CHAPTER 1

Introduction

Copulas are functions that link joint distribution functions to their univariate marginal distribution functions. This connection is given by Sklar's Theorem which states that if H is an n-dimensional joint distribution function with continuous univariate marginal distribution functions $F_1, F_2, \ldots, F_n$, then there exists a unique n-copula $C : [0,1]^n \to [0,1]$ such that

$$H(x_1, x_2, \ldots, x_n) = C(F_1(x_1), F_2(x_2), \ldots, F_n(x_n)) \tag{1.1}$$

for all $(x_1, x_2, \ldots, x_n) \in [-\infty, \infty]^n$.

Classical examples of copulas are the *independence copula* $\Pi^n(x_1, x_2, \ldots, x_n) = x_1 x_2 \cdots x_n$, the *comonotonicity copula* $M^n(x_1, x_2, \ldots, x_n) = \min(x_1, x_2, \ldots, x_n)$, and the *countermonotonicity copula* $W^2(x_1, x_2) = \max(x_1 + x_2 - 1, 0)$.

If the marginal distribution functions $F_1, F_2, \ldots, F_n$ of the joint distribution function H are non-continuous, then there are infinitely many copulas C satisfying Equation (1.1). Note that C is uniquely determined on $Ran(F_1) \times \cdots \times Ran(F_n)$.

Roughly speaking, an n-subcopula is a restriction of an n-copula on $\prod_{i=1}^{n} A_i$ where A_i are subsets of $[0,1]$ containing 0 and 1 for all i. In other words, an extension of an n-subcopula S is an n-copula C which agrees with S on its domain.

Since all C satisfying Equation (1.1) is uniquely determined on $Ran(F_1) \times \cdots \times Ran(F_n)$, there is a unique subcopula S whose domain is $Ran(F_1) \times \cdots \times Ran(F_n)$ satisfying

$$H(x_1, x_2, \ldots, x_n) = S(F_1(x_1), F_2(x_2), \ldots, F_n(x_n)) \tag{1.2}$$

for all $(x_1, x_2, \ldots, x_n) \in [-\infty, \infty]^n$. It can easily be seen that all copulas satisfying Equation (1.1) extend the unique subcopula S. Hence, we can consider the charecterization of all copulas satisfying Equation (1.1) as the subcopula extension problem.

In 1974, Sklar [12] proved that any bivariate subcopula can be extended to a bivariate copula but, generally, the extension is not unique. The multivariate case was proved later by the same author [1] which is also not unique in general.

In 2012, Amo, Carrillo, and Fernndez-Snchez [2] introduced a constructive method, by means of doubly stochastic measures, to describe all bivariate copulas satisfying Equation (1.1). The method is called *E-process*. Moreover, Amo et al. discussed the idea of using n-stochastic measures to the multivariate case but they are unable to provide the formula for the extension. The difficulty of using the idea is how to construct suitable n-stochastic measures in order to obtain copulas satisfying Equation (1.1) since there are too many cases to consider. However, there are several researchers trying to extending subcopulas in various special cases. These publications relating to subcopula extension problem are discussed in details in Chapter 2.

In 2002, Carley [3] found the maximum and minimum extensions of a given finite bivariate subcopula. His results are given in Theorem 2.38 and Theorem 2.39 respectively.

In 2007, Klement, Kolesrov, Mesiar, and Sempi [6] defined horizontal b-section of a copula C by fixing the second coordinate of a bivariate copula C with a constant $b \in (0, 1)$. Even if the horizontal b-section of a given copula C is not a subcopula, we still consider it as a subcopula by extending its domain to $[0, 1] \times \{0, b, 1\}$. This is possible because a copula is grounded and has uniform marginals. Thus, this can be considered as a subcopula extension problem. Klement et al. provided a copula, the greatest copula, and the smallest copula that extend the horizontal b-sections of a given copula C. Their results are stated in Theorem 2.43, Theorem 2.44, and Theorem 2.45, respectively.

In 2007, Baets and Meyer [7] provided a method to construct a new bivariate copula from a given copula by redefining the given copula in a given rectangle. Their main result, Theorem 2.46, states that the new defined function must also be *2*-increasing in the given rectangle and coincides with the given copula at their boundaries.

In 2008, Siburg and Stoimenov [9] provided a new way of constructing n-copulas by scaling and gluing finitely many n-copulas. The gluing construction of two copulas is given in Theorem 2.48 and the gluing construction of n-copulas is given in Theorem 2.49.

In 2009, Durante, Saminger-Platz, and Sarkoci [4] provided a method to construct a new copula from a given copula and a collection of copulas. The given copula is considered as the background copula. For each copula in the given collection, it associates with a rectangle in the unit square. Thus, there is a collection of rectangles in the unit square associated with the given collection of copulas. If each pair of the rectangles in the collection of rectangles is either disjoint or has common points just on their boundaries, then a function defined as in Theorem 2.50 is a copula.

In 2013, Baets, Meyer, Fernndez-Snchez, and beda-Flores [8] proved the existence of a *3*-copula with some given values of a *3*-quasi-copula. In other words, Baets et al. stated in Theorem 2.57 that there is a *3*-copula agrees with a given *3*-quasi-copula at a given point. This is also true for the case of two given points as it is stated in Theorem

2.58.

In 2013, Gonzlez-Barrios and Hernndez-Cedillo [5] generalized the results of Baets and Meyer [7] to higher dimensions (see Theorem 2.59) and also provided a multivariate patchwork construction of n-copulas in n-boxes. In Theorem 2.60, Gonzlez-Barrios and Hernndez-Cedillo provided a *3*-copula constructed from given two *3*-copulas and a *3*-box R with $(1, 1, 1)$ as one of its vertices. Their result in *3*-dimensions is generalized to higher dimensions as in Theorem 2.61.

In this thesis, we characterize all multivariate copulas satisfying Equation (1.1) in the case of the ranges of all marginal distribution functions are discrete, equivalently, all copulas extending the unique discrete subcopula satisfying Equation (1.2). Our main result may be considered as an extension in higher dimensions of the result of Amo et al.[2] in the case of discrete random variables. Nevertheless, proofs are quite different. In this work, we do not use stochastic measures but instead proving the result directly from the definitions.

The organization of this thesis is as follows. In the following chapter, the preliminaries including subcopula extensions are diccussed. In Chapter 3, we present the form of all copulas satisfying Equation (1.1) in the case of the ranges of all marginal distribution functions are discrete. Based on our result, we illustrate in Chapter 4 one application of this work through copula approximations. Finally, we give concluding remarks in Chapter 5.