CHAPTER 3

Main results

Let $X_1, X_2, ..., X_n$ be discrete random variables with joint distribution function H and marginal distribution functions $F_1, F_2, ..., F_n$.

By Sklar's theorem, there always exists a copula C satisfying

$$H(x_1, x_2, ..., x_n) = C(F_1(x_1), F_2(x_2), ..., F_n(x_n)) \tag{3.1}$$

for all $x_1, x_2, ..., x_n \in [-\infty, \infty]$. Moreover, C is uniquely determined on $\prod_{i=1}^{n} Ran(F_i)$, i.e., if there is another copula C' satisfying Equation (3.1), then $C' = C$ on $\prod_{i=1}^{n} Ran(F_i)$. It follows that there exists a unique subcopula $S : \prod_{i=1}^{n} Ran(F_i) \to \mathbb{I}$ such that

$$H(x_1, x_2, ..., x_n) = S(F_1(x_1), F_2(x_2), ..., F_n(x_n)). \tag{3.2}$$

The fact that any subcopula S can be extended to a copula was guaranteed by Sklar [1]. Indeed, since C is uniquely determined on $\prod_{i=1}^{n} Ran(F_i)$, the unique subcopula S is defined on $\prod_{i=1}^{n} Ran(F_i)$ by $S = C$ for any copula C satisfying Equation (3.1).

Our main result is to figure out all copulas satisfying Equation (3.1), equivalently, all copulas extending the unique subcopula S satisfying Equation (3.2).

Since all X_i are discrete, it follows that all $Ran(F_i)$ are discrete subsets of $\mathbb{I}$ containing 0 and 1. For the reason of convenience, denote $A_i := Ran(F_i)$ and $c^+ := \inf\{b \in A_i \mid b > c\}$ for all $c \in A_i$.

Denote also $T_{\overrightarrow{a}} := \prod_{i=1}^{n} [a_i, a_i^+]$ and $\beta_{\overrightarrow{a}} := V_S\left(T_{\overrightarrow{a}}\right)$ for every $\overrightarrow{a} = (a_1, a_2, ..., a_n) \in \prod_{i=1}^{n} A_i$. Recall that $\overrightarrow{a} \leq \overrightarrow{b}$ if and only if $a_i \leq b_i$ for all i.

The main result can be stated using all these notations.

Theorem 3.1. *Let $X_1, X_2, ..., X_n$ be discrete random variables with joint distribution function H and marginal distribution functions $F_1, F_2, ..., F_n$. Then C is an n-copula satisfying (3.1) if and only if C can be expressed in the form*

$$C\left(\overrightarrow{x}\right) = \sum_{\overrightarrow{k} \leq \overrightarrow{a}} \beta_{\overrightarrow{k}} C_{\overrightarrow{k}} \left(F_{1,\overrightarrow{k}}\left(\frac{x_1 - k_1}{k_1^+ - k_1}\right), ..., F_{n,\overrightarrow{k}}\left(\frac{x_n - k_n}{k_n^+ - k_n}\right)\right) \tag{3.3}$$

for every $\overrightarrow{a} \in \prod_{i=1}^{n} A_i$, $\overrightarrow{x} = (x_1, x_2, ..., x_n) \in T_{\overrightarrow{a}}$, *and the summation is taken over all* $\overrightarrow{k} \in \prod_{i=1}^{n} A_i$. *Here,* $C_{\overrightarrow{k}}$ *are copulas and* $F_{i,\overrightarrow{k}} : [-\infty, \infty] \to \mathbb{I}$ *are one-dimensional distribution functions with support on* $\mathbb{I}$ *satisfying*

$$x = \frac{1}{a^+ - a} \sum_{\overrightarrow{b}} \beta_{\overrightarrow{b}} F_{i,\overrightarrow{b}}(x) \tag{3.4}$$

for all $i \in \{1, 2, ..., n\}$, $a \in A_i$, *and all* $x \in \mathbb{I}$, *where the summation is taken over all* $\overrightarrow{b} = (b_1, b_2, ..., b_n) \in \prod_{i=1}^{n} A_i$ *such that* $b_i = a$.

With this result, we are able to obtain the main result of [2] in the discrete case as a corollary.

Corollary 3.2. *Let* H *be a bivariate discrete distribution function in* $[-\infty, \infty]^2$, *with given marginal* F *and* G *and* $S : Ran\,(F) \times Ran\,(G) \to \mathbb{I}$ *be its associated bivariate subcopula. Then,* C *is a bivariate copula satisfying* $C(F(x), G(y)) = H(x, y)$ *for all* $x, y \in \mathbb{R}$ *if and only if* C *can be expressed in the form*

$$\begin{aligned} C(x_1, x_2) =& S(a_1, a_2) + \beta_{(a_1,a_2)} C_{(a_1,a_2)} \left(F_{1,(a_1,a_2)} \left(\frac{x_1 - a_1}{a_1^+ - a_1} \right), F_{2,(a_1,a_2)} \left(\frac{x_2 - a_2}{a_2^+ - a_2} \right) \right) \\ &+ \sum_{k_1 < a_1} \beta_{(k_1,a_2)} F_{2,(k_1,a_2)} \left(\frac{x_2 - a_2}{a_2^+ - a_2} \right) + \sum_{k_2 < a_2} \beta_{(a_1,k_2)} F_{1,(a_1,k_2)} \left(\frac{x_1 - a_1}{a_1^+ - a_1} \right) \end{aligned} \tag{3.5}$$

for all $(x_1, x_2) \in [a_1, a_1^+] \times [a_2, a_2^+]$ *with* $(a_1, a_2) \in Ran\,(F) \times Ran\,(G)$. *Here,* $C_{(k_1,k_2)}$ *are copulas,* $F_{1,(k_1,k_2)}$ *and* $F_{2,(k_1,k_2)}$ *are distribution functions satisfying Equation (3.4) for all* $(k_1, k_2) \in Ran\,(F) \times Ran\,(G)$.

Proof. Let $(a_1, a_2) \in Ran\,(F) \times Ran\,(G)$, and $(x_1, x_2) \in [a_1, a_1^+] \times [a_2, a_2^+]$. By Theorem 3.1, C is a bivariate copula extending S if and only if C can be expressed in the form

$$\begin{aligned} C(x_1, x_2) =& \sum_{(k_1,k_2) \le (a_1,a_2)} \beta_{(k_1,k_2)} C_{(k_1,k_2)} \left(F_{1,(k_1,k_2)} \left(\frac{x_1 - k_1}{k_1^+ - k_1} \right), F_{2,(k_1,k_2)} \left(\frac{x_2 - k_2}{k_2^+ - k_2} \right) \right) \\ =& \sum_{k_1 < a_1, k_2 < a_2} \beta_{(k_1,k_2)} C_{(k_1,k_2)} \left(F_{1,(k_1,k_2)} \left(\frac{x_1 - k_1}{k_1^+ - k_1} \right), F_{2,(k_1,k_2)} \left(\frac{x_2 - k_2}{k_2^+ - k_2} \right) \right) \\ &+ \beta_{(a_1,a_2)} C_{(a_1,a_2)} \left(F_{1,(a_1,a_2)} \left(\frac{x_1 - a_1}{a_1^+ - a_1} \right), F_{2,(a_1,a_2)} \left(\frac{x_2 - a_2}{a_2^+ - a_2} \right) \right) \\ &+ \sum_{k_1 < a_1} \beta_{(k_1,a_2)} C_{(k_1,a_2)} \left(F_{1,(k_1,a_2)} \left(\frac{x_1 - k_1}{k_1^+ - k_1} \right), F_{2,(k_1,a_2)} \left(\frac{x_2 - a_2}{a_2^+ - a_2} \right) \right) \\ &+ \sum_{k_2 < a_2} \beta_{(a_1,k_2)} C_{(a_1,k_2)} \left(F_{1,(a_1,k_2)} \left(\frac{x_1 - a_1}{a_1^+ - a_1} \right), F_{2,(a_1,k_2)} \left(\frac{x_2 - k_2}{k_2^+ - k_2} \right) \right) \end{aligned}$$

for all $x_1, x_2 \in T_{(a_1,a_2)}$. Since $F_{i,(k_1,k_2)}\left(\frac{x_i-k_i}{k_i^+-k_i}\right) = 1$ whenever $k_i < a_i$ and $C_{(k_1,k_2)}$ are copulas, it follows that

$$\begin{aligned}
C(x_1,x_2) =& \sum_{k_1<a_1,k_2<a_2} \beta_{(k_1,k_2)} C_{(k_1,k_2)}(1,1) \\
&+ \beta_{(a_1,a_2)} C_{(a_1,a_2)}\left(F_{1,(a_1,a_2)}\left(\frac{x_1-a_1}{a_1^+-a_1}\right), F_{2,(a_1,a_2)}\left(\frac{x_2-a_2}{a_2^+-a_2}\right)\right) \\
&+ \sum_{k_1<a_1} \beta_{(k_1,a_2)} C_{(k_1,a_2)}\left(1, F_{2,(k_1,a_2)}\left(\frac{x_2-a_2}{a_2^+-a_2}\right)\right) \\
&+ \sum_{k_2<a_2} \beta_{(a_1,k_2)} C_{(a_1,k_2)}\left(F_{1,(a_1,k_2)}\left(\frac{x_1-a_1}{a_1^+-a_1}\right), 1\right) \\
=& \sum_{k_1<a_1,k_2<a_2} \beta_{(k_1,k_2)} \\
&+ \beta_{(a_1,a_2)} C_{(a_1,a_2)}\left(F_{1,(a_1,a_2)}\left(\frac{x_1-a_1}{a_1^+-a_1}\right), F_{2,(a_1,a_2)}\left(\frac{x_2-a_2}{a_2^+-a_2}\right)\right) \\
&+ \sum_{k_1<a_1} \beta_{(k_1,a_2)} F_{2,(k_1,a_2)}\left(\frac{x_2-a_2}{a_2^+-a_2}\right) + \sum_{k_2<a_2} \beta_{(a_1,k_2)} F_{1,(a_1,k_2)}\left(\frac{x_1-a_1}{a_1^+-a_1}\right).
\end{aligned}$$

Since $\sum_{k_1<a_1,k_2<a_2} \beta_{(k_1,k_2)} = V_S([0,a_1]\times[0,a_2]) = S(a_1,a_2)$, it follows that

$$\begin{aligned}
C(x_1,x_2) =& S(a_1,a_2) \\
&+ \beta_{(a_1,a_2)} C_{(a_1,a_2)}\left(F_{1,(a_1,a_2)}\left(\frac{x_1-a_1}{a_1^+-a_1}\right), F_{2,(a_1,a_2)}\left(\frac{x_2-a_2}{a_2^+-a_2}\right)\right) \\
&+ \sum_{k_1<a_1} \beta_{(k_1,a_2)} F_{2,(k_1,a_2)}\left(\frac{x_2-a_2}{a_2^+-a_2}\right) \\
&+ \sum_{k_2<a_2} \beta_{(a_1,k_2)} F_{1,(a_1,k_2)}\left(\frac{x_1-a_1}{a_1^+-a_1}\right)
\end{aligned}$$

as desired.

□

The proof of Theorem 3.1 is divided into two parts: the necessary part and sufficiency part. The necessary part is divided into several lemmas given follow.

Lemma 3.3. *Let S be an n-subcopula and $\vec{a} = (a_1, a_2, \ldots, a_n)$, $\vec{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{I}^n$ be such that $\vec{a} \leq \vec{b}$. Then*

$$\begin{aligned}
&V_S([a_1,b_1]\times\cdots\times[a_i,y_i]\times\cdots\times[a_n,b_n]) \\
&-V_S([a_1,b_1]\times\cdots\times[a_i,x_i]\times\cdots\times[a_n,b_n]) \\
&=V_S([a_1,b_1]\times\cdots\times[x_i,y_i]\times\cdots\times[a_n,b_n]),
\end{aligned}$$

whenever $0 \leq a_i \leq x_i < y_i \leq 1$ for all $i = 1, 2, \ldots, n$.

Proof. For each i and $v_k \in \{a_k, b_k\}$ such that $k \neq i$, we have $N(v_1, \ldots, a_i, \ldots, v_n) = N(v_1, \ldots, b_i, \ldots, v_n) + 1$. Thus,

$$
\begin{aligned}
& V_S\left([a_1, b_1] \times \cdots \times [a_i, b_i] \times \cdots \times [a_n, b_n]\right) \\
= & \sum_{\vec{u} \in \prod_{k=1}^{n} \{a_i, b_i\}} (-1)^{N\left(\vec{u}\right)} S\left(\vec{u}\right) \\
= & \sum_{u_k \in \{a_k, b_k\}, \forall k \neq i} \Big[(-1)^{N(u_1, \ldots, b_i, \ldots, u_n)} S(u_1, \ldots, b_i, \ldots, u_n) \\
& + (-1)^{N(u_1, \ldots, a_i, \ldots, u_n)} S(u_1, \ldots, a_i, \ldots, u_n)\Big] \\
= & \sum_{u_k \in \{a_k, b_k\}, \forall k \neq i} \Big[(-1)^{N(u_1, \ldots, b_i, \ldots, u_n)} S(u_1, \ldots, b_i, \ldots, u_n) \\
& + (-1)^{N(u_1, \ldots, b_i, \ldots, u_n)+1} S(u_1, \ldots, a_i, \ldots, u_n)\Big].
\end{aligned}
$$

Since $N(u_1,\ldots,x_i,\ldots,u_n) = N(u_1,\ldots,y_i,\ldots,u_n)$, it follows that

$$\begin{aligned}
&V_S([a_1,b_1]\times\cdots\times[a_i,y_i]\times\cdots\times[a_n,b_n]) \\
&\quad - V_S([a_1,b_1]\times\cdots\times[a_i,x_i]\times\cdots\times[a_n,b_n]) \\
=& \sum_{u_k\in\{a_k,b_k\},\forall k\neq i} \Big[(-1)^{N(u_1,\ldots,y_i,\ldots,u_n)} S(u_1,\ldots,y_i,\ldots,u_n) \\
&\quad + (-1)^{N(u_1,\ldots,a_i,\ldots,u_n)} S(u_1,\ldots,a_i,\ldots,u_n)\Big] \\
&\quad - \sum_{u_k\in\{a_k,b_k\},\forall k\neq i} \Big[(-1)^{N(u_1,\ldots,x_i,\ldots,u_n)} S(u_1,\ldots,x_i,\ldots,u_n) \\
&\quad + (-1)^{N(u_1,\ldots,a_i,\ldots,u_n)} S(u_1,\ldots,a_i,\ldots,u_n)\Big] \\
=& \sum_{u_k\in\{a_k,b_k\},\forall k\neq i} (-1)^{N(u_1,\ldots,y_i,\ldots,u_n)} S(u_1,\ldots,y_i,\ldots,u_n) \\
&\quad - \sum_{v_k\in\{a_k,b_k\},\forall k\neq i} (-1)^{N(v_1,\ldots,x_i,\ldots,v_n)} S(v_1,\ldots,x_i,\ldots,v_n) \\
=& \sum_{u_k\in\{a_k,b_k\},\forall k\neq i} (-1)^{N(u_1,\ldots,y_i,\ldots,u_n)} S(u_1,\ldots,y_i,\ldots,u_n) \\
&\quad + \sum_{v_k\in\{a_k,b_k\},\forall k\neq i} (-1)(-1)^{N(v_1,\ldots,x_i,\ldots,v_n)} S(v_1,\ldots,x_i,\ldots,v_n) \\
=& \sum_{u_k\in\{a_k,b_k\},\forall k\neq i} (-1)^{N(u_1,\ldots,y_i,\ldots,u_n)} S(u_1,\ldots,y_i,\ldots,u_n) \\
&\quad + \sum_{v_k\in\{a_k,b_k\},\forall k\neq i} (-1)^{N(v_1,\ldots,x_i,\ldots,v_n)+1} S(v_1,\ldots,x_i,\ldots,v_n) \\
=& \sum_{u_k\in\{a_k,b_k\},\forall k\neq i} \Big[(-1)^{N(u_1,\ldots,y_i,\ldots,u_n)} S(u_1,\ldots,y_i,\ldots,u_n) \\
&\quad + (-1)^{N(u_1,\ldots,x_i,\ldots,u_n)+1} S(u_1,\ldots,x_i,\ldots,u_n)\Big] \\
=& \sum_{u_k\in\{a_k,b_k\},\forall k\neq i} \Big[(-1)^{N(u_1,\ldots,y_i,\ldots,u_n)} S(u_1,\ldots,y_i,\ldots,u_n) \\
&\quad + (-1)^{N(u_1,\ldots,y_i,\ldots,u_n)+1} S(u_1,\ldots,x_i,\ldots,u_n)\Big] \\
=& V_S([a_1,b_1]\times\cdots\times[x_i,y_i]\times\cdots\times[a_n,b_n])
\end{aligned}$$

as desired. □

Lemma 3.4. *Let C be an n-copula, $\left[\vec{a},\vec{b}\right]$ be an n-box in $\mathbb{I}^n$ and $\beta = V_C\left(\left[\vec{a},\vec{b}\right]\right) \neq 0$. If $H:\mathbb{I}^n\to\mathbb{I}$ is defined by*

$$H\left(\vec{x}\right) = \frac{V_C\left(\prod_{i=1}^{n}[a_i,(b_i-a_i)x_i+a_i]\right)}{\beta}$$

for all $\vec{x}=(x_1,x_2,...,x_n)\in\mathbb{I}^n$, then H is n-increasing.

Proof. Let $\left[\overrightarrow{r}, \overrightarrow{s}\right]$ be an n-box in $\mathbb{I}^n$. Consider

$$
\begin{aligned}
& V_H\left(\left[\overrightarrow{r}, \overrightarrow{s}\right]\right) \\
=& \triangle_{r_1}^{s_1} \ldots \triangle_{r_{n-1}}^{s_{n-1}} \triangle_{r_n}^{s_n} H \\
=& \triangle_{r_1}^{s_1} \ldots \triangle_{r_{n-1}}^{s_{n-1}} \left[H\left(t_1, t_2, \ldots, t_{n-1}, s_n\right) - H\left(t_1, t_2, \ldots, t_{n-1}, r_n\right)\right] \\
=& \triangle_{r_1}^{s_1} \ldots \triangle_{r_{n-1}}^{s_{n-1}} \left[\frac{V_C\left(\prod_{i=1}^{n-1}\left[a_i, \left(b_i - a_i\right) t_i + a_i\right] \times \left[a_n, \left(b_n - a_n\right) s_n + a_n\right]\right)}{\beta}\right. \\
& \left. - \frac{V_C\left(\prod_{i=1}^{n-1}\left[a_i, \left(b_i - a_i\right) t_i + a_i\right] \times \left[a_n, \left(b_n - a_n\right) r_n + a_n\right]\right)}{\beta}\right] \\
=& \triangle_{r_1}^{s_1} \ldots \triangle_{r_{n-1}}^{s_{n-1}} \left[\frac{V_C\left(\prod_{i=1}^{n-1}\left[a_i, \left(b_i - a_i\right) t_i + a_i\right] \times \left[\left(b_n - a_n\right) t_n + a_n, \left(b_n - a_n\right) s_n + a_n\right]\right)}{\beta}\right] \\
=& \triangle_{r_1}^{s_1} \ldots \triangle_{r_{n-2}}^{s_{n-2}} \left[\frac{V_C\left(\prod_{i=2}^{n-2}\left[a_i, \left(b_i - a_i\right) t_i + a_i\right] \times \prod_{i=n-1}^{n}\left[\left(b_i - a_i\right) r_i + a_i, \left(b_i - a_i\right) s_i + a_i\right]\right)}{\beta}\right] \\
& \vdots \\
=& \triangle_{r_1}^{s_1} \left[\frac{V_C\left(\left[a_1, \left(b_1 - a_1\right) t_1 + a_1\right] \times \prod_{i=2}^{n}\left[\left(b_i - a_i\right) r_i + a_i, \left(b_i - a_i\right) s_i + a_i\right]\right)}{\beta}\right] \\
=& \frac{V_C\left(\prod_{i=1}^{n}\left[\left(b_i - a_i\right) r_i + a_i, \left(b_i - a_i\right) s_i + a_i\right]\right)}{\beta} \\
\geq& 0.
\end{aligned}
$$

Therefore, H is n-increasing. □

Lemma 3.5. *The function $H : \mathbb{I}^n \to \mathbb{I}$ as defined in Lemma 3.4 is a joint distribution function.*

Proof. All that left is to show that H is grounded and continuous from the right. Let $\overrightarrow{x} \in \mathbb{I}^n$ such that $x_i = 0$ for some i. Since $x_i = 0$, we have $\left(b_i - a_i\right) x_i + a_i = a_i$.
Hence,

$$C\left(v_1, \ldots, a_i, \ldots, v_n\right) = C\left(v_1, \ldots, \left(b_i - a_i\right) x_i + a_i, \ldots, v_n\right)$$

where $v_k \in \{a_k, (b_k - a_k)\, x_k + a_k\}$ for all $k \neq i$. Since $N(v_1, \ldots, a_i, \ldots, v_n) = N(v_1, \ldots, (b_i - a_i)\, x_i + a_i, \ldots$
1, it follows that

$$\begin{aligned}
& V_C\left(\prod_{i=1}^{n} [a_i, (b_i - a_i)\, x_i + a_i]\right) \\
&= \sum_{\vec{v}; v_i = a_i} (-1)^{N\left(\vec{v}\right)} C\left(\vec{v}\right) + \sum_{\vec{u}; u_i = (b_i - a_i) x_i + a_i} (-1)^{N\left(\vec{u}\right)} C\left(\vec{u}\right) \\
&= \sum_{\vec{v}; v_i = a_i} (-1)^{N\left(\vec{v}\right)} C\left(\vec{v}\right) + \sum_{\vec{u}; u_i = a_i} (-1)^{N\left(\vec{u}\right)} C\left(\vec{u}\right) \\
&= \sum_{\vec{v}; v_i = a_i} (-1)^{N\left(\vec{u}\right)+1} C\left(\vec{v}\right) + \sum_{\vec{u}; u_i = a_i} (-1)^{N\left(\vec{u}\right)} C\left(\vec{u}\right) \\
&= \sum_{\vec{u}; u_i = a_i} (-1)^{N\left(\vec{u}\right)+1} C\left(\vec{u}\right) + \sum_{\vec{u}; u_i = a_i} (-1)^{N\left(\vec{u}\right)} C\left(\vec{u}\right) \\
&= \sum_{\vec{u}; u_i = a_i} (-1)^{N\left(\vec{u}\right)} \left[C\left(\vec{v}\right) - C\left(\vec{v}\right)\right] \\
&= 0.
\end{aligned}$$

Hence,

$$\begin{aligned}
H\left(\vec{x}\right) &= \frac{V_C\left(\prod_{i=1}^{n} [a_i, (b_i - a_i)\, x_i + a_i]\right)}{\beta} \\
&= 0
\end{aligned}$$

for all $\vec{x} \in \mathbb{I}^n$ such that $x_i = 0$ for some i. It is obvious that

$$\begin{aligned}
H\left(\vec{1}\right) &= \frac{V_C\left(\prod_{i=1}^{n} [a_i, (b_i - a_i)\,(1) + a_i]\right)}{\beta} \\
&= \frac{V_C\left(\prod_{i=1}^{n} [a_i, b_i - a_i + a_i]\right)}{\beta} \\
&= \frac{V_C\left(\prod_{i=1}^{n} [a_i, b_i]\right)}{\beta} \\
&= 1.
\end{aligned}$$

Since C is continuous, we must have H is continuous which implies that H is continuous from the right.

Therefore, H is a joint distribution function. □

Lemma 3.6. *Let C be an* n*-copula and $\beta_{\overrightarrow{a}} = V_C\left(T_{\overrightarrow{a}}\right) \neq 0$, where $\overrightarrow{a} \in \prod_{i=1}^{n} A_i$. Let $H_{\overrightarrow{a}} : \mathbb{I}^n \to \mathbb{I}$ be defined by*

$$H_{\overrightarrow{a}}(\overrightarrow{x}) = \frac{V_C\left(\prod_{i=1}^{n}\left[a_i, \left(a_i^+ - a_i\right)x_i + a_i\right]\right)}{\beta_{\overrightarrow{a}}}$$

for all $\overrightarrow{x} \in T_{\overrightarrow{a}}$. If $F_{i,\overrightarrow{a}} : \mathbb{R} \to \mathbb{I}$ are marginal distribution functions of $H_{\overrightarrow{a}}$, then $F_{i,\overrightarrow{a}}$ are distribution functions with support lying in $\mathbb{I}$ satisfying Equation (3.4).

Proof. All that left is to show that $F_{i,\overrightarrow{a}}$ satisfies Equation (3.4). Fix $i \in \{1, 2, ..., n\}$. Since

$$F_{i,\overrightarrow{a}}(x) = \frac{V_C\left(\left(\prod_{j=1}^{i-1}\left[a_j, a_j^+\right]\right) \times \left[a_i, \left(a_i^+ - a_i\right)x + a_i\right] \times \left(\prod_{j=i+1}^{n}\left[a_j, a_j^+\right]\right)\right)}{\beta_{\overrightarrow{a}}},$$

it follows that

$$\begin{aligned}
&\frac{1}{a_i^+ - a_i}\sum_{\overrightarrow{b}}\beta_{\overrightarrow{b}}F_{i,\overrightarrow{b}}(x)\\
=&\frac{1}{a_i^+ - a_i}\sum_{\substack{b_j \in A_j\\ 1 \le j \le n-1, j \neq i}}\sum_{b_n \in A_n} V_C\left(\left(\prod_{j=1}^{i-1}\left[b_j, b_j^+\right]\right) \times \left[a_i, \left(a_i^+ - a_i\right)x + a_i\right]\right.\\
&\left.\times\left(\prod_{j=i+1}^{n-1}\left[b_j, b_j^+\right]\right) \times \left[b_n, b_n^+\right]\right)\\
=&\frac{1}{a_i^+ - a_i}\sum_{\substack{b_j \in A_j\\ 1 \le j \le n-1, j \neq i}} V_C\left(\left(\prod_{j=1}^{i-1}\left[b_j, b_j^+\right]\right) \times \left[a_i, \left(a_i^+ - a_i\right)x + a_i\right] \times \left(\prod_{j=i+1}^{n-1}\left[b_j, b_j^+\right]\right) \times \mathbb{I}\right)\\
=&\frac{1}{a_i^+ - a_i}\sum_{\substack{b_j \in A_j\\ 1 \le j \le n-2, j \neq i}}\sum_{b_{n-1} \in A_{n-1}} V_C\left(\left(\prod_{j=1}^{i-1}\left[b_j, b_j^+\right]\right) \times \left[a_i, \left(a_i^+ - a_i\right)x + a_i\right]\right.\\
&\left.\times\left(\prod_{j=i+1}^{n-2}\left[b_j, b_j^+\right]\right) \times \left[b_{n-1}, b_{n-1}^+\right] \times \mathbb{I}\right)\\
&\vdots\\
=&\frac{1}{a_i^+ - a_i} \cdot V_C\left(\mathbb{I}^{i-1} \times \left[a_i, \left(a_i^+ - a_i\right)x + a_i\right] \times \mathbb{I}^{n-i}\right)\\
=&\frac{1}{a_i^+ - a_i}\left(C\left(1, 1, ..., \left(a_i^+ - a_i\right)x + a_i, ..., 1, 1\right) - C\left(1, 1, ..., a_i, ..., 1, 1\right)\right)\\
=&\frac{1}{a_i^+ - a_i}\left(\left(a_i^+ - a_i\right)x + a_i - a_i\right)\\
=&x
\end{aligned}$$

which completes the proof. □

Theorem 3.7. *Let $S : \prod_{i=1}^{n} A_i \to \mathbb{I}$ be an n-subcopula. If C is an n-copula extending S, then C can be expressed as in Equation (3.3).*

Proof. Let $H_{\overrightarrow{a}}, F_{i,\overrightarrow{a}}$ be defined as in Lemma 3.6. Let $\overrightarrow{x} = (x_1, x_2, ..., x_n) \in T_{\overrightarrow{a}}$ and $\overrightarrow{x}^* = (x_1^*, x_2^*, ..., x_n^*) \in \mathbb{I}^n$ where

$$x_i^* = \begin{cases} 0 & \text{if } x_i < k_i, \\ \frac{x_i - k_i}{k_i^+ - k_i} & \text{if } k_i \leq x_i \leq k_i^+, \\ 1 & \text{if } x_i > k_i. \end{cases}$$

Then we obtain $F_{i,\overrightarrow{k}}(x_i^*) = F_{i,\overrightarrow{k}}\left(\frac{x_i - k_i}{k_i^+ - k_i}\right)$ for all $i = 1, 2, ..., n$. Since these $F_{i,\overrightarrow{k}}$ are marginal distribution functions of $H_{\overrightarrow{k}}$, there exists a copula $C_{\overrightarrow{k}}$ such that

$$\begin{aligned} C_{\overrightarrow{k}}\left(F_{1,\overrightarrow{k}}\left(\frac{x_1 - k_1}{k_1^+ - k_1}\right), ..., F_{n,\overrightarrow{k}}\left(\frac{x_n - k_n}{k_n^+ - k_n}\right)\right) &= C_{\overrightarrow{k}}\left(F_{1,\overrightarrow{k}}(x_1^*), ..., F_{n,\overrightarrow{k}}(x_n^*)\right) \\ &= H_{\overrightarrow{k}}(x_1^*, ..., x_n^*) \end{aligned}$$

by Sklar's Theorem. Therefore,

$$
\begin{aligned}
&\sum_{\vec{k}\leq\vec{a}} \beta_{\vec{k}} C_{\vec{k}} \left(F_{1,\vec{k}} \left(\frac{x_1 - k_1}{k_1^+ - k_1} \right), ..., F_{n,\vec{k}} \left(\frac{x_n - k_n}{k_n^+ - k_n} \right) \right) \\
=&\sum_{\vec{k}\leq\vec{a}} \beta_{\vec{k}} H_{\vec{k}} \left(x_1^*, ..., x_n^* \right) \\
=&\sum_{\vec{k}\leq\vec{a}} V_C \left(\prod_{i=1}^{n} \left[k_i, \left(k_i^+ - k_i \right) x_i^* + k_i \right] \right) \\
=&\sum_{\substack{k_i \leq a_i \\ 1\leq i\leq n-1}} \sum_{k_n \leq a_n} V_C \left(\left(\prod_{i=1}^{n-1} \left[k_i, \left(k_i^+ - k_i \right) x_i^* + k_i \right] \right) \times \left[k_n, \left(k_n^+ - k_n \right) x_n^* + k_n \right] \right) \\
=&\sum_{\substack{k_i \leq a_i \\ 1\leq i\leq n-1}} V_C \left(\left(\prod_{i=1}^{n-1} \left[k_i, \left(k_i^+ - k_i \right) x_i^* + k_i \right] \right) \times [0, x_n] \right) \\
=&\sum_{\substack{k_i \leq a_i \\ 1\leq i\leq n-2}} \sum_{k_{n-1} \leq a_{n-1}} V_C \left(\prod_{i=1}^{n-2} \left[k_i, \left(k_i^+ - k_i \right) x_i^* + k_i \right] \right. \\
&\left. \times \left[k_{n-1}, \left(k_{n-1}^+ - k_{n-1} \right) x_{n-1}^* + k_{n-1} \right] \times [0, x_n] \right) \\
=&\sum_{\substack{k_i \leq a_i \\ 1\leq i\leq n-2}} V_C \left(\prod_{i=1}^{n-2} \left[k_i, \left(k_i^+ - k_i \right) x_i^* + k_i \right] \times \prod_{i=n-1}^{n} [0, x_i] \right) \\
&\vdots \\
=&V_C \left(\prod_{i=1}^{n} [0, x_i] \right) \\
=&C \left(\vec{x} \right)
\end{aligned}
$$

as desired. □

We have already proved the necessary part of the main theorem, that is, if C is an n-copula satisfying (3.1), then C can be expressed in the form

$$
C\left(\vec{x}\right) = \sum_{\vec{k}\leq\vec{a}} \beta_{\vec{k}} C_{\vec{k}} \left(F_{1,\vec{k}} \left(\frac{x_1 - k_1}{k_1^+ - k_1} \right), ..., F_{n,\vec{k}} \left(\frac{x_n - k_n}{k_n^+ - k_n} \right) \right)
$$

for every $\vec{a} \in \prod_{i=1}^{n} A_i$, $\vec{x} = (x_1, x_2, ..., x_n) \in T_{\vec{a}}$, and the summation is taken over all $\vec{k} \in \prod_{i=1}^{n} A_i$. Here, $C_{\vec{k}}$ are copulas and $F_{i,\vec{k}} : [-\infty, \infty] \to \mathbb{I}$ are one-dimensional distribution functions with support on $\mathbb{I}$ satisfying

$$
x = \frac{1}{a^+ - a} \sum_{\vec{b}} \beta_{\vec{b}} F_{i,\vec{b}} (x)
$$

for all $i \in \{1, 2, ..., n\}$, $a \in A_i$, and all $x \in \mathbb{I}$ where the summation is taken over all $\overrightarrow{b} = (b_1, b_2, ..., b_n) \in \prod_{i=1}^{n} A_i$ such that $b_i = a$.

Next, we prove the sufficiency part.

Lemma 3.8. *The function C defined by Equation (3.3) is well-defined.*

Proof. The only place that need to be checked is when $\overrightarrow{b} \in T_{\overrightarrow{a}} \cap T_{\overrightarrow{c}}$ for some $\overrightarrow{a}, \overrightarrow{c} \in \prod_{i=1}^{n} A_i$. Without loss of generality, we may assume that $c_i = a_i$ except for a fixed $j \in \{1, 2, ..., n\}$ where $c_j = a_j^+$ instead. Then $b_j = c_j = a_j^+$. Consider,

$$
\begin{aligned}
&\sum_{\overrightarrow{k} \leq \overrightarrow{c}} \beta_{\overrightarrow{k}} C_{\overrightarrow{k}} \left(F_{1,\overrightarrow{k}} \left(\frac{b_1 - k_1}{k_1^+ - k_1} \right), ..., F_{n,\overrightarrow{k}} \left(\frac{b_n - k_n}{k_n^+ - k_n} \right) \right) \\
&= \sum_{\overrightarrow{k} \leq \overrightarrow{a}} \beta_{\overrightarrow{k}} C_{\overrightarrow{k}} \left(F_{1,\overrightarrow{k}} \left(\frac{b_1 - k_1}{k_1^+ - k_1} \right), ..., F_{n,\overrightarrow{k}} \left(\frac{b_n - k_n}{k_n^+ - k_n} \right) \right) \\
&+ \beta_{\overrightarrow{c}} C_{\overrightarrow{c}} \left(F_{1,\overrightarrow{c}} \left(\frac{b_1 - a_1}{a_1^+ - a_1} \right), ..., F_{j,\overrightarrow{c}} \left(\frac{a_j^+ - a_j^+}{a_j^{++} - a_j^+} \right), ..., F_{n,\overrightarrow{c}} \left(\frac{b_n - a_n}{a_n^+ - a_n} \right) \right) \\
&= \sum_{\overrightarrow{k} \leq \overrightarrow{a}} \beta_{\overrightarrow{k}} C_{\overrightarrow{k}} \left(F_{1,\overrightarrow{k}} \left(\frac{b_1 - k_1}{k_1^+ - k_1} \right), ..., F_{n,\overrightarrow{k}} \left(\frac{b_n - k_n}{k_n^+ - k_n} \right) \right) \\
&+ \beta_{\overrightarrow{c}} C_{\overrightarrow{c}} \left(F_{1,\overrightarrow{c}} \left(\frac{b_1 - a_1}{a_1^+ - a_1} \right), ..., F_{j,\overrightarrow{c}} (0), ..., F_{n,\overrightarrow{c}} \left(\frac{b_n - a_n}{a_n^+ - a_n} \right) \right) \\
&= \sum_{\overrightarrow{k} \leq \overrightarrow{a}} \beta_{\overrightarrow{k}} C_{\overrightarrow{k}} \left(F_{1,\overrightarrow{k}} \left(\frac{b_1 - k_1}{k_1^+ - k_1} \right), ..., F_{n,\overrightarrow{k}} \left(\frac{b_n - k_n}{k_n^+ - k_n} \right) \right).
\end{aligned}
$$

Therefore, C is well-defined. □

Lemma 3.9. *Let C be defined as in Equation (3.3), then C is n-increasing.*

Proof. Let $\overrightarrow{a} \in \prod A_i$ and $B := \prod_{i=1}^{n} [c_i, d_i]$, where $[c_i, d_i] \subseteq \mathbb{I}$ for all i. Let $\left[c_{i,\overrightarrow{a}}, d_{i,\overrightarrow{a}}\right] := [c_i, d_i] \cap [a_i, a_i^+] \subseteq [a_i, a_i^+]$. Then, $\prod_{i=1}^{n} \left[c_{i,\overrightarrow{a}}, d_{i,\overrightarrow{a}}\right] \subseteq T_{\overrightarrow{a}}$. Next, we show that

$$
\triangle_{c_{1,\overrightarrow{a}}}^{d_{1,\overrightarrow{a}}} ... \triangle_{c_{n-1,\overrightarrow{a}}}^{d_{n-1,\overrightarrow{a}}} \triangle_{c_{n,\overrightarrow{a}}}^{d_{n,\overrightarrow{a}}} C = \beta_{\overrightarrow{a}} \triangle_{F_{1,\overrightarrow{a}}\left(\frac{c_{1,\overrightarrow{a}} - a_1}{a_1^+ - a_1}\right)}^{F_{1,\overrightarrow{a}}\left(\frac{d_{1,\overrightarrow{a}} - a_1}{a_1^+ - a_1}\right)} \cdots \triangle_{F_{1,\overrightarrow{a}}\left(\frac{c_{n,\overrightarrow{a}} - a_n}{a_n^+ - a_n}\right)}^{F_{n,\overrightarrow{a}}\left(\frac{d_{n,\overrightarrow{a}} - a_n}{a_n^+ - a_n}\right)} C_{\overrightarrow{a}}
$$

by induction. For the basis, we have

$$
\begin{aligned}
&\triangle_{c_{n,\vec{a}}}^{d_{n,\vec{a}}} C\left(t_1,\dots,t_{n-1}\right)\\
&= C\left(t_1,t_2,...,t_{n-1},d_{n,\vec{a}}\right) - C\left(t_1,t_2,...,t_{n-1},c_{n,\vec{a}}\right)\\
&=\sum_{\vec{k}\le\vec{a}}\beta_{\vec{k}}C_{\vec{k}}\left(F_{1,\vec{k}}\left(\frac{t_1-k_1}{k_1^+-k_1}\right),...,F_{n-1,\vec{k}}\left(\frac{t_{n-1}-k_{n-1}}{k_{n-1}^+-k_{n-1}}\right),F_{n,\vec{k}}\left(\frac{d_{n,\vec{a}}-k_n}{k_n^+-k_n}\right)\right)\\
&\quad-\sum_{\vec{k}\le\vec{a}}\beta_{\vec{k}}C_{\vec{k}}\left(F_{1,\vec{k}}\left(\frac{t_1-k_1}{k_1^+-k_1}\right),...,F_{n-1,\vec{k}}\left(\frac{t_{n-1}-k_{n-1}}{k_{n-1}^+-k_{n-1}}\right),F_{n,\vec{k}}\left(\frac{c_{n,\vec{a}}-k_n}{k_n^+-k_n}\right)\right)\\
&=\beta_{\vec{a}}C_{\vec{a}}\left(F_{1,\vec{a}}\left(\frac{t_1-a_1}{a_1^+-a_1}\right),...,F_{n-1,\vec{a}}\left(\frac{t_{n-1}-a_{n-1}}{a_{n-1}^+-a_{n-1}}\right),F_{n,\vec{k}}\left(\frac{d_{n,\vec{a}}-a_n}{a_n^+-a_n}\right)\right)\\
&\quad-\beta_{\vec{a}}C_{\vec{a}}\left(F_{1,\vec{a}}\left(\frac{t_1-a_1}{a_1^+-a_1}\right),...,F_{n-1,\vec{a}}\left(\frac{t_{n-1}-a_{n-1}}{a_{n-1}^+-a_{n-1}}\right),F_{n,\vec{k}}\left(\frac{c_{n,\vec{a}}-a_n}{a_n^+-a_n}\right)\right)\\
&=\beta_{\vec{a}}\triangle_{F_{n,\vec{k}}\left(\frac{c_{n,\vec{a}}-a_n}{a_n^+-a_n}\right)}^{F_{n,\vec{k}}\left(\frac{d_{n,\vec{a}}-a_n}{a_n^+-a_n}\right)}C_{\vec{a}}\left(F_{1,\vec{a}}\left(\frac{t_1-a_1}{a_1^+-a_1}\right),...,F_{n-1,\vec{a}}\left(\frac{t_{n-1}-a_{n-1}}{a_{n-1}^+-a_{n-1}}\right)\right)
\end{aligned}
$$

For the inductive step, let $r<n$. Assume that

$$
\begin{aligned}
&\triangle_{c_{r+1,\vec{a}}}^{d_{r+1,\vec{a}}}...\triangle_{c_{n,\vec{a}}}^{d_{n,\vec{a}}}C\left(t_1,\dots,t_r\right)\\
&=\beta_{\vec{a}}\triangle_{F_{r+1,\vec{a}}\left(\frac{c_{r+1,\vec{a}}-a_{r+1}}{a_{r+1}^+-a_{r+1}}\right)}^{F_{r+1,\vec{a}}\left(\frac{d_{r+1,\vec{a}}-a_{r+1}}{a_{r+1}^+-a_{r+1}}\right)}\cdots\triangle_{F_{n,\vec{a}}\left(\frac{c_{n,\vec{a}}-a_n}{a_n^+-a_n}\right)}^{F_{n,\vec{a}}\left(\frac{d_{n,\vec{a}}-a_n}{a_n^+-a_n}\right)}C_{\vec{a}}\left(F_{1,\vec{a}}\left(\frac{t_1-a_1}{a_1^+-a_1}\right),...,F_{r,\vec{a}}\left(\frac{t_r-a_r}{a_r^+-a_r}\right)\right).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\triangle_{c_{r,\vec{a}}}^{d_{r,\vec{a}}}\triangle_{c_{r+1,\vec{a}}}^{d_{r+1,\vec{a}}}...\triangle_{c_{n,\vec{a}}}^{d_{n,\vec{a}}}C\left(t_1,\dots,t_{r-1}\right)\\
&=\triangle_{c_{r+1,\vec{a}}}^{d_{r+1,\vec{a}}}...\triangle_{c_{n,\vec{a}}}^{d_{n,\vec{a}}}C\left(t_1,\dots,t_{r-1},d_{r,\vec{a}}\right)-\triangle_{c_{r+1,\vec{a}}}^{d_{r+1,\vec{a}}}...\triangle_{c_{n,\vec{a}}}^{d_{n,\vec{a}}}C\left(t_1,\dots,t_{r-1},c_{r,\vec{a}}\right)\\
&=\beta_{\vec{a}}\triangle_{F_{r+1,\vec{a}}\left(\frac{c_{r+1,\vec{a}}-a_{r+1}}{a_{r+1}^+-a_{r+1}}\right)}^{F_{r+1,\vec{a}}\left(\frac{d_{r+1,\vec{a}}-a_{r+1}}{a_{r+1}^+-a_{r+1}}\right)}\cdots\triangle_{F_{n,\vec{a}}\left(\frac{c_{n,\vec{a}}-a_n}{a_n^+-a_n}\right)}^{F_{n,\vec{a}}\left(\frac{d_{n,\vec{a}}-a_n}{a_n^+-a_n}\right)}C_{\vec{a}}\left(F_{1,\vec{a}}\left(\frac{t_1-a_1}{a_1^+-a_1}\right),...,F_{r,\vec{a}}\left(\frac{d_{r,\vec{a}}-a_r}{a_r^+-a_r}\right)\right)\\
&\quad-\beta_{\vec{a}}\triangle_{F_{r+1,\vec{a}}\left(\frac{c_{r+1,\vec{a}}-a_{r+1}}{a_{r+1}^+-a_{r+1}}\right)}^{F_{r+1,\vec{a}}\left(\frac{d_{r+1,\vec{a}}-a_{r+1}}{a_{r+1}^+-a_{r+1}}\right)}\cdots\triangle_{F_{n,\vec{a}}\left(\frac{c_{n,\vec{a}}-a_n}{a_n^+-a_n}\right)}^{F_{n,\vec{a}}\left(\frac{d_{n,\vec{a}}-a_n}{a_n^+-a_n}\right)}C_{\vec{a}}\left(F_{1,\vec{a}}\left(\frac{t_1-a_1}{a_1^+-a_1}\right),...,F_{r,\vec{a}}\left(\frac{c_{r,\vec{a}}-a_r}{a_r^+-a_r}\right)\right)\\
&=\beta_{\vec{a}}\triangle_{F_{r,\vec{a}}\left(\frac{c_{r,\vec{a}}-a_r}{a_r^+-a_r}\right)}^{F_{r,\vec{a}}\left(\frac{d_{r,\vec{a}}-a_r}{a_r^+-a_r}\right)}\triangle_{F_{r+1,\vec{a}}\left(\frac{c_{r+1,\vec{a}}-a_{r+1}}{a_{r+1}^+-a_{r+1}}\right)}^{F_{r+1,\vec{a}}\left(\frac{d_{r+1,\vec{a}}-a_{r+1}}{a_{r+1}^+-a_{r+1}}\right)}\cdots\triangle_{F_{n,\vec{a}}\left(\frac{c_{n,\vec{a}}-a_n}{a_n^+-a_n}\right)}^{F_{n,\vec{a}}\left(\frac{d_{n,\vec{a}}-a_n}{a_n^+-a_n}\right)}C_{\vec{a}}\left(F_{1,\vec{a}}\left(\frac{t_1-a_1}{a_1^+-a_1}\right),...\right.\\
&\quad\left.,F_{r-1,\vec{a}}\left(\frac{t_{r-1,\vec{a}}-a_{r-1}}{a_{r-1}^+-a_{r-1}}\right)\right).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
V_C\left(\prod_{i=1}^{n}\left[c_{i,\vec{a}}, d_{i,\vec{a}}\right]\right) =& \triangle_{c_{1,\vec{a}}}^{d_{1,\vec{a}}} \dots \triangle_{c_{n-1,\vec{a}}}^{d_{n-1,\vec{a}}} \triangle_{c_{n,\vec{a}}}^{d_{n,\vec{a}}} C \\
=& \beta_{\vec{a}} \triangle_{F_{1,\vec{a}}\left(\frac{c_{1,\vec{a}}-a_1}{a_1^+-a_1}\right)}^{F_{1,\vec{a}}\left(\frac{d_{1,\vec{a}}-a_1}{a_1^+-a_1}\right)} \cdots \triangle_{F_{1,\vec{a}}\left(\frac{c_{n,\vec{a}}-a_n}{a_n^+-a_n}\right)}^{F_{n,\vec{a}}\left(\frac{d_{n,\vec{a}}-a_n}{a_n^+-a_n}\right)} C_{\vec{a}} \\
=& \beta_{\vec{a}} \left(\triangle_{F_{1,\vec{a}}\left(\frac{c_{1,\vec{a}}-a_1}{a_1^+-a_1}\right)}^{F_{1,\vec{a}}\left(\frac{d_{1,\vec{a}}-a_1}{a_1^+-a_1}\right)} \cdots \triangle_{F_{1,\vec{a}}\left(\frac{c_{n,\vec{a}}-a_n}{a_n^+-a_n}\right)}^{F_{n,\vec{a}}\left(\frac{d_{n,\vec{a}}-a_n}{a_n^+-a_n}\right)} C_{\vec{a}} \right) \\
=& \beta_{\vec{a}} V_{C_{\vec{a}}}\left(\prod_{i=1}^{n}\left[F_{i,\vec{a}}\left(\frac{c_{i,\vec{a}}-a_i}{a_i^+-a_i}\right), F_{i,\vec{a}}\left(\frac{d_{i,\vec{a}}-a_i}{a_i^+-a_i}\right)\right]\right) \\
\geq& \, 0.
\end{aligned}
$$

Since $B = \bigcup\limits_{\vec{a}\in\prod A_i}\left(T_{\vec{a}} \cap \prod\limits_{i=1}^{n}[c_i, d_i]\right) = \bigcup\limits_{\vec{a}\in\prod A_i}\left(\prod\limits_{i=1}^{n}\left[c_{i,\vec{a}}, d_{i,\vec{a}}\right]\right)$, we have

$$
\begin{aligned}
V_C(B) =& V_C\left(\bigcup_{\vec{a}\in\prod A_i}\left(T_{\vec{a}} \cap \prod_{i=1}^{n}[c_i, d_i]\right)\right) \\
=& \sum_{\vec{a}\in\prod A_i} V_C\left(\prod_{i=1}^{n}\left[c_{i,\vec{a}}, d_{i,\vec{a}}\right]\right) \\
\geq& \, 0.
\end{aligned}
$$

Therefore, C is n-increasing. □

Lemma 3.10. *Let $S : \prod\limits_{i=1}^{n} A_i \to \mathbb{I}$ be an n-subcopula and C be defined by Equation (3.3), then C is a copula extending S.*

Proof. The proof is divided into two parts. The first one is to show that C extends S and the last one is to show that C is a copula.

(1.) The function C extends S.

Let $\vec{a} \in \prod\limits_{i=1}^{n} A_i$. The proof of this part is divided into two cases.

Case 1. If $a_i = 0$ for some i, then $F_{i,\vec{k}}\left(\frac{a_i-k_i}{k_i^+-k_i}\right) = 0$ for all $\vec{k}$. Since $C_{\vec{k}}$ are copulas and

S is a subcopula,

$$\begin{aligned}
C\left(\vec{a}\right) &= \sum_{\vec{k}\leq\vec{a}} \beta_{\vec{k}} C_{\vec{k}}\left(F_{1,\vec{k}}\left(\frac{a_1-k_1}{k_1^+-k_1}\right),...,F_{i,\vec{k}}\left(\frac{a_i-k_i}{k_i^+-k_i}\right),...,F_{n,\vec{k}}\left(\frac{a_n-k_n}{k_n^+-k_n}\right)\right) \\
&= \sum_{\vec{k}\leq\vec{a}} \beta_{\vec{k}} C_{\vec{k}}\left(F_{1,\vec{k}}\left(\frac{a_1-k_1}{k_1^+-k_1}\right),...,0,...,F_{n,\vec{k}}\left(\frac{a_n-k_n}{k_n^+-k_n}\right)\right) \\
&= 0 \\
&= S\left(\vec{a}\right).
\end{aligned}$$

Case 2. If $a_i \neq 0$ for all $i = 1, 2, .., n$. Without loss of generality, one can assume that $a_i = b_i^+$, where $b_i := \sup\{c \in A_i | c < a_i\}$ for all i.
Since S is defined on $\prod_{i=1}^{n} A_i$ and $\beta_{\vec{k}} = V_S\left(\prod_{i=1}^{n}\left[k_i, k_i^+\right]\right)$, it follows that $S\left(\vec{a}\right) = V_S\left(\prod_{i=1}^{n}\left[0, b_i^+\right]\right) = \sum_{\vec{k}\leq\vec{b}} \beta_{\vec{k}}$.
For any $\vec{k} \leq \vec{b}$, we have $k_i < k_i^+ \leq b_i^+ = a_i$ for all i, then it follows that $\frac{a_i-k_i}{k_i^+-k_i} \geq 1$ which implies that $F_{i,\vec{k}}\left(\frac{a_i-k_i}{k_i^+-k_i}\right) = 1$ for i. Thus,

$$\begin{aligned}
C\left(\vec{a}\right) &= \sum_{\vec{k}\leq\vec{b}} \beta_{\vec{k}} C_{\vec{k}}\left(F_{1,\vec{k}}\left(\frac{a_1-k_1}{k_1^+-k_1}\right),...,F_{n,\vec{k}}\left(\frac{a_n-k_n}{k_n^+-k_n}\right)\right) \\
&= \sum_{\vec{k}\leq\vec{b}} \beta_{\vec{k}} C_{\vec{k}}\left(F_{1,\vec{k}}(1),...,F_{n,\vec{k}}(1)\right) \\
&= \sum_{\vec{k}\leq\vec{b}} \beta_{\vec{k}} \\
&= S\left(\vec{a}\right).
\end{aligned}$$

Therefore, C extends S.

(2) C is a copula.

It is obvious that the domain of C is $\mathbb{I}^n$. By Lemma 3.9, we already have that C is n-increasing. It remains to prove that C is grounded and it has uniform marginals.

Let $\vec{x} \in \mathbb{I}^n$ with $x_i = 0$ for some i. Since $C_{\vec{k}}$ are copulas,

$$\begin{aligned}
C\left(\vec{x}\right) &= \sum \beta_{\vec{k}} C_{\vec{k}}\left(F_{1,\vec{k}}\left(\frac{x_1-k_1}{k_1^+-k_1}\right),...,F_{i,\vec{k}}\left(\frac{0-k_i}{k_i^+-k_i}\right),...,F_{n,\vec{k}}\left(\frac{x_n-k_n}{k_n^+-k_n}\right)\right) \\
&= \sum \beta_{\vec{k}} C_{\vec{k}}\left(F_{1,\vec{k}}\left(\frac{x_1-k_1}{k_1^+-k_1}\right),...,0,...,F_{n,\vec{k}}\left(\frac{x_n-k_n}{k_n^+-k_n}\right)\right) \\
&= 0.
\end{aligned}$$

Therefore, C is grounded.

Let $\overrightarrow{x} \in \mathbb{I}^n$ be such that $x_j = 1$ for all $j \neq i$ when $i = 1, 2, ..., n$ fixed. Then $\overrightarrow{x} \in T_{\overrightarrow{a}}$ where $\overrightarrow{a} = \{1_1^-, 1_2^-, \ldots, a_i, \ldots, 1_{n-1}^-, 1_n^-\}$ and $1_j^- := \sup\{b \in A_j | b < 1\}$. Now,

$$
\begin{aligned}
C\left(\overrightarrow{x}\right) &= \sum_{\overrightarrow{k} \leq \overrightarrow{a}} \beta_{\overrightarrow{k}} C_{\overrightarrow{k}} \left(F_{1,\overrightarrow{k}}\left(\frac{1-k_1}{k_1^+ - k_1}\right), ..., F_{i,\overrightarrow{k}}\left(\frac{x_i - k_i}{k_i^+ - k_i}\right), ..., F_{n,\overrightarrow{k}}\left(\frac{1-k_n}{k_n^+ - k_n}\right)\right) \\
&= \sum_{\overrightarrow{k} \leq \overrightarrow{a}} \beta_{\overrightarrow{k}} C_{\overrightarrow{k}} \left(1, ..., F_{i,\overrightarrow{k}}\left(\frac{x_i - k_i}{k_i^+ - k_i}\right), ..., 1\right) \\
&= \sum_{\overrightarrow{k} \leq \overrightarrow{a}} \beta_{\overrightarrow{k}} F_{i,\overrightarrow{k}}\left(\frac{x_i - k_i}{k_i^+ - k_i}\right) \\
&= \sum_{a \leq a_i} \left(\sum_{\overrightarrow{k} \leq \overrightarrow{a}, k_i = a} \beta_{\overrightarrow{k}} F_{i,\overrightarrow{k}}\left(\frac{x_i - k_i}{k_i^+ - k_i}\right)\right) \\
&= \left(\sum_{a < a_i} \left(\sum_{\overrightarrow{k} \leq \overrightarrow{a}, k_i = a} \beta_{\overrightarrow{k}} F_{i,\overrightarrow{k}}\left(\frac{x_i - a}{a^+ - a}\right)\right)\right) + \sum_{\overrightarrow{k} \leq \overrightarrow{a}, k_i = a_i} \beta_{\overrightarrow{k}} F_{i,\overrightarrow{k}}\left(\frac{x_i - a_i}{a_i^+ - a_i}\right) \\
&= \left(\sum_{a < a_i} \left(\sum_{\overrightarrow{k} \leq \overrightarrow{a}, k_i = a} \beta_{\overrightarrow{k}} \right)\right) + \sum_{\overrightarrow{k} \leq \overrightarrow{a}, k_i = a_i} \beta_{\overrightarrow{k}} F_{i,\overrightarrow{k}}\left(\frac{x_i - a_i}{a_i^+ - a_i}\right) \\
&= \left(\sum_{a < a_i} \left(\sum_{\overrightarrow{k} \leq \overrightarrow{a}, k_i = a} V_S\left(\prod_{j=1}^{n} \left[k_j, k_j^+\right]\right)\right)\right) + \left(\frac{x_i - a_i}{a_i^+ - a_i}\right)(a_i^+ - a_i) \\
&= \left(\sum_{a < a_i} \left(\sum_{\overrightarrow{k} \leq \overrightarrow{a}, k_i = a} V_S\left(\prod_{j=1}^{i-1} [k_i, k_i^+] \times [a, a^+] \times \prod_{j=i+1}^{n} [k_i, k_i^+]\right)\right)\right) + (x_i - a_i) \\
&= \left(\sum_{a < a_i} V_S \left(\prod_{j=1}^{i-1} [0,1] \times [a, a^+] \times \prod_{j=i+1}^{n} [0,1]\right)\right) + (x_i - a_i) \\
&= \left(\sum_{a < a_i} \left(S\left(1, 1, ..., a^+, ..., 1, 1\right) - S\left(1, 1, ..., a, ..., 1, 1\right)\right)\right) + (x_i - a_i) \\
&= \left(\sum_{a < a_i} (a^+ - a)\right) + (x_i - a_i) \\
&= (a_i - 0) + (x_i - a_i) \\
&= x_i.
\end{aligned}
$$

Therefore, C has uniform marginals, which completes the proof. □

Example 3.11. If $Ran\,(F) = \left\{0, \frac{1}{2}, 1\right\}$, $Ran\,(G) = \left\{0, \frac{1}{2}, 1\right\}$ and $Ran\,(H) = \left\{0, \frac{1}{2}, 1\right\}$ and the subcopula $S : Ran\,(F) \times Ran\,(G) \times Ran\,(H) \to \mathbb{I}$ defined by $S\,(a, b, c) = \min\{a, b, c\}$.

Then, a copula C extending S has the form

$$C(x,y,z)=\begin{cases}\frac{1}{2}C_1(2x,2y,2z) & \text{if } (x,y,z)\in\left[0,\frac{1}{2}\right]\times\left[0,\frac{1}{2}\right]\times\left[0,\frac{1}{2}\right]\\ \frac{1}{2}C_2(2y,2z) & \text{if } (x,y,z)\in\left[\frac{1}{2},1\right]\times\left[0,\frac{1}{2}\right]\times\left[0,\frac{1}{2}\right]\\ \frac{1}{2}C_3(2x,2z) & \text{if } (x,y,z)\in\left[0,\frac{1}{2}\right]\times\left[\frac{1}{2},1\right]\times\left[0,\frac{1}{2}\right]\\ \frac{1}{2}C_4(2x,2y) & \text{if } (x,y,z)\in\left[0,\frac{1}{2}\right]\times\left[0,\frac{1}{2}\right]\times\left[\frac{1}{2},1\right]\\ x & \text{if } (x,y,z)\in\left[0,\frac{1}{2}\right]\times\left[\frac{1}{2},1\right]\times\left[\frac{1}{2},1\right]\\ y & \text{if } (x,y,z)\in\left[\frac{1}{2},1\right]\times\left[0,\frac{1}{2}\right]\times\left[\frac{1}{2},1\right]\\ z & \text{if } (x,y,z)\in\left[\frac{1}{2},1\right]\times\left[\frac{1}{2},1\right]\times\left[0,\frac{1}{2}\right]\\ \frac{1}{2}+\frac{1}{2}C_5(2x-1,2y-1,2z-1) & \text{if } (x,y,z)\in\left[\frac{1}{2},1\right]\times\left[\frac{1}{2},1\right]\times\left[\frac{1}{2},1\right]\end{cases}$$

where C_1, C_2, C_3, C_4, and C_5 are any copulas.