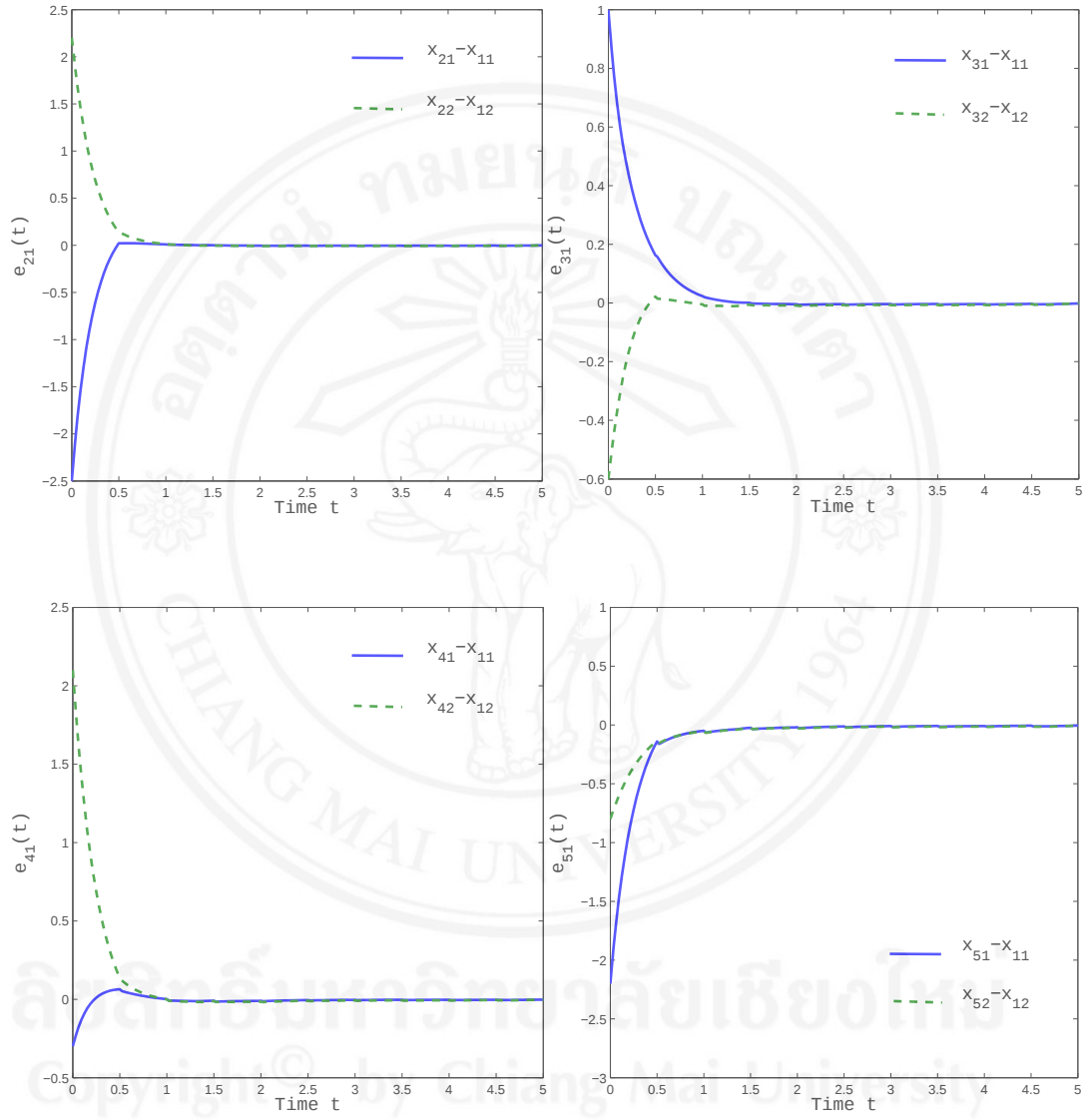


Figure 4.2: Synchronization errors trajectories, $e_{i1}(t) = x_i(t) - x_1(t)$, $i = 2, 3, 4, 5$.