CHAPTER 2

Preliminaries

Let $[-\infty, \infty]$ denote the set of extended real numbers.

For any $a, b \in [-\infty, \infty]$ in which $a \leq b$, denote $(a, b] := \{x \in [-\infty, \infty] \,|a < x \leq b\}$, $[a, b] := \{x \in [-\infty, \infty] \,|a \leq x \leq b\}$, and $\mathbb{I} := [0, 1]$.

If $\overrightarrow{a} \in [-\infty, \infty]^n$, then it means that $\overrightarrow{a} = (a_1, a_2, \ldots, a_n)$ where $a_i \in [-\infty, \infty]$ for all i.

For any real number k, denote $\overrightarrow{k} := (k, k, \ldots, k)$.

We will write $\overrightarrow{a} \leq \overrightarrow{b}$ whenever $a_i \leq b_i$ for all i and $\overrightarrow{a} < \overrightarrow{b}$ whenever $a_i < b_i$ for all i.

For $\overrightarrow{a} \leq \overrightarrow{b}$, denote $\left(\overrightarrow{a}, \overrightarrow{b}\right] := \prod_{i=1}^{n} (a_i, b_i]$ and an *n-box* $\left[\overrightarrow{a}, \overrightarrow{b}\right] := \prod_{i=1}^{n} [a_i, b_i]$.

2.1 Distribution Functions

Definition 2.1. A subset A of $[-\infty, \infty]$ is said to be *discrete* if

$$\inf_{y \in A \setminus \{x\}} |x - y| > 0$$

for all $x \in A$.

Note that any discrete subset of $[-\infty, \infty]$ is countable.

Definition 2.2. Let $f : [a, b] \to [-\infty, \infty]$. Then f is said to be *nondecreasing* if

$$f(x) \leq f(y)$$

whenever $a \leq x \leq y \leq b$.

Definition 2.3. Let $f : [a, b] \to \mathbb{I}$. Then f is said to be *continuous from the right* if for every $x \in [a, b]$ and every $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon$$

whenever $a \leq x \leq y < x + \delta \leq b$.

Definition 2.4. Let $f : [-\infty, \infty] \to \mathbb{I}$. Then f is called a *distribution function* if it satisfies the following properties

1. $\lim_{x \to -\infty} f(x) = 0$ and $\lim_{x \to \infty} f(x) = 1$,

2. f is nondecreasing, and

3. f is continuous from the right.

If a function $f : \mathbb{I} \to \mathbb{I}$ is nondecreasing, continuous from the right, $f(0) = 0$, and $f(1) = 1$, then the function f can be extended to a distribution function by additionally defining $f(x) = 0$ when $x < 0$ and $f(x) = 1$ when $x > 1$.

Example 2.5. For each $a \in (-\infty, \infty)$, a function $\delta_a : [-\infty, \infty] \to \{0, 1\}$ defined by

$$\delta_a(x) = \begin{cases} 0 & \text{if } x \in [-\infty, a), \\ 1 & \text{if } x \in [a, \infty] \end{cases}$$

is a distribution function.

Proof. Let $a \in (-\infty, \infty)$.

1. It is obvious that $\delta_a(-\infty) = 0$ and $\lim_{x \to \infty} \delta_a(x) = 1$.

2. Next, we show that f is nondecreasing. Let $x \leq y$. If $x = y$, there is nothing to prove. Assume that $x < y$. There are three cases to consider.

   Case 1. $a \leq x < y$
   In this case, $\delta_a(x) = 1 = \delta_a(y)$.

   Case 2. $x < a \leq y$
   In this case, $0 = \epsilon_a(x) < \delta_a(y) = 1$.

   Case 3. $x < y < a$
   In this case, $\delta_a(x) = 0 = \delta_a(y)$.

   Therefore, $\delta_a(x) \leq \delta_a(y)$ whenever $x \leq y$.

3. Last, we show that δ_a is continuous from the right.

   Let $x \in [-\infty, \infty]$ and $\epsilon > 0$. There are two cases to consider.

   Case 1. $x \geq a$
   Choose $\delta = \epsilon > 0$.
   For any $y \in [x, x + \delta)$, $|\delta_a(y) - \delta_a(x)| = |1 - 1| = 0 < \epsilon$.

Case 2. $x < a$

Choose $\delta = \frac{a-x}{2} > 0.$

For any $y \in [x, x+\delta)$, $|\delta_a(y) - \delta_a(x)| = |0-0| = 0 < \epsilon.$

Thus, δ_a is continuous from the right.

By 1., 2., and 3., δ_a is a distribution function.

□

Definition 2.6. For any $H : \Theta \times \mathbb{R} \to \mathbb{R}$, define the *difference* $\triangle_a^b H : \Theta \to \mathbb{R}$ by setting

$$\triangle_a^b H(\theta) = H(\theta, b) - H(\theta, a)$$

for all $\theta \in \Theta$ and $a, b \in \mathbb{R}$ in which $a \le b$.

Definition 2.7. Let $A_i \subseteq [-\infty, \infty]$ for all $i = 1, 2, \ldots, n$. For any $H : \prod_{i=1}^{n} A_i \to \mathbb{I}$, define V_H by setting

$$V_H\left(\prod_{i=1}^{n}(a_i, b_i]\right) := \triangle_{a_1}^{b_1} \cdots \triangle_{a_n}^{b_n} H$$

for all $a_i, b_i \in A_i$ in which $a_i \le b_i$.

Example 2.8. Let $H : [-\infty, \infty]^2 \to \mathbb{I}$. We have $\triangle_{a_2}^{b_2} H(t) = H(t, b_2) - H(t, a_2)$ and

$$\begin{aligned} V_H((a_1, b_1] \times (a_2, b_2]) &= \triangle_{a_1}^{b_1} \triangle_{a_2}^{b_2} H \\ &= \triangle_{a_1}^{b_1}\left(\triangle_{a_2}^{b_2} H\right) \\ &= \triangle_{a_2}^{b_2} H(b_1) - \triangle_{a_2}^{b_2} H(a_1) \\ &= (H(b_1, b_2) - H(b_1, a_2)) - (H(a_1, b_2) - H(a_1, a_2)) \\ &= H(b_1, b_2) - H(b_1, a_2) - H(a_1, b_2) + H(a_1, a_2) \end{aligned}$$

for all $a_1, a_2, b_1, b_2 \in [-\infty, \infty]$ in which $a_1 \le b_1$ and $a_2 \le b_2$.

Proposition 2.9. *Let $A_i \subseteq [-\infty, \infty]$ for all $i = 1, 2, \ldots, n$. For any $H : \prod_{i=1}^{n} A_i \to \mathbb{I}$ and all $a_i, b_i \in A_i$ in which $a_i \le b_i$,*

$$V_H\left(\prod_{i=1}^{n}(a_i, b_i]\right) := \sum_{\vec{v} \in \prod_{i=1}^{n}\{a_i, b_i\}} (-1)^{N\left(\vec{v}\right)} H\left(\vec{v}\right),$$

where $N\left(\vec{v}\right)$ is the number of i such that $v_i = a_i$.

Definition 2.10. Let $A_i \subseteq [-\infty,\infty]$ for all $i = 1,2,\ldots,n$. A function $H : \prod_{i=1}^{n} A_i \to \mathbb{I}$ is said to be *n-increasing* if

$$V_H\left(\prod_{i=1}^{n}(a_i, b_i]\right) \geq 0,$$

for all $a_i, b_i \in A_i$ in which $a_i \leq b_i$.

Definition 2.11. Let $A_i \subseteq [-\infty,\infty]$ for all $i = 1,2,\ldots,n$. A function $H : \prod_{i=1}^{n} A_i \to \mathbb{I}$ is said to be *continuous from the right in each argument* if, for each $k = 1,2,\ldots,n$ and each $\epsilon > 0$, there exists $\delta > 0$ such that

$$|H(x_1, x_2, \ldots, x_k, \ldots, x_n) - H(x_1, x_2, \ldots, y_k, \ldots, x_n)| < \epsilon$$

for all $(x_1, x_2, \ldots, x_n) \in \prod_{i=1}^{n} A_i$ and all $y_k \in A_k \cap (x_k, x_k + \delta]$.

Definition 2.12. Let $H : [-\infty,\infty]^n \to \mathbb{I}$. Then H is called a *joint distribution function* if it satisfies the following properties

1. $\lim\limits_{x_i \to -\infty, \exists i} H\left(\overrightarrow{x}\right) = 0,$
2. $\lim\limits_{x_i \to \infty, \forall i} H\left(\overrightarrow{x}\right) = 1,$
3. H is n-increasing, and
4. H is continuous from the right in each argument.

Let H be a joint distribution function. For each $i = 1, 2, ..., n$, the function $F_i : [-\infty,\infty] \to \mathbb{I}$ defined by $F_i(x_i) = H\left(\overrightarrow{x}\right)$, where all the coordinates of $\overrightarrow{x}$ are equal to ∞ except possibly x_i, is called a *marginal distribution function* of H.

Remark 2.13. *Every marginal distribution function is a distribution function.*

Example 2.14. A function $\delta_0 : [-\infty,\infty]^2 \to \mathbb{I}$ defined by

$$\delta_0(x,y) = \begin{cases} 1 & \text{if } x \geq 0, y \geq 0, \\ 0 & \text{otherwise} \end{cases}$$

is a joint distribution function.

Proof. Let $x, y \in [-\infty,\infty]$. It is obvious that

1. $\delta_0(x,-\infty) = 0$ and $\delta_0(-\infty, y) = 0$, and
2. $\lim\limits_{x\to\infty}\lim\limits_{y\to\infty} \delta_0(x,y) = \lim\limits_{y\to\infty}\lim\limits_{x\to\infty} \delta_0(x,y) = 1.$

Next, we show that δ_0 is *2*-increasing. Let $a, b, c, d \in [-\infty, \infty]$ be such that $a \leq b$ and $c \leq d$. To show that $V_{\delta_0}((a, b] \times (c, d]) \geq 0$, there are nine cases to consider.

Case 1. $b \geq a \geq 0$ and $d \geq c \geq 0$

In this case,

$$\begin{aligned} V_{\delta_0}((a, b] \times (c, d]) &= \delta_0(b, d) + \delta_0(a, c) - \delta_0(b, c) - \delta_0(a, d) \\ &= 1 + 1 - 1 - 1 \\ &= 0. \end{aligned}$$

Case 2. $b \geq a \geq 0$ and $d \geq 0 \geq c$

In this case,

$$\begin{aligned} V_{\delta_0}((a, b] \times (c, d]) &= \delta_0(b, d) + \delta_0(a, c) - \delta_0(b, c) - \delta_0(a, d) \\ &= 1 + 0 - 0 - 1 \\ &= 0. \end{aligned}$$

Case 3. $b \geq a \geq 0$ and $0 \geq d \geq c$

In this case,

$$\begin{aligned} V_{\delta_0}((a, b] \times (c, d]) &= \delta_0(b, d) + \delta_0(a, c) - \delta_0(b, c) - \delta_0(a, d) \\ &= 0 + 0 - 0 - 0 \\ &= 0. \end{aligned}$$

Case 4. $b \geq 0 \geq a$ and $d \geq c \geq 0$

In this case,

$$\begin{aligned} V_{\delta_0}((a, b] \times (c, d]) &= \delta_0(b, d) + \delta_0(a, c) - \delta_0(b, c) - \delta_0(a, d) \\ &= 1 + 0 - 0 - 1 \\ &= 0. \end{aligned}$$

Case 5. $b \geq 0 \geq a$ and $d \geq 0 \geq c$

In this case,

$$\begin{aligned} V_{\delta_0}((a, b] \times (c, d]) &= \delta_0(b, d) + \delta_0(a, c) - \delta_0(b, c) - \delta_0(a, d) \\ &= 1 + 0 - 0 - 0 \\ &= 1 > 0. \end{aligned}$$

Case 6. $b \geq 0 \geq a$ and $0 \geq d \geq c$

In this case,

$$\begin{aligned} V_{\delta_0}\left((a,b] \times (c,d]\right) &= \delta_0\left(b,d\right) + \delta_0\left(a,c\right) - \delta_0\left(b,c\right) - \delta_0\left(a,d\right) \\ &= 0 + 0 - 0 - 0 \\ &= 0. \end{aligned}$$

Case 7. $0 \geq b \geq a$ and $d \geq 0 \geq c$

In this case,

$$\begin{aligned} V_{\delta_0}\left((a,b] \times (c,d]\right) &= \delta_0\left(b,d\right) + \delta_0\left(a,c\right) - \delta_0\left(b,c\right) - \delta_0\left(a,d\right) \\ &= 0 + 0 - 0 - 0 \\ &= 0. \end{aligned}$$

Case 8. $0 \geq b \geq a$ and $d \geq c \geq 0$

In this case,

$$\begin{aligned} V_{\delta_0}\left((a,b] \times (c,d]\right) &= \delta_0\left(b,d\right) + \delta_0\left(a,c\right) - \delta_0\left(b,c\right) - \delta_0\left(a,d\right) \\ &= 0 + 0 - 0 - 0 \\ &= 0. \end{aligned}$$

Case 9. $0 \geq b \geq a$ and $0 \geq d \geq c$

In this case,

$$\begin{aligned} V_{\delta_0}\left((a,b] \times (c,d]\right) &= \delta_0\left(b,d\right) + \delta_0\left(a,c\right) - \delta_0\left(b,c\right) - \delta_0\left(a,d\right) \\ &= 0 + 0 - 0 - 0 \\ &= 0. \end{aligned}$$

Thus, $V_{\delta_0}\left((a,b] \times (c,d]\right) \geq 0, \forall\,(a,b] \times (c,d] \subset [-\infty,\infty]^2.$

Finally, we show that δ_0 is continuous from the right in each argument.

Let $x, y \in [-\infty,\infty]$ and $\epsilon > 0$. There are four cases to consider.

Case 1. $x \geq 0, y \geq 0$

Choose $\delta = \epsilon > 0$.

For any $x^+ \in [x, x+\delta)\,,\ |\delta_0\left(x,y\right) - \delta_0\left(x^+,y\right)| = |1-1| = 0 < \epsilon.$

Case 2. $x < 0, y \geq 0$

Choose $\delta_1 = -\frac{x}{2} > 0$ and $\delta_2 = \epsilon > 0$.

For any $x^+ \in [x, x+\delta_1)\,,\ |\delta_0\left(x,y\right) - \delta_0\left(x^+,y\right)| = |0-0| = 0 < \epsilon$

and for any $y^+ \in [y, y+\delta_2)\,,\ |\delta_0\left(x,y^+\right) - \delta_0\left(x,y\right)| = |0-0| = 0 < \epsilon.$

Case 3. $x \geq 0, y < 0$

Choose $\delta_1 = \epsilon > 0$ and $\delta_2 = -\frac{y}{2} > 0$.

For any $x^+ \in [x, x + \delta_1)$, $|\delta_0(x, y) - \delta_0(x^+, y)| = |0 - 0| = 0 < \epsilon$

and for any $y^+ \in [y, y + \delta_2)$, $|\delta_0(x, y^+) - \delta_0(x, y)| = |0 - 0| = 0 < \epsilon$.

Case 4. $x < 0, y < 0$

Choose $\delta_1 = -\frac{x}{2} > 0$ and $\delta_2 = -\frac{y}{2} > 0$.

For any $x^+ \in [x, x + \delta_1)$, $|\delta_0(x, y) - \delta_0(x^+, y)| = |0 - 0| = 0 < \epsilon$

and for any $y^+ \in [y, y + \delta_2)$, $|\delta_0(x, y^+) - \delta_0(x, y)| = |0 - 0| = 0 < \epsilon$.

Thus, δ_0 is continuous from the right in each argument.

Therefore, δ_0 is a joint distribution.

If $F : [-\infty, \infty] \to \mathbb{I}$ is defined by

$$F(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0 \end{cases}$$

and $G : [-\infty, \infty] \to \mathbb{I}$ is defined by

$$G(y) = \begin{cases} 1 & \text{if } y \geq 0, \\ 0 & \text{if } y < 0 \end{cases}$$

then F and G are marginal distribution functions of δ_0.

□

Theorem 2.15. *Let $H : \mathbb{I}^n \to \mathbb{I}$ be a function satisfying the following properties*

1. *$H\left(\overrightarrow{x}\right) = 0$ whenever at least one coordinate of $\overrightarrow{x}$ is equal to 0,*
2. *$H\left(\overrightarrow{1}\right) = 1$,*
3. *H is n-increasing, and*
4. *H is continuous from the right in each argument.*

Define $\hat{H} : [-\infty, \infty]^n \to \mathbb{I}$ by

$$\hat{H}\left(\overrightarrow{x}\right) = H\left((x_1 \vee 0) \wedge 1, (x_2 \vee 0) \wedge 1, \ldots, (x_n \vee 0) \wedge 1\right)$$

for each $\overrightarrow{x} = (x_1, x_2, \ldots, x_n) \in [-\infty, \infty]^n$. Then $\hat{H}$ is a joint distribution function and coincides with H on $\mathbb{I}^n$.

Proof. Let $\overrightarrow{x} = (x_1, x_2, \ldots, x_n) \in [-\infty, \infty]^n$ and $\delta : [-\infty, \infty] \to \mathbb{I}$ be defined by $\delta(x) = (x \vee 0) \wedge 1$. We have

$$\begin{aligned}\hat{H}\left(\overrightarrow{x}\right) &= H\left((x_1 \vee 0) \wedge 1, (x_2 \vee 0) \wedge 1, \ldots, (x_n \vee 0) \wedge 1\right) \\ &= H\left(x_1, x_2, \ldots, x_n\right) \\ &= H\left(\overrightarrow{x}\right)\end{aligned}$$

for any $\overrightarrow{x} \in \mathbb{I}^n$. Thus, $\hat{H}$ coincides with H on $\mathbb{I}^n$.

Next, we show that $\hat{H}$ is a joint distribution function.

1. If there is at least one coordinate of $\overrightarrow{x}$ which is equal to $-\infty$, says $x_k = -\infty$, then

$$\begin{aligned}\hat{H}\left(\overrightarrow{x}\right) &= H\left(\delta(x_1), \delta(x_2), \ldots, \delta(x_k), \ldots, \delta(x_n)\right) \\ &= H\left(\delta(x_1), \delta(x_2), \ldots, \delta(-\infty), \ldots, \delta(x_n)\right) \\ &= H\left(\delta(x_1), \delta(x_2), \ldots, 0, \ldots, \delta(x_n)\right) \\ &= 0.\end{aligned}$$

2. Let $x_i \to \infty$ for all $i = 1, 2, \ldots, n$. We have $x_i > 1$ for all i. It follows that

$$\begin{aligned}\lim_{x_i \to \infty, \forall i} \hat{H}\left(x_1, x_2, \ldots, x_n\right) &= \lim_{x_i \to \infty, \forall i} H\left(\delta(x_1), \delta(x_2), \ldots, \delta(x_n)\right) \\ &= \lim_{x_i \to \infty, \forall i} H\left(1, 1, \ldots, 1\right) \\ &= H\left(1, 1, \ldots, 1\right) \\ &= 1.\end{aligned}$$

3. Let $\left(\overrightarrow{a}, \overrightarrow{b}\right] \subset [-\infty, \infty]^n$. Since $\overrightarrow{0} \leq \delta\overrightarrow{(a)} \leq \delta\overrightarrow{(b)} \leq \overrightarrow{1}$, it follows that

$$\begin{aligned}V_{\hat{H}}\left(\left(\overrightarrow{a}, \overrightarrow{b}\right]\right) &= \sum_{(u_1, u_2, \ldots, u_n) \in \prod_{i=1}^{n} \{a_i, b_i\}} (-1)^{N(u_1, u_2, \ldots, u_n)} \hat{H}\left(u_1, u_2, \ldots, u_n\right) \\ &= \sum_{(u_1, u_2, \ldots, u_n) \in \prod_{i=1}^{n} \{a_i, b_i\}} (-1)^{N(u_1, u_2, \ldots, u_n)} H\left(\delta(u_1), \delta(u_2), \ldots, \delta(u_n)\right) \\ &= \sum_{(v_1, v_2, \ldots, v_n) \in \prod_{i=1}^{n} \{\delta(a_i), \delta(b_i)\}} (-1)^{N(v_1, v_2, \ldots, v_n)} H\left(v_1, v_2, \ldots, v_n\right) \\ &= V_H\left(\left(\delta\overrightarrow{(a)}, \delta\overrightarrow{(b)}\right]\right) \geq 0.\end{aligned}$$

4. Next, we show that $\hat{H}$ is continuous from the right in each argument. Let $i \in \{1, 2, \ldots, n\}$ and $\epsilon > 0$. There are three cases to consider.

Case 1. $x_i < 0$

By choosing $\gamma = -x_i > 0$, we have

$$\begin{aligned}
&\left|\hat{H}\left(x_1, x_2, \ldots, x_i, \ldots, x_n\right) - \hat{H}\left(x_1, x_2, \ldots, y_i, \ldots, x_n\right)\right| \\
=&\left|H\left(\delta\left(x_1\right), \delta\left(x_2\right), \ldots, \delta\left(x_i\right), \ldots, \delta\left(x_n\right)\right)\right. \\
&\left.-H\left(\delta\left(x_1\right), \delta\left(x_2\right), \ldots, \delta\left(y_i\right), \ldots, \delta\left(x_n\right)\right)\right| \\
=&\left|H\left(\delta\left(x_1\right), \delta\left(x_2\right), \ldots, 0, \ldots, \delta\left(x_n\right)\right)\right. \\
&\left.-H\left(\delta\left(x_1\right), \delta\left(x_2\right), \ldots 0, \ldots, \delta\left(x_n\right)\right)\right| \\
=&0 \\
<&\epsilon
\end{aligned}$$

for all $y_i \in [x_i, x_i + \gamma) = [x_i, x_i + (-x_i)) = [x_i, 0)$.

Case 2. $0 \leq x_i \leq 1$

Since $\hat{H} = H$ on $\mathbb{I}$ and H is continuous from the right in each argument, there is nothing to prove in this case.

Case 3. $x_i > 1$

By choosing $\gamma = \epsilon > 0$, we have

$$\begin{aligned}
&\left|\hat{H}\left(x_1, x_2, \ldots, x_i, \ldots, x_n\right) - \hat{H}\left(x_1, x_2, \ldots, y_i, \ldots, x_n\right)\right| \\
=&\left|H\left(\delta\left(x_1\right), \delta\left(x_2\right), \ldots, \delta\left(x_i\right), \ldots, \delta\left(x_n\right)\right)\right. \\
&\left.-H\left(\delta\left(x_1\right), \delta\left(x_2\right), \ldots, \delta\left(y_i\right), \ldots, \delta\left(x_n\right)\right)\right| \\
=&\left|H\left(\delta\left(x_1\right), \delta\left(x_2\right), \ldots, 1, \ldots, \delta\left(x_n\right)\right)\right. \\
&\left.-H\left(\delta\left(x_1\right), \delta\left(x_2\right), \ldots 1, \ldots, \delta\left(x_n\right)\right)\right| \\
=&0 \\
<&\epsilon
\end{aligned}$$

for all $y_i \in [x_i, x_i + \gamma) = [x_i, x_i + \epsilon)$.

By 1. - 4., we conclude that $\hat{H}$ is a joint distribution function and coincides with H on $\mathbb{I}^n$.

□

A function on $\mathbb{I}^n$ with the properties 1. - 4. in Theorem 2.15 can be considered as a joint distribution function because it can be extended to a joint distribution function by Theorem 2.15.

2.2 Probability Measures

Definition 2.16. Let Ω be a nonempty set and 2^{Ω} denote the power set of Ω. A class $\Sigma \subseteq 2^{\Omega}$ is called a *σ-algebra* on Ω if it satisfies the following properties

1. $\emptyset \in \Sigma$,
2. if $E \in \Sigma$, then $E^{C} = \Omega \setminus E \in \Sigma$, and
3. if $E_1, E_2, E_3, \ldots \in \Sigma$, then $\bigcup\limits_{k=1}^{\infty} E_k \in \Sigma$.

The ordered pair (Ω, Σ) is called a *measurable space* and the elements of Σ are called *measurable sets*.

Let Ω be a nonempty set. For any $\Lambda \subseteq 2^{\Omega}$, denote the intersection of all σ-algebras containing Λ by $\sigma(\Lambda)$. Note that $\sigma(\Lambda)$ is the smallest σ-algebra containing Λ.

Definition 2.17. Let $\Omega \subseteq [-\infty, \infty]^{n}$ where $n \in \mathbb{N}$ and $\mathscr{O}$ be the set of all open subsets of Ω. Then $\sigma(\mathscr{O})$ is called *the Borel σ-algebra on* Ω which specifically be denoted by $\mathscr{B}(\Omega)$. The elements of $\mathscr{B}(\Omega)$ are called *Borel sets*.

Theorem 2.18. *Let $\Omega \subseteq [-\infty, \infty]^{n}$ where $n \in \mathbb{N}$ and*

$$\Lambda = \left\{ \left(\prod_{i=1}^{n} (a_i, b_i] \right) \cap \Omega \mid -\infty \leq a_i \leq b_i \leq \infty \text{ for all } i = 1, 2, \ldots, n \right\}.$$

Then $\mathscr{B}(\Omega) = \sigma(\Lambda)$.

Definition 2.19. Let Ω be a nonempty set and Σ be a σ-algebra on Ω. A function $\mu : \Sigma \to \mathbb{I}$ is called a *probability measure* if it satisfies the following properties

1. $\mu(\emptyset) = 0$ and $\mu(\Omega) = 1$, and
2. for any countable collection $\{E_i\}_{i \in \mathscr{I}}$ of elements in Σ such that $E_j \cap E_k = \emptyset$ when $j, k \in \mathscr{I}$ and $j \neq k$, $\mu\left(\bigcup\limits_{i \in \mathscr{I}} E_i \right) = \sum\limits_{i \in \mathscr{I}} \mu(E_i)$.

Definition 2.20. Let $\mathscr{B}(\mathbb{I})$ be the Borel σ-algebra on $\mathbb{I}$,

$$\Gamma := \left\{ D \subseteq \mathbb{I} \mid D = \bigcup_{k=1}^{n} (a_k, b_k] \text{ for some } n \in \mathbb{N} \text{ such that } (a_i, b_i] \cap (a_j, b_j] = \emptyset \text{ whenever } i \neq j \right\} \cup \{\emptyset\},$$

and $\tau : \Gamma \to \mathbb{I}$ be defined by $\tau(\emptyset) = 0$ and

$$\tau\left(\bigcup_{k=1}^{n} (a_k, b_k] \right) := \sum_{k=1}^{n} b_k - a_k$$

where $\bigcup\limits_{k=1}^{n}(a_k, b_k] \neq \emptyset$. A function $\lambda : \mathscr{B}(\mathbb{I}) \to \mathbb{I}$ defined by

$$\lambda(A) := \inf\left\{\sum_{n=1}^{\infty}\tau(D_n) \,|\, A \subseteq \sum_{n=1}^{\infty}D_n, (D_n)_{n=1}^{\infty} \subset \Gamma\right\}$$

is called the *Lebesgue measure on* $\mathbb{I}$.

Remark 2.21. *λ is a probability measure on $\mathbb{I}$.*

Theorem 2.22. *Let Ω be a nonempty set and $\Lambda \subseteq 2^{\Omega}$ be nonempty and closed under finite intersections. If P_1 and P_2 are probability measures on $\sigma(\Lambda)$ such that $P_1 = P_2$ on Λ, then $P_1 = P_2$ on $\sigma(\Lambda)$.*

Theorem 2.23. *Let $A_i \subseteq [-\infty, \infty]$ for all $i = 1, 2, \ldots, n$. Let $H : \prod\limits_{i=1}^{n} A_i \to \mathbb{I}$ and*

$$S := \left\{\left(\vec{a}, \vec{b}\right] \subset \prod_{i=1}^{n} A_i \mid -\infty < a_i \leq b_i < \infty \text{ for all } i = 1, 2, \ldots, n\right\}.$$

Then $V_H : S \to \mathbb{I}$ defined as in Definition 2.7 can be extended to a probability measure on $\mathscr{B}\left(\prod\limits_{i=1}^{n} A_i\right)$.

If H is continuous, then the measure V_H satisfies

$$V_H\left(\left[\vec{a}, \vec{b}\right]\right) = V_H\left(\left(\vec{a}, \vec{b}\right]\right),$$

and hence we may define

$$V_H\left(\prod_{i=1}^{n}[a_i, b_i]\right) := \sum_{\vec{v} \in \prod_{i=1}^{n}\{a_i, b_i\}} (-1)^{N\left(\vec{v}\right)} H\left(\vec{v}\right)$$

directly where $N\left(\vec{v}\right)$ is the number of i such that $v_i = a_i$.

Definition 2.24. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. A *random variable* is a Borel measurable function from Ω to $\mathbb{R}$. A *random vector* is a Borel measurable function from Ω to $\mathbb{R}^n$.

Definition 2.25. For any random variable X defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, its *distribution function* is a function F_X defined by

$$F_X(x) = \mathbb{P}(X \leq x)$$

for all $x \in \mathbb{R}$.

Definition 2.26. A random variable X defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ is said to be *discrete* if there exists a discrete subset A of $\mathbb{R}$ such that

$$\mathbb{P}(X \in A) = 1,$$

that is, the support of the distribution function of X is discrete.

Definition 2.27. For any random vector $(X_1, \ldots, X_n)$ defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, its *(joint) distribution function* is a function H defined by

$$H(x_1, x_2, \ldots, x_n) = \mathbb{P}(X_1 \leq x_1, \ldots, X_n \leq x_n)$$

for all $x_i \in \mathbb{R}$ where $i = 1, 2, \ldots, n$.

Definition 2.28. A random vector $(X_1, X_2, \ldots, X_n)$ is said to be *discrete* if X_i is discrete for all $i = 1, 2, \ldots, n$.

2.3 Subcopulas and Copulas

Definition 2.29. Let A_i be subsets of $\mathbb{I}$ containing 0 and 1 for all $i = 1, 2, \ldots, n$. Then $S : \prod_{i=1}^{n} A_i \to \mathbb{I}$ is called an *n-subcopula* (or, *subcopula*, for brevity) if it satisfies the following properties

1. S is *grounded*, i.e., $S\left(\overrightarrow{u}\right) = 0$ if $\overrightarrow{u}$ has at least one coordinate which is equal to 0,
2. *S has uniform marginals*, i.e., $S\left(\overrightarrow{u}\right) = u_k$ if all the coordinates of $\overrightarrow{u}$ are equal to 1 except possibly u_k, and
3. S is *n-increasing.*

A subcopula whose domain is $\mathbb{I}^n$ is called an *n-copula* (or, *copula*, for brevity).

Theorem 2.30. *Every subcopula is uniformly continuous on its domain.*

Definition 2.31. Let S be an n-subcopola. Then an n-copula C is called an *extension* of S if

$$C\left(\overrightarrow{x}\right) = S\left(\overrightarrow{x}\right) \tag{2.1}$$

for all $\overrightarrow{x}$ in the domain of S.

Theorem 2.32. *[1, Appendix 1] An extension of any n-subcopula always exists, that is, given any n-subcopula S, there is an n-copula C satisfying Equation (2.1).*

What follows are a few simple but important examples of copulas. Let M^n, Π^n and W^n be given by

$$\begin{aligned} M^n\left(\overrightarrow{u}\right) &= \min\left(u_1, u_2, \ldots, u_n\right), \\ \Pi^n\left(\overrightarrow{u}\right) &= u_1 u_2 \cdots u_n, \\ W^n\left(\overrightarrow{u}\right) &= \max\left(u_1 + u_2 + \cdots + u_n - n + 1, 0\right) \end{aligned} \tag{2.2}$$

for all $\overrightarrow{u} \in (u_1, u_2, \ldots, u_n) \in \mathbb{I}^n$. The function M^n and Π^n are n-copulas for all $n \geq 2$. The function W^n is a copula when $n = 2$ and fails to be an n-copula for any $n > 2$.

Example 2.33. The function M^n defined by Equation (2.2) is a copula.

Proof. Let $\overrightarrow{u} \in \mathbb{I}^n$.

1. If $\overrightarrow{u}$ has at least one coordinate which is equal to 0, then $M^n\left(\overrightarrow{u}\right) = 0$.

2. If all the coordinates of $\overrightarrow{u}$ are equal to 1 except possibly u_k, then $M^n\left(\overrightarrow{u}\right) = u_k$ since $0 \leq u_k < 1$.

3. Next, we show that M^n is n-increasing by proving that

$$V_{M^n}\left(\left[\overrightarrow{a}, \overrightarrow{b}\right]\right) = \max\left(\min\left(b_1, b_2, \ldots, b_n\right) - \max\left(a_1, a_2, \ldots, a_n\right), 0\right)$$

for all n-box $B := \left[\overrightarrow{a}, \overrightarrow{b}\right] \subseteq \mathbb{I}^n$.

Let $\left[\overrightarrow{a}, \overrightarrow{b}\right]$ be an n-box. Rearrange $a_1, a_2, \ldots, a_n$ in ascending order and rename them with $r_i's$ so that $r_1 \leq r_2 \leq \cdots \leq r_n = \max\left(a_1, a_2, \ldots, a_n\right)$. Let $s_i := b_j$ whenever $r_i = a_j$. We have $r_i \leq s_i$ for all $i = 1, 2, \ldots, n$ and $\min\left(b_1, b_2, \ldots, b_n\right) = s_1 \leq s_2 \leq \cdots \leq s_n$. Note that,

$$\begin{aligned} V_{M^n}\left(\left[\overrightarrow{a}, \overrightarrow{b}\right]\right) &= \sum_{\overrightarrow{v} \in \prod_{i=1}^{n}\{a_i, b_i\}} (-1)^{N\left(\overrightarrow{v}\right)} M^n\left(\overrightarrow{v}\right) \\ &= \sum_{\forall i, v_i \in \{a_i, b_i\}} (-1)^{N\left(\overrightarrow{v}\right)} \min\left(v_1, v_2, \ldots, v_n\right) \\ &= \sum_{\forall i, u_i \in \{r_i, s_i\}} (-1)^{N\left(\overrightarrow{u}\right)} \min\left(u_1, u_2, \ldots, u_n\right) \\ &= \sum_{\overrightarrow{u} \in \prod_{i=1}^{n}\{r_i, s_i\}} (-1)^{N\left(\overrightarrow{u}\right)} M^n\left(\overrightarrow{u}\right) \\ &= V_{M^n}\left(\left[\overrightarrow{r}, \overrightarrow{s}\right]\right). \end{aligned}$$

For each $k = 1, 2, \ldots, n$, let $\mathbb{V}_k$ be the set of all elements of $\prod_{i=1}^{n} \{r_i, s_i\}$ such that the first appearance of r_i's is at the kth coordinate.
Note that $\prod_{i=1}^{n} \{r_i, s_i\} = \left(\bigcup_{k=1}^{n-1} \mathbb{V}_k\right) \cup \{(s_1, s_2, \ldots, r_n), (s_1, s_2, \ldots, s_n)\}$.
Let $\overrightarrow{v} \in \mathbb{V}_k$, and, for each $j = k+1, k+2, \ldots, n$, let $\overrightarrow{v_{rj}}$ and $\overrightarrow{v_{sj}}$ be the elements of $\mathbb{V}_k$ such that all coordinates of $\overrightarrow{v_{rj}}$ and $\overrightarrow{v_{sj}}$ are the same except for the jth coordinate which the jth coordinate of $\overrightarrow{v_{rj}}$ is r_j and that of $\overrightarrow{v_{sj}}$ is s_j.
Since $r_k \leq r_j \leq s_j$ for all $j > k$ and $\min(s_1, s_2, \ldots, s_n) \leq s_j$ for all j, it follows that if there exists $i < k$ such that $\min(s_1, s_2, \ldots, s_n) \leq s_i \leq r_k$, then $M^n\left(\overrightarrow{v}\right) = \min(s_1, s_2, \ldots, s_n)$. Hence,

$$M^n\left(\overrightarrow{v}\right) = \begin{cases} \min(s_1, s_2, \ldots, s_n) & \text{if } \min(s_1, s_2, \ldots, s_n) \leq r_k \\ r_k & \text{otherwise.} \end{cases}$$

Then,

$$M^n\left(\overrightarrow{v_{rj}}\right) = M^n\left(\overrightarrow{v_{sj}}\right)$$

for all $j = k+1, k+2, \ldots, n$. Since $\overrightarrow{v_{rj}}$ and $\overrightarrow{v_{sj}}$ are different at exactly one coordinate, it follows that $(-1)^{N\left(\overrightarrow{v_{rj}}\right)} + (-1)^{N\left(\overrightarrow{v_{sj}}\right)} = 0$. Hence,

$$(-1)^{N\left(\overrightarrow{v_{rj}}\right)} M^n\left(\overrightarrow{v_{rj}}\right) + (-1)^{N\left(\overrightarrow{v_{sj}}\right)} M^n\left(\overrightarrow{v_{sj}}\right) = 0.$$

Thus, $\sum_{\overrightarrow{v} \in \mathbb{V}_k} (-1)^{N\left(\overrightarrow{v}\right)} M^n\left(\overrightarrow{v}\right) = 0$ for each k.
Since $\prod_{i=1}^{n} \{r_i, s_i\} = \left(\bigcup_{k=1}^{n-1} \mathbb{V}_k\right) \cup \{(s_1, s_2, \ldots, r_n), (s_1, s_2, \ldots, s_n)\}$, it follows that

$$\begin{aligned}
V_{M^n}\left(\left[\overrightarrow{r}, \overrightarrow{s}\right]\right) &= \sum_{\overrightarrow{v} \in \prod_{i=1}^{n} \{r_i, s_i\}} (-1)^{N\left(\overrightarrow{v}\right)} M^n\left(\overrightarrow{v}\right) \\
&= \sum_{\overrightarrow{v} \in \left(\bigcup_{k=1}^{n-1} \mathbb{V}_k\right) \cup \{(s_1, s_2, \ldots, r_n), (s_1, s_2, \ldots, s_n)\}} (-1)^{N\left(\overrightarrow{v}\right)} M^n\left(\overrightarrow{v}\right) \\
&= \sum_{\overrightarrow{v} \in \bigcup_{k=1}^{n-1} \mathbb{V}_k} (-1)^{N\left(\overrightarrow{v}\right)} M^n\left(\overrightarrow{v}\right) \\
&\quad + (-1)^{N((s_1, s_2, \ldots, r_n))} M^n((s_1, s_2, \ldots, r_n)) \\
&\quad + (-1)^{N((s_1, s_2, \ldots, s_n))} M^n((s_1, s_2, \ldots, s_n)) \\
&= 0 + (-1)^{(1)} M^n((s_1, s_2, \ldots, r_n)) + (-1)^{(0)} M^n((s_1, s_2, \ldots, s_n)) \\
&= M^n((s_1, s_2, \ldots, s_n)) - M^n((s_1, s_2, \ldots, r_n)) \\
&= \min(s_1, s_2, \ldots, s_n) - \min(s_1, s_2, \ldots, r_n).
\end{aligned}$$

Thus, we have $V_{M^n}\left(\left[\overrightarrow{r}, \overrightarrow{s}\right]\right)$ depends on r_n. There are two cases to consider.

Case 1. $\min(s_1, s_2, \ldots, s_n) < r_n$

Then $\min(s_1, s_2, \ldots, s_n) = \min(s_1, s_2, \ldots, r_n)$.

Thus,

$$\begin{aligned} V_{M^n}\left(\left[\overrightarrow{r}, \overrightarrow{s}\right]\right) &= \min(s_1, s_2, \ldots, s_n) - \min(s_1, s_2, \ldots, r_n) \\ &= 0 \\ &= \max(\min(s_1, s_2, \ldots, s_n) - r_n, 0) \\ &= \max(\min(b_1, b_2, \ldots, b_n) - \max(a_1, a_2, \ldots, a_n), 0). \end{aligned}$$

Case 2. $\min(s_1, s_2, \ldots, s_n) \geq r_n$

Since $s_i \geq \min(s_1, s_2, \ldots, s_n) \geq r_n$ for all i, it follows that

$$\begin{aligned} \min(s_1, s_2, \ldots, r_n) &= r_n \\ &= \max(r_1, r_2, \ldots, r_n) \\ &= \max(a_1, a_2, \ldots, a_n). \end{aligned}$$

Thus,

$$\begin{aligned} V_{M^n}\left(\left[\overrightarrow{r}, \overrightarrow{s}\right]\right) &= \min(s_1, s_2, \ldots, s_n) - \min(s_1, s_2, \ldots, r_n) \\ &= \min(s_1, s_2, \ldots, s_n) - r_n \\ &= \min(b_1, b_2, \ldots, b_n) - \max(a_1, a_2, \ldots, a_n) \\ &= \max(\min(b_1, b_2, \ldots, b_n) - \max(a_1, a_2, \ldots, a_n), 0). \end{aligned}$$

Therefore,

$$\begin{aligned} V_{M^n}\left(\left[\overrightarrow{a}, \overrightarrow{b}\right]\right) &= V_{M^n}\left(\left[\overrightarrow{r}, \overrightarrow{s}\right]\right) \\ &= \max(\min(b_1, b_2, \ldots, b_n) - \max(a_1, a_2, \ldots, a_n), 0) \\ &\geq 0. \end{aligned}$$

By 1.-3., we conclude that M^n is a copula.

□

Example 2.34. The function $\Pi^n : \mathbb{I}^n \to \mathbb{I}$ defined by Equation (2.2) is an n-copula.

Proof. Let $\overrightarrow{u} = (u_1, u_2, \ldots, u_n) \in \mathbb{I}^n$.

1. If $\overrightarrow{u}$ has at least one coordinate which is equal to 0, then $\Pi^n\left(\overrightarrow{u}\right) = 0$.

2. If all the coordinates of $\overrightarrow{u}$ are equal to 1 except possibly u_k, then $\Pi^n\left(\overrightarrow{u}\right) = u_k$.

3. Next, we show that Π^n is n-increasing. Let $\left[\overrightarrow{a}, \overrightarrow{b}\right]$ be an n-box. Since $b_i - a_i \geq 0$ for all i and

$$\begin{aligned}
(b_1 - a_1)(b_2 - a_2)\cdots(b_n - a_n) &= \sum_{\forall i=1,2,\ldots,n; v_i \in \{b_i, -a_i\}} v_1 v_2 \cdots v_n \\
&= \sum_{\forall i=1,2,\ldots,n; v_i \in \{b_i, -a_i\}} \Pi^n(v_1, v_2, \cdots, v_n) \\
&= \sum_{\overrightarrow{v} \in \prod_{i=1}^{n} \{b_i, -a_i\}} \Pi^n\left(\overrightarrow{v}\right) \\
&= \sum_{\overrightarrow{v} \in \prod_{i=1}^{n} \{b_i, a_i\}} (-1)^{N\left(\overrightarrow{v}\right)} \Pi^n\left(\overrightarrow{v}\right) \\
&= V_{\Pi^n}\left(\left[\overrightarrow{a}, \overrightarrow{b}\right]\right),
\end{aligned}$$

it follows that $V_{\Pi^n}\left(\left[\overrightarrow{a}, \overrightarrow{b}\right]\right) = (b_1 - a_1)(b_2 - a_2)\cdots(b_n - a_n) \geq 0$. Hence, Π^n is n-increasing.

By 1.-3., Π^n is a copula.

□

Example 2.35. The function $W^n : \mathbb{I}^n \to \mathbb{I}$ defined by Equation (2.2) is a copula when $n = 2$ but it fails to be an n-copula for any $n > 2$.

Proof. Let $u, v \in \mathbb{I}$, then

1.

$$\begin{aligned}
W^2(u, 0) &= \max(u + 0 - 1, 0) \\
&= \max(u - 1, 0) \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
W^2(0, v) &= \max(0 + v - 1, 0) \\
&= \max(v - 1, 0) \\
&= 0,
\end{aligned}$$

2.

$$\begin{aligned} W^2(u,1) &= \max(u+1-1,0) \\ &= \max(u,0) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} W^2(1,v) &= \max(1+v-1,0) \\ &= \max(v,0) \\ &= v. \end{aligned}$$

3. Next, we show that W^2 is *2*-increasing. Let $m,n,s,t \in \mathbb{I}$ be such that $m \leq n$ and $s \leq t$. There are nine cases to consider.

Case 1. $m+s-1 < m+t-1 < 0$ and $n+s-1 < n+t-1 < 0$

Thus,

$$\begin{aligned} V_{W^2}([m,n]\times[s,t]) &= W^2(n,t)+W^2(m,s)-W^2(n,s)-W^2(m,t) \\ &= \max(n+t-1,0)+\max(m+s-1,0) \\ &\quad -\max(n+s-1,0)-\max(m+t-1,0) \\ &= 0+0-0-0 \\ &= 0. \end{aligned}$$

Case 2. $m+s-1 < m+t-1 < 0$ and $n+s-1 < 0 \leq n+t-1$

Thus,

$$\begin{aligned} V_{W^2}([m,n]\times[s,t]) &= W^2(n,t)+W^2(m,s)-W^2(n,s)-W^2(m,t) \\ &= \max(n+t-1,0)+\max(m+s-1,0) \\ &\quad -\max(n+s-1,0)-\max(m+t-1,0) \\ &= n+t-1 \geq 0. \end{aligned}$$

Case 3. $m+s-1 < m+t-1 < 0$ and $0 \leq n+s-1 < n+t-1$

Thus,

$$\begin{aligned} V_{W^2}([m,n]\times[s,t]) &= W^2(n,t)+W^2(m,s)-W^2(n,s)-W^2(m,t) \\ &= \max(n+t-1,0)+\max(m+s-1,0) \\ &\quad -\max(n+s-1,0)-\max(m+t-1,0) \\ &= (n+t-1)+0-(n+s-1)-0 \\ &= t-s \geq 0. \end{aligned}$$

Case 4. $m+s-1<0\le m+t-1$ and $n+s-1<n+t-1<0$

Since $m<n$, it follows that $0\le m+t-1<n+t-1<0$ which is a contradiction. Thus, this case is impossible.

Case 5. $m+s-1<0\le m+t-1$ and $0\le n+s-1\le n+t-1$

Thus,

$$\begin{aligned}V_{W^2}\left([m,n]\times[s,t]\right) &= W^2\left(n,t\right)+W^2\left(m,s\right)-W^2\left(n,s\right)-W^2\left(m,t\right)\\ &= \max\left(n+t-1,0\right)+\max\left(m+s-1,0\right)\\ &\quad-\max\left(n+s-1,0\right)-\max\left(m+t-1,0\right)\\ &= (n+t-1)-(n+s-1)-0-(m+t-1)\\ &= 1-s-m>0.\end{aligned}$$

Case 6. $m+s-1<0\le m+t-1$ and $n+s-1<0\le n+t-1$

Thus,

$$\begin{aligned}V_{W^2}\left([m,n]\times[s,t]\right) &= W^2\left(n,t\right)+W^2\left(m,s\right)-W^2\left(n,s\right)-W^2\left(m,t\right)\\ &= \max\left(n+t-1,0\right)+\max\left(m+s-1,0\right)\\ &\quad-\max\left(n+s-1,0\right)-\max\left(m+t-1,0\right)\\ &= (n+t-1)-(m+t-1)\\ &= n-m>0.\end{aligned}$$

Case 7. $0\le m+s-1<m+t-1$ and $n+s-1<n+t-1<0$

Since $m<n$, it follows that $0\le m+s-1<n+s-1<0$ which is a contradiction. Thus, this case is impossible.

Case 8. $0\le m+s-1<m+t-1$ and $n+s-1<0\le n+t-1$

Since $m<n$, it follows that $0\le m+s-1<n+s-1<0$ which is a contradiction. Thus, this case is impossible.

Case 9. $0\le m+s-1<m+t-1$ and $0\le n+s-1<n+t-1$

Thus,

$$\begin{aligned}V_{W^2}\left([m,n]\times[s,t]\right) &= W^2\left(n,t\right)+W^2\left(m,s\right)-W^2\left(n,s\right)-W^2\left(m,t\right)\\ &= \max\left(n+t-1,0\right)+\max\left(m+s-1,0\right)\\ &\quad-\max\left(n+s-1,0\right)-\max\left(m+t-1,0\right)\\ &= (n+t-1)+(m+s-1)\\ &\quad-(n+s-1)-(m+t-1)\\ &= 0.\end{aligned}$$

By all of the nine cases, $V_{W^2}([m,n]\times[s,t]) \geq 0, \forall [m,n], [s,t] \subseteq \mathbb{I}$.
Therefore, W^2 is a copula.

Next, we show that

$$V_{W^n}\left(\left[\overrightarrow{\frac{1}{2}}, \overrightarrow{1}\right]\right) = 1 - \frac{n}{2}$$

for any $n > 2$ and hence, W^n fails to be an n-copula.
Since $\overrightarrow{v} = (v_1, v_2, \ldots, v_n) \in \prod_{i=1}^{n} \left\{\frac{1}{2}, 1\right\}$, we have

$$\begin{aligned}
W^n\left(\overrightarrow{v}\right) &= \max\left(v_1 + v_2 + \cdots + v_n - n + 1, 0\right) \\
&= \max\left(N\left(\overrightarrow{v}\right)\left(\frac{1}{2}\right) + \left(n - N\left(\overrightarrow{v}\right)\right)(1) - n + 1, 0\right) \\
&= \max\left(\frac{N\left(\overrightarrow{v}\right)}{2} + n - N\left(\overrightarrow{v}\right) - n + 1, 0\right) \\
&= \max\left(1 - \frac{N\left(\overrightarrow{v}\right)}{2}, 0\right).
\end{aligned}$$

There are three cases to consider since $W^n\left(\overrightarrow{v}\right)$ depends on $N\left(\overrightarrow{v}\right)$.

Case 1. $N\left(\overrightarrow{v}\right) \geq 2$

By dividing both sides of the inequality by 2, we have $\frac{N\left(\overrightarrow{v}\right)}{2} \geq 1$, and then $1 - \frac{N\left(\overrightarrow{v}\right)}{2} \leq 0$

Thus, $(-1)^{N\left(\overrightarrow{v}\right)} W^n\left(\overrightarrow{v}\right) = (-1)^{N\left(\overrightarrow{v}\right)}(0) = 0$.

Case 2. $N\left(\overrightarrow{v}\right) = 1$

Since $N\left(\overrightarrow{v}\right) = 1$, it follows that all the coordinates of $\overrightarrow{v}$ are equal to 1 except possiply v_k which is equal to $\frac{1}{2}$.

Thus, $W^n\left(\overrightarrow{v}\right) = \max\left(1 - \frac{N\left(\overrightarrow{v}\right)}{2}, 0\right) = \max\left(1 - \frac{1}{2}, 0\right) = \frac{1}{2}$.

Then, $(-1)^{N\left(\overrightarrow{v}\right)} W^n\left(\overrightarrow{v}\right) = (-1)^1\left(\frac{1}{2}\right) = -\frac{1}{2}$.

Case 3. $N\left(\overrightarrow{v}\right) = 0$

Since $N\left(\overrightarrow{v}\right) = 0$, it follows that $\overrightarrow{v} = \overrightarrow{1}$.

Thus, we have $W^n\left(\overrightarrow{1}\right) = \max\left(1 - \frac{0}{2}, 0\right) = 1$.

Then, $(-1)^{N\left(\overrightarrow{v}\right)} W^n\left(\overrightarrow{1}\right) = (-1)^0(1) = 1$.

Since $V_{W^n}\left(\left[\overrightarrow{\tfrac{1}{2}}, \overrightarrow{1}\right]\right) = \sum_{\overrightarrow{v} \in \prod_{i=1}^{n}\{a_i,b_i\}} (-1)^{N\left(\overrightarrow{v}\right)} W^n\left(\overrightarrow{v}\right)$ and $n > 2$, it follows that

$$
\begin{aligned}
V_{W^n}\left(\left[\overrightarrow{\frac{1}{2}}, \overrightarrow{1}\right]\right) &= 0 + \left(-\frac{1}{2}\right)(n) + 1 \\
&= 1 - \frac{n}{2} \\
&< 0.
\end{aligned}
$$

Therefore, W^n is not n-increasing for $n > 2$. Hence, it fails to be an n-copula for $n > 2$.

□

Theorem 2.36. *[10, Theorem 2.10.12] If S is an n-subcopula, then*

$$W^n\left(\overrightarrow{u}\right) \leq S\left(\overrightarrow{u}\right) \leq M^n\left(\overrightarrow{u}\right) \tag{2.3}$$

for every $\overrightarrow{u}$ in the domain of S.

The functions W^n and M^n are known as the *Fréchet-Hoeffding bounds.*

By Theorem 2.36, we have

$$W^n\left(\overrightarrow{u}\right) \leq C\left(\overrightarrow{u}\right) \leq M^n\left(\overrightarrow{u}\right) \tag{2.4}$$

for any copula C and all $\overrightarrow{u} \in \mathbb{I}^n$ since any copula is a subcopula.

Theorem 2.37. *(Sklar's Theorem) Let H be a joint distribution function with marginal distribution functions $F_1, F_2, ..., F_n$. Then there exists an n-copula C such that*

$$H(x_1, x_2, ..., x_n) = C(F_1(x_1), F_2(x_2), ..., F_n(x_n)) \tag{2.5}$$

for all $\overrightarrow{x} = (x_1, x_2, \ldots, x_n) \in [-\infty, \infty]^n$. Moreover, C is uniquely determined on $Ran(F_1) \times Ran(F_2) \times ... \times Ran(F_n)$, where $Ran(F_i)$ denote the range of F_i, for all $i \in \{1, 2, ..., n\}$. Conversely, if C is an n-copula and $F_1, F_2, ..., F_n$ are distribution functions, then the function H defined by equation (2.5) is a joint distribution functions with marginal distribution functions $F_1, F_2, ..., F_n$.

2.4 Subcopula Extensions

In 2002, Carley [3] found the maximum and minimum extensions of a given finite bivariate subcopula as Theorem 2.38 and Theorem 2.39 below.

Theorem 2.38. *[3, Theorem 1] Let S be a bivariate subcopula whose domain is $\{a_0, a_1, \ldots, a_m\} \times \{b_0, b_1, \ldots, b_n\}$, where $0 = a_0 < a_1 < \cdots < a_m = 1$ and $0 = b_0 < b_1 < \cdots < b_n = 1$ and the blocks of $\mathbb{I}^2$ associated with S be the rectangles of the form $B_{ij} = [a_{i-1}, a_i] \times [b_{j-1}, b_j]$, $1 \le i \le m$, $1 \le j \le n$. Then, $M_S : \mathbb{I}^2 \to \mathbb{I}$ defined by*

$$M_S(x,y) = \sum_{i,j} \max\left(\min\left(x - \alpha_{ij}, y - \beta_{ij}, V_S(B_{ij})\right), 0\right),$$

where

$$\begin{aligned} \alpha_{11} &= 0, & \beta_{11} &= 0, \\ \alpha_{i,j+1} &= \alpha_{ij} + V_S(B_{ij}), & \beta_{i+1,j} &= \beta_{ij} + V_S(B_{ij}), \\ \alpha_{i+1,1} &= \alpha_{im} + V_S(B_{im}), & \beta_{1,j+1} &= \beta_{nj} + V_S(B_{nj}) \end{aligned}$$

is the maximum extension of S.

Theorem 2.39. *[3, Theorem 2] Let S be a bivariate subcopula whose domain is $\{a_0, a_1, \ldots, a_m\} \times \{b_0, b_1, \ldots, b_n\}$, where $0 = a_0 < a_1 < \cdots < a_m = 1$ and $0 = b_0 < b_1 < \cdots < b_n = 1$ and the blocks of $\mathbb{I}^2$ associated with S be the rectangles of the form $B_{ij} = [a_{i-1}, a_i] \times [b_{j-1}, b_j]$, $1 \le i \le m$, $1 \le j \le n$. Then, $W_S : \mathbb{I}^2 \to \mathbb{I}$ defined by*

$$W_S(x,y) = \sum_{i,j} \max\left[\min\left(x - \gamma_{ij}, V_S(B_{ij})\right) + \min\left(y - \delta_{ij}, V_S(B_{ij})\right) - V_S(B_{ij}), 0\right],$$

where

$$\begin{aligned} \gamma_{1m} &= 0, & \delta_{n1} &= 0, \\ \gamma_{i,m-j} &= \gamma_{i,m-j+1} + V_S(B_{i,m-j+1}), & \delta_{n-i,j} &= \delta_{n-i+1,j} + V_S(B_{n-i+1,j}), \\ \gamma_{i,m} &= \gamma_{1,i-1} + V_S(B_{1,i-1}), & \delta_{n,j} &= \delta_{1,j-1} + V_S(B_{1,j-1}) \end{aligned}$$

is the minimum extension of S.

Example 2.40. Let $S : \left\{0, \frac{1}{2}, 1\right\} \times \left\{0, \frac{3}{4}, 1\right\} \to \mathbb{I}$ be defined by $S(a,b) = ab$. All corresponding constants related to M_S and W_S are given in the figure 2.1. Thus, we have

$$\begin{aligned} M_S(x,y) = {} & \max\left(\min\left(x, y, \frac{3}{8}\right), 0\right) \\ & + \max\left(\min\left(x - \frac{1}{2}, y - \frac{3}{8}, \frac{3}{8}\right), 0\right) \\ & + \max\left(\min\left(x - \frac{3}{8}, y - \frac{3}{4}, \frac{1}{8}\right), 0\right) \\ & + \max\left(\min\left(x - \frac{7}{8}, y - \frac{7}{8}, \frac{1}{8}\right), 0\right) \end{aligned}$$

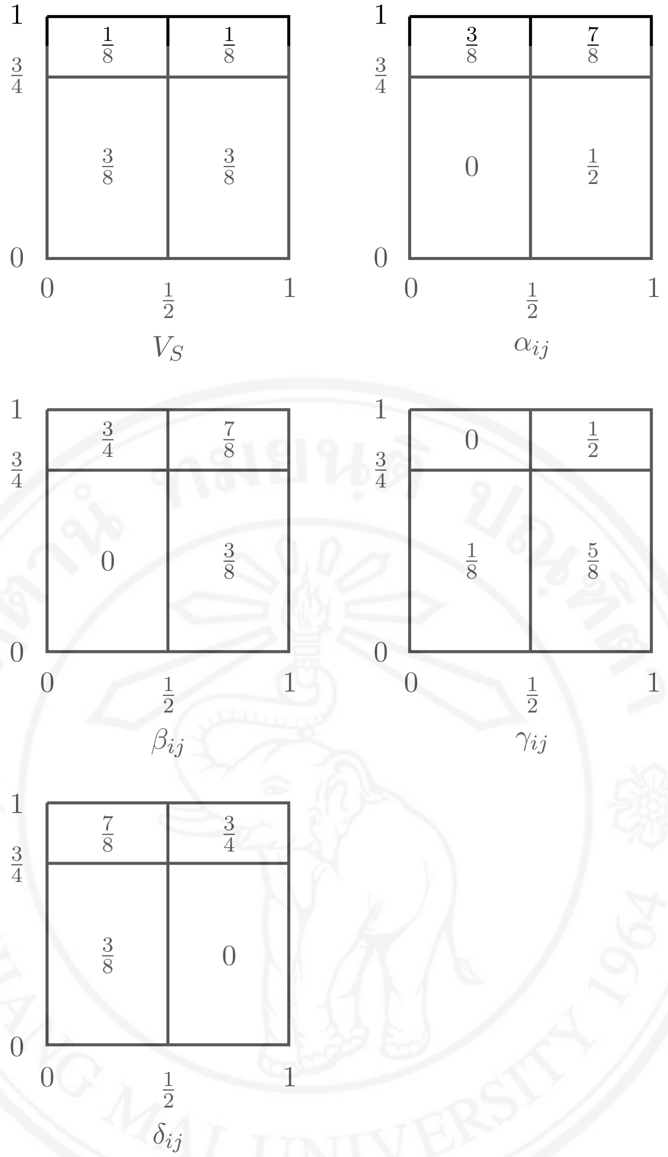

Figure 2.1: The corresponding constants related to M_S and W_S

is the maximum extension of S and

$$\begin{aligned} W_S(x,y) &= \max\left[\min\left(x-\frac{1}{8},\frac{3}{8}\right)+\min\left(y-\frac{3}{8},\frac{3}{8}\right)-\frac{3}{8},0\right] \\ &\quad +\max\left[\min\left(x-\frac{5}{8},\frac{3}{8}\right)+\min\left(y,\frac{3}{8}\right)-\frac{3}{8},0\right] \\ &\quad +\max\left[\min\left(x,\frac{1}{8}\right)+\min\left(y-\frac{7}{8},\frac{1}{8}\right)-\frac{1}{8},0\right] \\ &\quad +\max\left[\min\left(x-\frac{1}{2},\frac{1}{8}\right)+\min\left(y-\frac{3}{4},\frac{1}{8}\right)-\frac{1}{8},0\right] \end{aligned}$$

is the minimum extension of S.

In 2007, Klement, Kolesrov, Mesiar, and Sempi [6] defined horizontal b-section of the copula C by fixing the second coordinate of a bivariate copula C with a constant $b \in (0,1)$. The precise definition of horizontal b-section is given as follows.

Definition 2.41. Let $b \in (0,1)$ be a fixed number. The *horizontal b-section* of a bivariate copula C is the function $h_{C,b} : \mathbb{I} \to \mathbb{I}$ given by

$$h_{C,b}(x) = C(x,b)$$

for all $x \in \mathbb{I}$.

Remark 2.42. *All horizontal b-sections of copulas are nondecreasing and 1-Lipschitz functions.*

Let W^2 and M^2 be the Fréchet-Hoeffding lower and upper bounds of bivariate copulas as previously defined by Equation (2.2). Thus,

$$h_{W^2,b} \leq h_{C,b} \leq h_{M^2,b} \tag{2.6}$$

for any bivariate copula C. Klement et al. [6] considered nondecreasing and 1-Lipschitz functions h such that

$$\max(x+b-1,0) \leq h(x) \leq \min(x,b) \tag{2.7}$$

for a fixed b in $(0,1)$ and all $x \in \mathbb{I}$ and provided a copula, the greatest copula, and the smallest copula such that their horizontal b-sections coincide with h.

More precisely, let $\mathscr{H}_b$ be the set of all nondecreasing, 1-Lipschitz functions h satisfying the same bounds as in Inequality (2.6) for a fixed $b \in (0,1)$. Then a copula, the greatest copula, and the smallest copula such that their horizontal b-sections coincide with h are given in Theorem 2.43, Theorem 2.44, and Theorem 2.45, respectively.

Theorem 2.43. *[6, Proposition 2.1] Let $b \in (0,1)$ and $h \in \mathscr{H}_b$. Then the function $\widetilde{C}_h : \mathbb{I}^2 \to \mathbb{I}$ defined by*

$$\widetilde{C}_h(x,y) = \begin{cases} \frac{yh(x)}{b} & \textit{if } y \leq b, \\ \frac{(1-y)h(x)+(y-b)x}{1-b} & \textit{otherwise} \end{cases}$$

is a copula such that $h_{\widetilde{C}_h,b} = h$.

Theorem 2.44. *[6, Theorem 3.1] Let $b \in (0,1)$ and $h \in \mathscr{H}_b$. Then the function $\overline{C}_h : \mathbb{I}^2 \to \mathbb{I}$ defined by*

$$\overline{C}_h(x,y) = \begin{cases} y & \text{if } y \leq h(x), \\ h(x) & \text{if } h(x) < y \leq b, \\ y - b + h(x) & \text{if } b < y \leq x + b - h(x), \\ x & \text{otherwise} \end{cases}$$

is the greatest copula such that $h_{\overline{C}_h, b} = h$.

Theorem 2.45. *[6, Theorem 3.2] Let $b \in (0,1)$ and $h \in \mathscr{H}_b$. Then the function $\underline{C}_h : \mathbb{I}^2 \to \mathbb{I}$ defined by*

$$\underline{C}_h(x,y) = \begin{cases} y & \text{if } y \leq b - h(x), \\ y - b + h(x) & \text{if } b - h(x) < y \leq b, \\ h(x) & \text{if } b < y \leq 1 - x + h(x), \\ x + y - 1 & \text{otherwise} \end{cases}$$

is the smallest copula such that $h_{\underline{C}_h, b} = h$.

This can be considered as a subcopula extension problem. Let $S : \mathbb{I} \times \{0, b, 1\} \to \mathbb{I}$ be defined by

$$S(x,y) = \begin{cases} 0 & \text{if } y = 0, \\ h(x) & \text{if } y = b, \\ x & \text{if } y = 1. \end{cases}$$

It is not hard to show that S is a bivariate subcopula and the copulas given in Theorem 2.43, Theorem 2.44, and Theorem 2.45 are actually the extensions of S.

In 2007, Baets and Meyer [7] provided a method to construct a new bivariate copula from a given copula by redefining the old given copula in a given rectangle. The following theorem tells that the new defined function is a copula if and only if it is *2*-increasing in the given rectangle and coincides with the old one at the boundaries.

Theorem 2.46. *[7, Proposition 7] Consider a copula C, a rectangle $[u,u'] \times [v,v'] \subseteq \mathbb{I}^2$, and a $[u,u'] \times [v,v'] \to \mathbb{I}$ mapping D. Let $Q : \mathbb{I}^2 \to \mathbb{I}$ be defined by*

$$Q(x,y) = \begin{cases} D(x,y) & \text{if } (x,y) \in [u,u'] \times [v,v'], \\ C(x,y) & \text{otherwise.} \end{cases}$$

Thus, Q is a copula if and only if C and D coincide on the boundaries of $[u,u'] \times [v,v']$ and D is 2-increasing on $[u,u'] \times [v,v']$.

This can be considered as a subcopula extension problem. Let $S : (\{0,1\} \cup [u,u']) \times (\{0,1\} \cup [v,v']) \to \mathbb{I}$ be defined by

$$S(x,y) = \begin{cases} 0 & \text{if } x = 0 \text{ or } y = 0, \\ y & \text{if } x = 1, \\ x & \text{if } y = 1, \\ D(x,y) & \text{if } (x,y) \in [u,u'] \times [v,v']. \end{cases}$$

Then S is a bivariate subcopula such that $S_{|[u,u'] \times [v,v']} = D$ and one of its extensions is Q.

In addition, Baets and Meyer provided a copula construction method by considering a given copula as a background copula and a collection of copulas as foreground copulas. The method is given in the Theorem 2.47 below.

Theorem 2.47. *[7, Theorem 2] Consider two collections* $((u_i, u_i'))_{i \in \mathscr{I}}$ *and* $\left(\left(v_j, v_j'\right)\right)_{j \in \mathscr{J}}$ *of nonempty pairwise disjoint open subintervals of* $(0,1)$. *Consider a copula* C^b, *called background copula, a collection* $\left(C_{i,j}^f\right)_{i \in \mathscr{I}, j \in \mathscr{J}}$ *of copulas, called foreground copulas, and a collection* $\left(\lambda\left(u_i, u_i', v_j, v_j'\right)\right)_{i \in \mathscr{I}, j \in \mathscr{J}}$ *of positive multipliers. For any* $i \in \mathscr{I}$ *and* $j \in \mathscr{J}$, *define the* $[u_i, u_i'] \times \left[v_j, v_j'\right] \to \mathbb{R}$ *mapping* $D_{i,j}^b$ *by*

$$D_{i,j}^b(x,y) = C^b(x,y) - \lambda\left(u_i, u_i', v_j, v_j'\right) C^b\left(\frac{x-u_i}{u_i'-u_i}, \frac{y-v_j}{v_j'-v_j}\right)$$

and $Q : \mathbb{I}^2 \to \mathbb{I}$ *by*

$$Q(x,y) = \begin{cases} \lambda\left(u_i, u_i', v_j, v_j'\right) C_{i,j}^f\left(\frac{x-u_i}{u_i'-u_i}, \frac{y-v_j}{v_j'-v_j}\right) & \\ +D_{i,j}^b(x,y) & \text{if } (x,y) \in [u_i, u_i'] \times \left[v_j, v_j'\right], \\ C^b(x,y) & \text{otherwise.} \end{cases}$$

If for all $i \in \mathscr{I}$ *and* $j \in \mathscr{J}$ *it holds that* $D_{i,j}^b$ *is 2-increasing on* $[u_i, u_i'] \times \left[v_j, v_j'\right]$, *then* Q *is a copula.*

This can be considered as a subcopula extension problem.
Let $S : \left(\{0,1\} \cup \left(\underset{i \in \mathscr{I}}{\cup} [u_i, u_i']\right)\right) \times \left(\{0,1\} \cup \left(\underset{j \in \mathscr{J}}{\cup} \left[v_j, v_j'\right]\right)\right) \to \mathbb{I}$ be defined by $S = Q$ on its domain. Then S is a subcopula such that

$$S(x,y) = \lambda\left(u_i, u_i', v_j, v_j'\right) C_{i,j}^f\left(\frac{x-u_i}{u_i'-u_i}, \frac{y-v_j}{v_j'-v_j}\right) + D_{i,j}^b(x,y)$$

for all $(x,y) \in [u_i, u_i'] \times \left[v_j, v_j'\right]$ and one of its extensions is Q.

In 2008, Siburg and Stoimenov [9] provided a new way of constructing n-copulas by scaling and gluing finitely many n-copulas. Firstly, they illustrated the gluing construction in its most basic form, gluing two copulas.

Theorem 2.48. *[9, Theorem 2.1] For any two n-copulas C_1, C_2, any index $i \in \{1, 2, \ldots, n\}$, and any number $\theta \in (0, 1)$. Partition the unit cube as:*

$$\mathbb{I}^n = (\mathbb{I} \times \cdots \times [0, \theta] \times \cdots \times \mathbb{I}) \cup (\mathbb{I} \times \cdots \times [\theta, 1] \times \cdots \times \mathbb{I}),$$

and then define $C_1 \otimes_{x_i=\theta} C_2 : \mathbb{I}^n \to \mathbb{I}$ by setting

$$(C_1 \otimes_{x_i=\theta} C_2)(x_1, \ldots, x_i, \ldots, x_n) = \theta C_1 \left(x_1, \ldots, \tfrac{x_i}{\theta}, \ldots, x_n\right) \tag{2.8}$$

if $0 \le x_i \le \theta$, and

$$\begin{aligned} &(C_1 \otimes_{x_i=\theta} C_2)(x_1, \ldots, x_i, \ldots, x_n) \\ &= (1-\theta) C_2 \left(x_1, \ldots, \frac{x_i - \theta}{1-\theta}, \ldots, x_n\right) + \theta C_1 (x_1, \ldots, 1, \ldots, x_n) \end{aligned} \tag{2.9}$$

if $\theta \le x_i \le 1$. Thus, $C_1 \otimes_{x_i=\theta} C_2$ is an n-copula.

In Theorem 2.49 below, Siburg and Stoimenov provided the gluing method for the general case of finitely many copulas. This can also be realized by sequentially gluing two copulas as described in the previous theorem.

Theorem 2.49. *[9, Theorem 2.2] Fix any $i \in \{1, 2, \ldots, n\}$ and number θ_k such that $0 = \theta_0 < \theta_1 < \cdots < \theta_N = 1$, and let $C_1, \ldots, C_N$ be n-copulas. Partition the unit cube as:*

$$\mathbb{I}^n = \bigcup_{k=1}^{n} \mathbb{I} \times \cdots \times [\theta_{k-1}, \theta_k] \times \cdots \times \mathbb{I}$$

and then define by $\underset{x_i=\theta_k}{\otimes} C_k : \mathbb{I}^n \to \mathbb{I}$ by

$$\begin{aligned} \left(\underset{x_i=\theta_k}{\otimes} C_k\right)(x_1, \ldots, x_i, \ldots x_n) &= (\theta_k - \theta_{k-1}) C_k \left(x_1, \ldots, \frac{x_i - \theta_{k-1}}{\theta_k - \theta_{k-1}}, \ldots x_n\right) \\ &\quad + \theta_{k-1} C_{k-1} (x_1, \ldots, 1, \ldots, x_n), \end{aligned}$$

if $x_i \in [\theta_{k-1}, \theta_k]$ with $1 \le k \le N$. Thus, the function $\underset{x_i=\theta_k}{\otimes} C_k$ is an n-copula.

This can be considered as a subcopula extension problem. Let $C_1, \ldots, C_N$ be n-copulas. For a fixed $i \in \{1, 2, \ldots, n\}$ and number θ_k such that $0 = \theta_0 < \theta_1 < \cdots < \theta_N = 1$, let $S : \mathbb{I} \times \cdots \times \{\theta_0, \theta_1, \ldots, \theta_{N-1}, \theta_N\} \times \ldots \times \mathbb{I} \to \mathbb{I}$ be defined by

$$\begin{aligned} S(x_1, \ldots, x_i, \ldots x_n) &= (\theta_k - \theta_{k-1}) C_k \left(x_1, \ldots, \frac{x_i - \theta_{k-1}}{\theta_k - \theta_{k-1}}, \ldots x_n\right) \\ &\quad + \theta_{k-1} C_{k-1} (x_1, \ldots, 1, \ldots, x_n), \end{aligned}$$

if $x_i = \theta_k$ with $1 \leq k \leq N$, and

$$S(x_1, \ldots, x_i, \ldots, x_n) = 0,$$

if $x_i = \theta_0 = 0$. Then S is an n-subcopula and one of its extensions is $\underset{x_i=\theta_k}{\otimes} C_k$.

In 2009, Durante, Saminger-Platz, and Sarkoci [4] provided a method to construct a new copula from a given copula (which is considered as the background copoula) and a given collection of copulas. For each copula in the given collection, it associates with a rectangle in the unit square such that each pair of the rectangles is either disjoint or has common points just on their boundaries.

For each rectangle in the associated collection of rectangles, the desired new copula is given by redefining the background copula as in the Theorem 2.50 below.

Theorem 2.50. *[4, Theorem 2.2] Let* $\{C_i\}_{i\in\mathscr{I}}$ *be a collection of copulas and let* $\left\{R_i = \left[a_1^i, a_2^i\right] \times \left[b_1^i, b_2^i\right]\right\}_{i\in\mathscr{I}}$ *be a collection of rectangles* R_i *in* $\mathbb{I}^2$ *with boundaries* ∂R_i *such that* $R_i \cap R_j \subseteq \partial R_i \cap \partial R_j$*, for every* $i \neq j$*, i.e.,* R_i *and* R_j *have common points just on their boundaries. Let* C *be a copula and put* $\lambda_i := V_C(R_i)$*. Let* $\widetilde{C} : \mathbb{I}^2 \to \mathbb{I}$ *be defined by*

$$\widetilde{C}(x,y) = \begin{cases} \lambda_i C_i\left(\frac{V_C\left(\left[a_1^i, x\right]\times\left[b_1^i, b_2^i\right]\right)}{\lambda_i}, \frac{V_C\left(\left[a_1^i, a_2^i\right]\times\left[b_1^i, y\right]\right)}{\lambda_i}\right) & \\ +C\left(x, b_1^i\right) + C\left(a_1^i, y\right) - C\left(a_1^i, b_1^i\right) & \text{if } (x,y) \in R_i \text{ with } \lambda_i \neq 0, \\ C(x,y) & \text{otherwise} \end{cases}$$

for every $(x,y) \in \mathbb{I}^2$*. Then* $\widetilde{C}$ *is a copula.*

This also can be considered as a subcopula extension problem by letting $S : \left(\{0,1\} \cup \left(\underset{i\in\mathscr{I}}{\cup} \left[a_1^i, a_2^i\right]\right)\right) \times \left(\{0,1\} \cup \left(\underset{i\in\mathscr{I}}{\cup} \left[b_1^i, b_2^i\right]\right)\right) \to \mathbb{I}$ be defined by $S = \widetilde{C}$ on its domain. This function S is a subcopula such that

$$\begin{aligned} S(x,y) &= \lambda_i C_i\left(\frac{V_C\left(\left(a_1^i, x\right] \times \left(b_1^i, b_2^i\right]\right)}{\lambda_i}, \frac{V_C\left(\left(a_1^i, a_2^i\right] \times \left(b_1^i, y\right]\right)}{\lambda_i}\right) \\ &\quad +C\left(x, b_1^i\right) + C\left(a_1^i, y\right) - C\left(a_1^i, b_1^i\right) \end{aligned}$$

for all $(x,y) \in R_i$ with $\lambda_i \neq 0$ and one of its extensions is $\widetilde{C}$.

In 2012, Amo, Carrillo, and Fernndez-Snchez [2] characterized all bivariate copulas associated with non-continuous random variables. More precisely, let $H : [-\infty, \infty]^2 \to \mathbb{I}$ be a joint distribution function with marginal distribution functions F and G and $S : \overline{Ran(F)} \times \overline{Ran(G)} \to \mathbb{I}$ be the unique subcopula satisfying

$$S(F(x), G(y)) = H(x,y) \tag{2.10}$$

for all $x, y \in [-\infty, \infty]$.

Since F is nondecreasing and continuous from the right, it follows that the connected components of $Ran\,(F)$ are either an interval or a singleton.

Let S_1 be the family constituted by the closures of connected components of $Ran\,(F)$, P_1 be the class of elements in S_1 which are singletons, and $D_1 := S_1 \setminus P_1$.

The complement in $\mathbb{I}$ of the union of all elements of S_1 is a union of disjoint open intervals. Let O_1 be the family of all the closure of these disjoint open intervals.

With Γ as an index set, write $T := \{T_t := [a_t, b_t] \,|T_t \in D_1 \cup O_1\}_{t \in \Gamma}$.

Similarly, there exists the corresponding sets S_2, P_2, D_2, O_2 and $J := \{J_j := [c_j, d_j] \,|J_j \in D_2 \cup O_2\}_{j \in \Lambda}$ with an index set Λ for the distribution function G.

For any $T_t \in O_1$, we select a family of distribution functions $F_{tj} : \mathbb{I} \to \mathbb{I}$ satisfying

$$x = \frac{1}{b_t - a_t} \sum_j \beta_{tj} F_{tj}\,(x) \tag{2.11}$$

for all $x \in \mathbb{I}$, where $\beta_{tj} := V_S\,([a_t, b_t] \times [c_j, d_j])$.

For each $J_j \in O_2$, we select a family of distribution function $G_{tj} : \mathbb{I} \to \mathbb{I}$ satisfying

$$x = \frac{1}{d_j - c_j} \sum_t \beta_{tj} G_{tj}\,(x) \tag{2.12}$$

for all $x \in \mathbb{I}$, where $\beta_{tj} := V_S\,([a_t, b_t] \times [c_j, d_j])$.

In the case of $T_t \in D_1$ and $\beta_{tj} \neq 0$, F_{tj} is defined by

$$F_{tj}\,(x) := \frac{1}{\beta_{tj}} V_S\,([a_t, (b_t - a_t)\,x + a_t] \times [c_j, d_j])\,. \tag{2.13}$$

In the case of $J_j \in D_2$ and $\beta_{tj} \neq 0$, G_{tj} is defined by

$$G_{tj}\,(y) := \frac{1}{\beta_{tj}} V_S\,([a_t, b_t] \times [c_j, (d_j - c_j)\,y + c_j])\,. \tag{2.14}$$

With the above notations, the characterization of all bivariate copulas associated with non-continuous random variables can be presented as follows.

Theorem 2.51. *[2, Theorem 4] Let $H : [-\infty, \infty]^2 \to \mathbb{I}$ be a joint distribution function with given marginal distribution functions F and G. Then, C is a copula satisfying the equation*

$$C\,(F\,(x)\,, G\,(y)) = H\,(x, y) \tag{2.15}$$

if and only if C can be expressed in the form

$$C\,(x, y) = S\,(x, y)\; when\; (x, y) \in \overline{Ran\,(F)} \times \overline{Ran\,(G)}$$

and

$$\begin{aligned}C(x,y) = & S(a_t, c_j) + \beta_{tj} C_{tj}\left(F_{tj}\left(\frac{x-a_t}{b_t-a_t}\right), G_{tj}\left(\frac{y-c_j}{d_j-c_j}\right)\right) \\ & + \sum_{t' \in S_t} \beta_{t'j} G_{t'j}\left(\frac{y-c_j}{d_j-c_j}\right) \\ & + \sum_{j' \in Z_j} \beta_{tj'} F_{tj'}\left(\frac{x-a_t}{b_t-a_t}\right)\end{aligned} \tag{2.16}$$

when $(x,y) \notin \overline{Ran(G)} \times \overline{Ran(G)}$ *and* $(x,y) \in T_t \times J_j$*, where* C_{tj} *are copulas,* F_{tj} *and* G_{tj} *are distribution functions satisfying Equations (2.11)-(2.14) with* $S_t := \{t' | a_{t'} < a_t\}$ *and* $Z_j := \{j' | c_{j'} < c_j\}$*.*

In fact, all copulas satisfying Equation (2.15) are extensions of the unique subcopula S satisfying Equation (2.10).

Furthermore, in [2], Amo et al. also described the upper and lower bounds of the set of all copulas extending the unique subcopula S, that is, the function

$$US(x,y) := \sup\{C(x,y) \mid C \text{ is a copula extending } S\},$$

and the function

$$LS(x,y) := \inf\{C(x,y) \mid C \text{ is a copula extending } S\}.$$

For any index $t \in \Gamma$, the interval T_t is divided into indexed subintervals (in Λ) in such a way that the interval $T_t^j := \left[a_t^j, b_t^j\right] \subset T_t$ is an interval of length $V_S(T_t \times J_j)$, and its lower extreme is given by $a_t + \sum_{c_{j'} < c_j} V_S(T_t \times J_{j'})$, i.e., $a_t^j = a_t + \sum_{c_{j'} < c_j} V_S(T_t \times J_{j'})$ and $b_t^j = a_t^j + V_S(T_t \times J_j)$. In the same manner, the interval J_j is also divided into indexed subintervals J_j^t, i.e., for any index $j \in \Lambda$ and any $t \in \Gamma$,

$$J_j^t := \left[c_j^t, d_j^t\right] = \left[c_j + \sum_{a_{t'} < a_t} V_S(T_{t'} \times J_j), c_j^t + V_S(T_t \times J_j)\right].$$

Then, F_{tj} is defined by

$$F_{tj}(x) = \begin{cases} 0 & \text{if } 0 \le x \le \frac{a_t^j - a_t}{b_t - a_t}, \\ \frac{b_t - a_t}{b_t^j - a_t^j} x + \frac{a_t - a_t^j}{b_t^j - a_t^j} & \text{if } \frac{a_t^j - a_t}{b_t - a_t} \le x \le \frac{b_t^j - a_t}{b_t - a_t}, \\ 1 & \text{if } \frac{b_t^j - a_t}{b_t - a_t} \le x \le 1, \end{cases} \tag{2.17}$$

and G_{tj} is defined by

$$G_{tj}(x) = \begin{cases} 0 & \text{if } 0 \le x \le \frac{c_j^t - c_j}{d_j - c_j}, \\ \frac{d_j - c_j}{d_j^t - c_j^t} x + \frac{c_j - c_j^t}{d_j^t - c_j^t} & \text{if } \frac{c_j^t - c_j}{d_j - c_j} \le x \le \frac{d_j^t - c_j}{d_j - c_j}, \\ 1 & \text{if } \frac{d_j^t - c_j}{d_j - c_j} \le x \le 1. \end{cases} \tag{2.18}$$

The copulas US and LS are given as in Theorem 2.52 and Theorem 2.53 below.

Theorem 2.52. *[2, Theorem 11] If we choose the function F_{tj} and G_{tj} defined by Equation (2.17) and Equation (2.18) respectively, and $C_{tj} = M^2$, then C defined by Equation (2.16) is the copula US.*

Theorem 2.53. *[2, Theorem 12] If we choose the function F_{tj} and G_{tj} defined by Equation (2.17) and Equation (2.18) respectively, and $C_{tj} = W^2$, then C defined by Equation (2.16) is the copula LS.*

Theorem 2.52 and 2.53 above include the result due to Carley in [3] as a particular case, that is, when the sets $Ran\,(F)$ and $Ran(G)$ are finite.

In 2013, Baets, Meyer, Fernndez-Snchez, and beda-Flores [8] proved the existence of a *3*-copula with a given value of a *3*-quasi-copula at a single point and that of a *3*-copula with given values of a *3*-quasi-copula at two points.

In the general definition of an n-quasi-copula ($n \geq 2$) the notion of increasing tracks was used.

Definition 2.54. [8] An *increasing n-track B* in $\mathbb{I}^n$ is any set of the form
$B = \{(F_1(t), F_2(t), \ldots, F_n(t))\,|t \in \mathbb{I}\}$, where F_i is a continuous distribution function such that $F_i(0) = 0$ and $F_i(1) = 1$ for $i = 1, 2, \ldots, n$.

The definition of an n-dimensional quasi-copula can be stated as follows.

Definition 2.55. [8] For any natural number $n \geq 2$, an *n-dimensional quasi-copula* (briefly, an *n-quasi-copula*) is a function $Q : \mathbb{I}^n \to \mathbb{I}$ such that for every inreasing n-track B in $\mathbb{I}^n$ there exists an n-copula C_B that coincides with Q on B, i.e., $Q\left(\overrightarrow{u}\right) = C_B\left(\overrightarrow{u}\right)$ whenever $\overrightarrow{u} \in B$.

An alternative characterization of n-quasi-copulas is given in Theorem 2.56 below.

Theorem 2.56. *A function $Q : \mathbb{I}^n \to \mathbb{I}$, $n \geq 2$, is an n-quasi-copula if and only if it is grounded, has uniform marginals and satisfies the following two conditions:*

1. *Q is increasing in each variable u_i, $i = 1, \ldots, n$: if $u_1, \ldots, u_n, v_i$ are in $\mathbb{I}$ and $u_i < v_i$, then $Q(u_1, \ldots, u_i, \ldots, u_n) \leq Q(u_1, \ldots, v_i, \ldots, u_n)$;*

2. *Q is 1-Lipschitz continuous: for every $\overrightarrow{u} = (u_1, u_2, \ldots, u_n)$ and $\overrightarrow{v} = (v_1, v_2, \ldots, v_n)$ in $\mathbb{I}^n$, it holds that $\left|Q\left(\overrightarrow{u}\right) - Q\left(\overrightarrow{v}\right)\right| \leq \sum_{i=1}^{n} |u_i - v_i|$.*

The next theorem states that there exists a copula which coincides with a given n-quasi-copula at any given point.

Theorem 2.57. *[8, Theorem 2] For any* $(u_1, u_2, u_3) \in \mathbb{I}^3$ *and any* 3*-quasi-copula* Q*, there exists a* 3*-copula* C*-which depends on* (u_1, u_2, u_3)*-such that* $C(u_1, u_2, u_3) = Q(u_1, u_2, u_3)$.

This is also true for the case of two points.

Theorem 2.58. *[8, Theorem 3] For any* $(u_1, u_2, u_3), (v_1, v_2, v_3) \in \mathbb{I}^3$ *and any* 3*-quasi-copula* Q*, there exists a* 3*-copula* C*-which depends on* (u_1, u_2, u_3) *and* (v_1, v_2, v_3)*-such that* $C(u_1, u_2, u_3) = Q(u_1, u_2, u_3)$ *and* $C(v_1, v_2, v_3) = Q(v_1, v_2, v_3)$.

In the case of two points, consider $S : \prod_{i=1}^{3} \{0, u_i, v_i, 1\} \to \mathbb{I}$ defined by

$$S(x, y, z) = \begin{cases} Q(u_1, u_2, u_3) & \text{if } (x, y, z) = (u_1, u_2, u_3), \\ Q(v_1, v_2, v_3) & \text{if } (x, y, z) = (v_1, v_2, v_3), \\ C(x, y, z) & \text{otherwise.} \end{cases}$$

Then S is a subcopula and C extends S.

In the case of one given point, it can also be considered in a similar way.

In 2013, Gonzlez-Barrios and Hernndez-Cedillo [5] generalized Theorem 2.46 above to higher dimensions as follows.

Theorem 2.59. *[5, Theorem 1.2] Let* $C : \mathbb{I}^n \to \mathbb{I}$ *be an* n*-copula, let* $R := \prod_{i=1}^{n} [u_i, v_i] \subset \mathbb{I}^n$ *be a non-trivial* n*-box. Let* $D : R \to \mathbb{I}$ *be a function. Define* $Q : \mathbb{I}^n \to \mathbb{I}$ *by*

$$Q\left(\overrightarrow{x}\right) = \begin{cases} D\left(\overrightarrow{x}\right) & \text{if } \overrightarrow{x} \in R, \\ C\left(\overrightarrow{x}\right) & \text{if } \overrightarrow{x} \in \mathbb{I}^n \setminus R. \end{cases}$$

Then, Q *is an* n*-copula if and only if* $D = C$ *on* $\delta(R)$*, the boundaries of* R*, and* D *is* n*-increasing.*

This can be considered as a subcopula extension problem. Let $S : \prod_{i=1}^{n} [0, u_i] \cup [v_i, 1] \to \mathbb{I}$ be defined by $S = C$ on its domain. Then S is a subcopula such that $S_{|R} = D$ and one of its extensions is the copula Q.

Gonzlez-Barrios and Hernndez-Cedillo also provided a multivariate patchwork construction of n-copulas in n-boxes. In Theorem 2.60 below, they started by taking a *3*-copula and a *3*-box R with $(1, 1, 1)$ as one of its vertices.

Theorem 2.60. *[5, Theorem 3.1] Let C and C_1 be two 3-copulas and let $R = [u_1, 1] \times [u_2, 1] \times [u_3, 1]$ where $0 < u_i < 1$ for $i \in \{1, 2, 3\}$ and define $\overrightarrow{0} = (0, 0, 0)$. Assume that $\lambda = V_C(R) > 0$, and for every $x_1 \in [u_1, 1]$, for every $x_2 \in [u_2, 1]$ and for every $x_3 \in [u_3, 1]$, define*

$$\begin{aligned} R_{x_1} &= [u_1, x_1] \times [u_2, 1] \times [u_3, 1], \\ R_{x_2} &= [u_1, 1] \times [u_2, x_2] \times [u_3, 1], \\ R_{x_3} &= [u_1, 1] \times [u_2, 1] \times [u_3, x_3]. \end{aligned}$$

Let $\tilde{C} : \mathbb{I}^3 \to \mathbb{I}$ be defined by

$$\tilde{C}\left(\overrightarrow{x}\right) = \begin{cases} \lambda C_1\left(\frac{V_C(R_{x_1})}{\lambda}, \frac{V_C(R_{x_2})}{\lambda}, \frac{V_C(R_{x_2})}{\lambda}\right) + V_C\left(\left[\overrightarrow{0}, \overrightarrow{x}\right] \setminus \left[\overrightarrow{u}, \overrightarrow{x}\right]\right) & \text{if } \overrightarrow{x} \in R, \\ C\left(\overrightarrow{x}\right) & \text{otherwise} \end{cases}$$

for all $\overrightarrow{x} = (x_1, x_2, x_3) \in \mathbb{I}^3$, where $\overrightarrow{u} = (u_1, u_2, u_3)$. Then $\tilde{C}$ is a 3-copula.

Theorem 2.60 is generalized to larger dimentions as follows.

Theorem 2.61. *[5, Theorem 3.4] For every $n \geq 3$, let C and C_1 be two n-copulas and let $R := \left[\overrightarrow{u}, \overrightarrow{1}\right]$ where $\overrightarrow{u} = (u_1, u_2, \ldots, u_n) \in [0, 1)^n$. Assume that $\lambda := V_C(R) > 0$, and for every $i \in \{1, \ldots, n\}$ and for every $x_i \in (u_i, 1]$ define $R_{x_i} := [u_1, 1] \times \ldots \times [u_{i-1}, 1] \times [u_i, x_i] \times [u_{i+1}, 1] \times \ldots \times [u_n, 1]$. Let $\left(C \underset{\overrightarrow{u}}{\uplus} C_1\right) : \mathbb{I}^n \to \mathbb{I}$ be defined by*

$$\left(C \underset{\overrightarrow{u}}{\uplus} C_1\right)\left(\overrightarrow{x}\right) = \begin{cases} \lambda C_1\left(\frac{V_C(R_{x_1})}{\lambda}, \ldots, \frac{V_C(R_{x_n})}{\lambda}\right) + V_C\left(\left[\overrightarrow{0}, \overrightarrow{x}\right] \setminus \left[\overrightarrow{u}, \overrightarrow{x}\right]\right) & \text{if } \overrightarrow{x} \in R, \\ C\left(\overrightarrow{x}\right) & \text{otherwise} \end{cases}$$

for all $\overrightarrow{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{I}^n$. Then $\left(C \underset{\overrightarrow{u}}{\uplus} C_1\right)$ is an n-copula.

What follows is an example for Theorem 2.61 where $n = 3$, $C = M^3$, $C_1 = \Pi^3$, and $R = \left[\frac{1}{2}, 1\right] \times \left[\frac{1}{2}, 1\right] \times \left[\frac{1}{2}, 1\right]$.

Example 2.62. Let $R = \left[\frac{1}{2}, 1\right] \times \left[\frac{1}{2}, 1\right] \times \left[\frac{1}{2}, 1\right]$. Then, $\left(M^3 \underset{\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)}{\uplus} \Pi^3\right)$ defined by

$$\left(M^3 \underset{\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)}{\uplus} \Pi^3\right)(x_1, x_2, x_3) = \begin{cases} 4x_1x_2x_3 - 2(x_1x_2 + x_1x_3 + x_2x_3) & \\ \quad + x_1 + x_2 + x_3 & \text{if } \overrightarrow{x} \in R, \\ \min(x_1, x_2, x_3) & \text{otherwise} \end{cases}$$

is a *3*-copula.

Let $S : \prod_{i=1}^{n} (\{0\} \cup [u_i, 1]) \to \mathbb{I}$ be defined by $S = \left(C \underset{\overrightarrow{u}}{\uplus} C_1 \right)$ on its domain. Then, S is a subcopula such that

$$S\left(\overrightarrow{x}\right) = \lambda C_1 \left(\frac{V_C\left(R_{x_1}\right)}{\lambda}, \ldots, \frac{V_C\left(R_{x_n}\right)}{\lambda} \right) + V_C\left(\left[\overrightarrow{0}, \overrightarrow{x}\right] \setminus \left[\overrightarrow{u}, \overrightarrow{x}\right] \right)$$

for all $\overrightarrow{x} \in R$ and one of its extensions is the copula $\left(C \underset{\overrightarrow{u}}{\uplus} C_1 \right)$.