CHAPTER 4

Copula Approximation

There are two concrete examples of approximations of a copula: the checkmin approximation and the checkerboard approximation [11].

For a given n-copula C and $m \in \mathbb{N}$, the checkmin approximation of copula C is the copula $A_m : \mathbb{I}^n \to \mathbb{I}$ defined by

$$A_m\left(\vec{x}\right) = \sum_{\vec{i}} m V_C\left(\left(\frac{\vec{i}}{m}, \frac{\vec{i}+\vec{1}}{m}\right]\right) \chi_{\left(\frac{\vec{i}}{m}, \vec{1}\right]}\left(\vec{x}\right) \min\left(x_1 - \frac{i_1}{m}, \ldots, x_n - \frac{i_n}{m}, \frac{1}{m}\right) \tag{4.1}$$

and the checkerboard approximation of a copula C is the copula $B_m : \mathbb{I}^n \to \mathbb{I}$ defined by

$$B_m\left(\vec{x}\right) = \sum_{\vec{i}} m^n V_C\left(\left(\frac{\vec{i}}{m}, \frac{\vec{i}+\vec{1}}{m}\right]\right) \chi_{\left(\frac{\vec{i}}{m}, \vec{1}\right]}\left(\vec{x}\right) \prod_{i=1}^{n} \min\left(x_k - \frac{i_k}{m}, \frac{1}{m}\right) \tag{4.2}$$

where $\vec{i} = (i_1, i_2, \ldots, i_n) \in \{0, 1, \ldots, m-1\}^n$, $\vec{1} = (1, 1, \ldots, 1)$, and χ_A is the characteristic function of the set A. Here, the copulas A_m and B_m are the approximations of C in the sense that $A_m = B_m = C$ on $\left\{0, \frac{1}{m}, \frac{2}{m} \ldots, \frac{m-1}{m}, 1\right\}^n$. Thus, $A_m \to C$ and $B_m \to C$ uniformly.

Let S be the restriction of C on $\left\{0, \frac{1}{m}, \frac{2}{m} \ldots, \frac{m-1}{m}, 1\right\}^n$. Obviously, A_m, B_m and C extend the same subcopula S which implies that they can be written as in the form in our main result.

Since we know that A_m and B_m converge to C uniformly, applying the main result will tell us how fast they converge.

What follows does not only answer the question that how fast A_m and B_m converge to C but also answer the question in the case of any approximation of a given copula C.

Theorem 4.1. *The function $F_{i,\vec{b}}$ satisfying Equation (3.4) is an $\left(\frac{a_i^+ - a_i}{\beta_{\vec{b}}}\right)$-Lipschitz function for all $\vec{b} \in \prod_{j=1}^{i-1} A_j \times \{a_i\} \times \prod_{j=i+1}^{n} A_j$ and $i = 1, 2, \ldots, n$.*

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Proof. Let $\overrightarrow{b'} \in \prod_{j=1}^{i-1} A_j \times \{a_i\} \times \prod_{j=i+1}^{n} A_j$. By Equation (3.4),

$$\beta_{\overrightarrow{b'}} F_{i,\overrightarrow{b'}}(x) = \left(a_i^+ - a_i\right) x - \left(\sum_{\overrightarrow{b}} \beta_{\overrightarrow{b}} F_{i,\overrightarrow{b}}(x) - \beta_{\overrightarrow{b'}} F_{i,\overrightarrow{b'}}(x)\right)$$

which implies that

$$F_{i,\overrightarrow{b'}}(x) = \frac{a_i^+ - a_i}{\beta_{\overrightarrow{b'}}} x - \frac{1}{\beta_{\overrightarrow{b'}}} \left(\sum_{\overrightarrow{b}} \beta_{\overrightarrow{b}} F_{i,\overrightarrow{b}}(x) - \beta_{\overrightarrow{b'}} F_{i,\overrightarrow{b'}}(x)\right).$$

Without loss of generality, we may assume that $y \leq x$. Consider

$$\begin{aligned}
\left|F_{i,\overrightarrow{b'}}(x) - F_{i,\overrightarrow{b'}}(y)\right| &= F_{i,\overrightarrow{b'}}(x) - F_{i,\overrightarrow{b'}}(y) \\
&= \frac{a_i^+ - a_i}{\beta_{\overrightarrow{b'}}} x - \frac{1}{\beta_{\overrightarrow{b'}}} \left(\left[\sum_{\overrightarrow{b}} \beta_{\overrightarrow{b}} F_{i,\overrightarrow{b}}(x)\right] - \beta_{\overrightarrow{b'}} F_{i,\overrightarrow{b'}}(x)\right) \\
&\quad - \left(\frac{a_i^+ - a_i}{\beta_{\overrightarrow{b'}}} y - \frac{1}{\beta_{\overrightarrow{b'}}} \left(\left[\sum_{\overrightarrow{b}} \beta_{\overrightarrow{b}} F_{i,\overrightarrow{b}}(y)\right] - \beta_{\overrightarrow{b'}} F_{i,\overrightarrow{b'}}(y)\right)\right) \\
&= \frac{a_i^+ - a_i}{\beta_{\overrightarrow{b'}}} (x - y) \\
&\quad - \frac{1}{\beta_{\overrightarrow{b'}}} \left[\left(\sum_{\overrightarrow{b}} \beta_{\overrightarrow{b}} F_{i,\overrightarrow{b}}(x)\right) - \beta_{\overrightarrow{b'}} F_{i,\overrightarrow{b'}}(x)\right. \\
&\quad \left. - \left(\sum_{\overrightarrow{b}} \beta_{\overrightarrow{b}} F_{i,\overrightarrow{b}}(y)\right) + \beta_{\overrightarrow{b'}} F_{i,\overrightarrow{b'}}(y)\right] \\
&= \frac{a_i^+ - a_i}{\beta_{\overrightarrow{b'}}} (x - y) \\
&\quad - \frac{1}{\beta_{\overrightarrow{b'}}} \left[\left(\sum_{\overrightarrow{b}} \left[\beta_{\overrightarrow{b}} F_{i,\overrightarrow{b}}(x) - \beta_{\overrightarrow{b}} F_{i,\overrightarrow{b}}(y)\right]\right)\right. \\
&\quad \left. - \left(\beta_{\overrightarrow{b'}} F_{i,\overrightarrow{b'}}(x) - \beta_{\overrightarrow{b'}} F_{i,\overrightarrow{b'}}(y)\right)\right].
\end{aligned}$$

Since $F_{i,\overrightarrow{b}}(x) - F_{i,\overrightarrow{b}}(y) \geq 0$ and $\beta_{\overrightarrow{b}} \geq 0$, we have $\beta_{\overrightarrow{b}} F_{i,\overrightarrow{b}}(x) - \beta_{\overrightarrow{b}} F_{i,\overrightarrow{b}}(y) \geq 0$ for all $\overrightarrow{b} \in \prod_{j=1}^{i-1} A_j \times \{a_i\} \times \prod_{j=i+1}^{n} A_j$.

Since $\overrightarrow{b'} \in \prod_{j=1}^{i-1} A_j \times \{a_i\} \times \prod_{j=i+1}^{n} A_j$ and $\beta_{\overrightarrow{b'}} \geq 0$, it follows that

$$\sum_{\overrightarrow{b}} \left[\beta_{\overrightarrow{b}} F_{i,\overrightarrow{b}}(x) - \beta_{\overrightarrow{b}} F_{i,\overrightarrow{b}}(y)\right] \geq \beta_{\overrightarrow{b'}} F_{i,\overrightarrow{b'}}(x) - \beta_{\overrightarrow{b'}} F_{i,\overrightarrow{b'}}(y) \geq 0$$

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which implies that

$$-\frac{1}{\beta_{\overrightarrow{b'}}}\left[\left(\sum_{\overrightarrow{b}}\left[\beta_{\overrightarrow{b}}F_{i,\overrightarrow{b}}(x)-\beta_{\overrightarrow{b}}F_{i,\overrightarrow{b}}(y)\right]\right)-\left(\beta_{\overrightarrow{b'}}F_{i,\overrightarrow{b'}}(x)-\beta_{\overrightarrow{b'}}F_{i,\overrightarrow{b'}}(y)\right)\right]\leq 0.$$

Hence,

$$\begin{aligned}\left|F_{i,\overrightarrow{b'}}(x)-F_{i,\overrightarrow{b'}}(y)\right| &\leq \frac{a_i^+-a_i}{\beta_{\overrightarrow{b'}}}(x-y)\\ &= \frac{a_i^+-a_i}{\beta_{\overrightarrow{b'}}}\left|x-y\right|.\end{aligned}$$

Therefore, $F_{i,\overrightarrow{b'}}$ is an $\left(\frac{a_i^+-a_i}{\beta_{\overrightarrow{b}}}\right)$-Lipschitz function. $\square$

Theorem 4.2. For any $\overrightarrow{b}=(b_1,\ldots,b_n)\in\prod_{i=1}^{n}A_i$, and any $i=1,2,\ldots,n$, *let* $\underline{F}_{i,\overrightarrow{b}}:[-\infty,\infty]\to\mathbb{I}$ *be defined by*

$$\underline{F}_{i,\overrightarrow{b}}(x)=\left(\left(\frac{b_i^+-b_i}{\beta_{\overrightarrow{b}}}x+1-\frac{b_i^+-b_i}{\beta_{\overrightarrow{b}}}\right)\vee 0\right)\wedge 1$$

and $\overline{F}_{i,\overrightarrow{b}}:[-\infty,\infty]\to\mathbb{I}$ *be defined by*

$$\overline{F}_{i,\overrightarrow{b}}(x)=\left(\left(\frac{b_i^+-b_i}{\beta_{\overrightarrow{b}}}x\right)\vee 0\right)\wedge 1.$$

Then $\underline{F}_{i,\overrightarrow{b}}\leq F_{i,\overrightarrow{b}}\leq\overline{F}_{i,\overrightarrow{b}}$ *for any* $F_{i,\overrightarrow{b}}$ *satisfying Equation (3.4).*

Proof. It is obvious that $\underline{F}_{i,\overrightarrow{b}}(x)=F_{i,\overrightarrow{b}}(x)=\overline{F}_{i,\overrightarrow{b}}(x)$ whenever $x\in[-\infty,0)\cup(1,\infty]$. Assume $0\leq x\leq 1$. By Theorem 4.1, we have

$$F_{i,\overrightarrow{b}}(x)=F_{i,\overrightarrow{b}}(x)-F_{i,\overrightarrow{b}}(0)\leq\frac{a_i^+-a_i}{\beta_{\overrightarrow{b'}}}(x-0)$$

and

$$F_{i,\overrightarrow{b}}(1)-F_{i,\overrightarrow{b}}(x)=1-F_{i,\overrightarrow{b}}(x)\leq\frac{a_i^+-a_i}{\beta_{\overrightarrow{b'}}}(1-x)$$

which implies that

$$\frac{a_i^+-a_i}{\beta_{\overrightarrow{b}}}x+1-\frac{a_i^+-a_i}{\beta_{\overrightarrow{b}}}\leq F_{i,\overrightarrow{b}}(x)\leq\frac{a_i^+-a_i}{\beta_{\overrightarrow{b}}}x$$

as desired. $\square$

Lemma 4.3. *Let* $S:\prod_{i=1}^{n}A_i\to\mathbb{I}$ *be a discrete subcopula. The function* $M_S:\mathbb{I}^n\to\mathbb{I}$ *defined by*

$$M_S\left(\overrightarrow{x}\right)=\sum_{\overrightarrow{k}\leq\overrightarrow{a}}\beta_{\overrightarrow{k}}M^n\left(\overline{F}_{1,\overrightarrow{k}}\left(\frac{x_1-k_1}{k_1^+-k_1}\right),\ldots,\overline{F}_{n,\overrightarrow{k}}\left(\frac{x_n-k_n}{k_n^+-k_n}\right)\right)$$

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whenever $x \in T_{\overrightarrow{a}}$*, where* $\overline{F}_{i,\overrightarrow{k}}$ *is defined as in Theorem 4.2, is an upper bound of the set* $\{C : C \text{ is a copula extending } S\}$.

Proof. Let S be a discrete subcopula and $\overrightarrow{x} \in T_{\overrightarrow{a}}$, $\overrightarrow{a} \in Dom(S)$. By the main theorem, all copulas C extending S can be expressed in the form

$$C\left(\overrightarrow{x}\right) = \sum_{\overrightarrow{k} \leq \overrightarrow{a}} \beta_{\overrightarrow{k}} C_{\overrightarrow{k}} \left(F_{1,\overrightarrow{k}} \left(\frac{x_1 - k_1}{k_1^+ - k_1} \right), ..., F_{n,\overrightarrow{k}} \left(\frac{x_n - k_n}{k_n^+ - k_n} \right) \right).$$

Since the copula M^n is an upper bound of all n-copulas and nondecreasing, it follows that

$$\begin{aligned} C\left(\overrightarrow{x}\right) &\leq \sum_{\overrightarrow{k} \leq \overrightarrow{a}} \beta_{\overrightarrow{k}} M^n \left(F_{1,\overrightarrow{k}} \left(\frac{x_1 - k_1}{k_1^+ - k_1} \right), ..., F_{n,\overrightarrow{k}} \left(\frac{x_n - k_n}{k_n^+ - k_n} \right) \right) \\ &\leq \sum_{\overrightarrow{k} \leq \overrightarrow{a}} \beta_{\overrightarrow{k}} M^n \left(\overline{F}_{1,\overrightarrow{k}} \left(\frac{x_1 - k_1}{k_1^+ - k_1} \right), ..., \overline{F}_{n,\overrightarrow{k}} \left(\frac{x_n - k_n}{k_n^+ - k_n} \right) \right) \\ &= M_S\left(\overrightarrow{x}\right), \end{aligned}$$

for all C extending S. □

Lemma 4.4. *Let* $S : \prod_{i=1}^{n} A_i \to \mathbb{I}$ *be a discrete subcopula. The function* $W_S : \mathbb{I}^n \to \mathbb{I}$ *defined by*

$$W_S\left(\overrightarrow{x}\right) = \sum_{\overrightarrow{k} \leq \overrightarrow{a}} \beta_{\overrightarrow{k}} W^n \left(\underline{F}_{1,\overrightarrow{k}} \left(\frac{x_1 - k_1}{k_1^+ - k_1} \right), \ldots, \underline{F}_{n,\overrightarrow{k}} \left(\frac{x_n - k_n}{k_n^+ - k_n} \right) \right)$$

whenever $x \in T_{\overrightarrow{a}}$*, where* $\underline{F}_{i,\overrightarrow{k}}$ *is defined as Theorem 4.2, is a lower bound of the set* $\{C : C \text{ is a copula extending } S\}$.

Proof. Let S be a discrete subcopula and $\overrightarrow{x} \in T_{\overrightarrow{a}}$, $\overrightarrow{a} \in Dom(S)$. By the main theorem, all copulas C extending S can be expressed in the form

$$C\left(\overrightarrow{x}\right) = \sum_{\overrightarrow{k} \leq \overrightarrow{a}} \beta_{\overrightarrow{k}} C_{\overrightarrow{k}} \left(F_{1,\overrightarrow{k}} \left(\frac{x_1 - k_1}{k_1^+ - k_1} \right), ..., F_{n,\overrightarrow{k}} \left(\frac{x_n - k_n}{k_n^+ - k_n} \right) \right).$$

Since W^n is a lower bound of all n-copulas and all copulas are nondecreasing, it follows that

$$\begin{aligned} C\left(\overrightarrow{x}\right) &\geq \sum_{\overrightarrow{k} \leq \overrightarrow{a}} \beta_{\overrightarrow{k}} C_{\overrightarrow{k}} \left(\underline{F}_{1,\overrightarrow{k}} \left(\frac{x_1 - k_1}{k_1^+ - k_1} \right), ..., \underline{F}_{n,\overrightarrow{k}} \left(\frac{x_n - k_n}{k_n^+ - k_n} \right) \right) \\ &\geq \sum_{\overrightarrow{k} \leq \overrightarrow{a}} \beta_{\overrightarrow{k}} W^n \left(\underline{F}_{1,\overrightarrow{k}} \left(\frac{x_1 - k_1}{k_1^+ - k_1} \right), ..., \underline{F}_{n,\overrightarrow{k}} \left(\frac{x_n - k_n}{k_n^+ - k_n} \right) \right) \\ &= W_S\left(\overrightarrow{x}\right) \end{aligned}$$

for all C extending S. □

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Theorem 4.5. *Let* $S : \prod_{i=1}^{n} A_i \to \mathbb{I}$ *be a discrete subcopula. If* C_1 *and* C_2 *are two* n*-copulas extending* S*, then*

$$\left|C_1\left(\overrightarrow{x}\right) - C_2\left(\overrightarrow{x}\right)\right| \leq \sum_{i=1}^{n} \left|a_i^+ - a_i\right|$$

whenever $\overrightarrow{x} \in T_{\overrightarrow{a}}$*, for some* $\overrightarrow{a} \in Dom\,(S)$*.*

Proof. Let $\overrightarrow{x} \in T_{\overrightarrow{a}}$, $\overrightarrow{a} \in \prod A_i$ and copulas C_1 and C_2 extend S. For any $\overrightarrow{k} < \overrightarrow{a}$, we have $k_i < a_i \leq x_i$ which implies that $k_i^+ \leq a_i \leq x_i$. Then $k_i^+ - k_i \leq a_i - k_i \leq x_i - k_i$ which implies that $\frac{x_i - k_i}{k_i^+ - k_i} \geq 1$. Hence,

$$\begin{aligned}
M^n\left(\overline{F}_{1,\overrightarrow{k}}\left(\frac{x_1 - k_1}{k_1^+ - k_1}\right), \ldots, \overline{F}_{n,\overrightarrow{k}}\left(\frac{x_n - k_n}{k_n^+ - k_n}\right)\right) &= M^n\left(1 \ldots, 1\right) \\
&= \min\left(1, \ldots, 1\right) \\
&= 1 \\
&= \max\left(1 + \ldots + 1 - n + 1, 0\right) \\
&= W^n\left(1, \ldots, 1\right) \\
&= W^n\left(\underline{F}_{1,\overrightarrow{k}}\left(\frac{x_1 - k_1}{k_1^+ - k_1}\right), \ldots\right. \\
&\quad \left., \underline{F}_{n,\overrightarrow{k}}\left(\frac{x_n - k_n}{k_n^+ - k_n}\right)\right)
\end{aligned}$$

in this case.

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Since $W_S \leq C_1 \leq M_S$ and $W_S \leq C_2 \leq M_S$,

$$\begin{aligned}
\left|C_1\left(\overrightarrow{x}\right) - C_2\left(\overrightarrow{x}\right)\right| &\leq \sum_{\overrightarrow{k} \leq \overrightarrow{a}} \beta_{\overrightarrow{k}} M^n \left(\overline{F}_{1,\overrightarrow{k}}\left(\frac{x_1 - k_1}{k_1^+ - k_1}\right), ..., \overline{F}_{n,\overrightarrow{k}}\left(\frac{x_n - k_n}{k_n^+ - k_n}\right)\right) \\
&\quad - \sum_{\overrightarrow{k} \leq \overrightarrow{a}} \beta_{\overrightarrow{k}} W^n \left(\underline{F}_{1,\overrightarrow{k}}\left(\frac{x_1 - k_1}{k_1^+ - k_1}\right), ..., \underline{F}_{n,\overrightarrow{k}}\left(\frac{x_n - k_n}{k_n^+ - k_n}\right)\right) \\
&= \sum_{\overrightarrow{k} < \overrightarrow{a}} \beta_{\overrightarrow{k}} + \sum_{\overrightarrow{k}; k_i = a_i, \exists i} \beta_{\overrightarrow{k}} M^n \left(\overline{F}_{1,\overrightarrow{k}}\left(\frac{x_1 - k_1}{k_1^+ - k_1}\right), \ldots\right. \\
&\quad \left., \overline{F}_{n,\overrightarrow{k}}\left(\frac{x_n - k_n}{k_n^+ - k_n}\right)\right) - \left(\sum_{\overrightarrow{k} < \overrightarrow{a}} \beta_{\overrightarrow{k}} + \sum_{\overrightarrow{k}; k_i = a_i, \exists i} \beta_{\overrightarrow{k}} W^n\right. \\
&\quad \left.\left(\underline{F}_{1,\overrightarrow{k}}\left(\frac{x_1 - k_1}{k_1^+ - k_1}\right), \ldots, \underline{F}_{n,\overrightarrow{k}}\left(\frac{x_n - k_n}{k_n^+ - k_n}\right)\right)\right) \\
&= \sum_{\overrightarrow{k}; k_i = a_i, \exists i} \beta_{\overrightarrow{k}} M^n \left(\overline{F}_{1,\overrightarrow{k}}\left(\frac{x_1 - k_1}{k_1^+ - k_1}\right), \ldots, \overline{F}_{n,\overrightarrow{k}}\left(\frac{x_n - k_n}{k_n^+ - k_n}\right)\right) \\
&\quad - \sum_{\overrightarrow{k}; k_i = a_i, \exists i} \beta_{\overrightarrow{k}} W^n \left(\underline{F}_{1,\overrightarrow{k}}\left(\frac{x_1 - k_1}{k_1^+ - k_1}\right), \ldots, \underline{F}_{n,\overrightarrow{k}}\left(\frac{x_n - k_n}{k_n^+ - k_n}\right)\right) \\
&= \sum_{\overrightarrow{k}; k_i = a_i, \exists i} \beta_{\overrightarrow{k}} \left[M^n \left(\overline{F}_{1,\overrightarrow{k}}\left(\frac{x_1 - k_1}{k_1^+ - k_1}\right), \ldots, \overline{F}_{n,\overrightarrow{k}}\left(\frac{x_n - k_n}{k_n^+ - k_n}\right)\right)\right. \\
&\quad \left. - W^n \left(\underline{F}_{1,\overrightarrow{k}}\left(\frac{x_1 - k_1}{k_1^+ - k_1}\right), \ldots, \underline{F}_{n,\overrightarrow{k}}\left(\frac{x_n - k_n}{k_n^+ - k_n}\right)\right)\right].
\end{aligned}$$

For each $\overrightarrow{k} \leq \overrightarrow{a}$,

$$\begin{aligned}
0 &\leq W^n \left(\underline{F}_{1,\overrightarrow{k}}\left(\frac{x_1 - k_1}{k_1^+ - k_1}\right), \ldots, \underline{F}_{n,\overrightarrow{k}}\left(\frac{x_n - k_n}{k_n^+ - k_n}\right)\right) \\
&\leq M^n \left(\underline{F}_{1,\overrightarrow{k}}\left(\frac{x_1 - k_1}{k_1^+ - k_1}\right), \ldots, \underline{F}_{n,\overrightarrow{k}}\left(\frac{x_n - k_n}{k_n^+ - k_n}\right)\right) \\
&\leq M^n \left(\overline{F}_{1,\overrightarrow{k}}\left(\frac{x_1 - k_1}{k_1^+ - k_1}\right), \ldots, \overline{F}_{n,\overrightarrow{k}}\left(\frac{x_n - k_n}{k_n^+ - k_n}\right)\right) \\
&\leq 1
\end{aligned}$$

which implies

$$\begin{aligned}
0 &\leq M^n \left(\overline{F}_{1,\overrightarrow{k}}\left(\frac{x_1 - k_1}{k_1^+ - k_1}\right), \ldots, \overline{F}_{n,\overrightarrow{k}}\left(\frac{x_n - k_n}{k_n^+ - k_n}\right)\right) \\
&\quad - W^n \left(\underline{F}_{1,\overrightarrow{k}}\left(\frac{x_1 - k_1}{k_1^+ - k_1}\right), \ldots, \underline{F}_{n,\overrightarrow{k}}\left(\frac{x_n - k_n}{k_n^+ - k_n}\right)\right) \\
&\leq 1.
\end{aligned}$$

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Therefore,

$$\begin{aligned}
\left|C_1\left(\vec{x}\right)-C_2\left(\vec{x}\right)\right| &\le \sum_{\vec{k};k_i=a_i,\exists i}\beta_{\vec{k}}\left[M^n\left(\overline{F}_{1,\vec{k}}\left(\frac{x_1-k_1}{k_1^+-k_1}\right),\ldots,\overline{F}_{n,\vec{k}}\left(\frac{x_n-k_n}{k_n^+-k_n}\right)\right)\right.\\
&\quad \left.-W^n\left(\underline{F}_{1,\vec{k}}\left(\frac{x_1-k_1}{k_1^+-k_1}\right),\ldots,\underline{F}_{n,\vec{k}}\left(\frac{x_n-k_n}{k_n^+-k_n}\right)\right)\right]\\
&\le \sum_{\vec{k};k_i=a_i,\exists i}\beta_{\vec{k}}\\
&\le \sum_{\vec{k};k_1=a_1}\beta_{\vec{k}}+\sum_{\vec{k};k_2=a_2}\beta_{\vec{k}}+\ldots+\sum_{\vec{k};k_n=a_n}\beta_{\vec{k}}\\
&= (a_1^+-a_1)+(a_2^+-a_2)+\ldots+(a_n^+-a_n)\\
&= \sum_{i=1}^{n}\left|a_i^+-a_i\right|
\end{aligned}$$

as desired. □

Corollary 4.6. *Let* A_m *be the checkmin approximation of an* n*-copula* C *defined as in Equation (4.1). Then*

$$\sup_{\vec{x}\in\mathbb{I}^n}\left|A_m\left(\vec{x}\right)-C\left(\vec{x}\right)\right|\le\frac{n}{m}.$$

Proof. Since A_m and C extend the same subcopula, we can apply Theorem 4.5 and the fact that $a_i^+-a_i=\frac{1}{m}$ to get $\left|A_m\left(\vec{x}\right)-C\left(\vec{x}\right)\right|\le\sum_{i=1}^{n}\frac{1}{m}=\frac{n}{m}$ for all $\vec{x}\in T_{\frac{\vec{i}}{m}}$. Thus, the corollary follows. □

Corollary 4.7. *Let* B_m *be the checkerboard approximation of an* n*-copula* C *defined as in Equation (4.2). Then*

$$\sup_{\vec{x}\in\mathbb{I}^n}\left|B_m\left(\vec{x}\right)-C\left(\vec{x}\right)\right|\le\frac{n}{m}.$$

Proof. Since B_m and C extend the same subcopula, we can apply Theorem 4.5 and the fact that $a_i^+-a_i=\frac{1}{m}$ to get $\left|B_m\left(\vec{x}\right)-C\left(\vec{x}\right)\right|\le\sum_{i=1}^{n}\frac{1}{m}=\frac{n}{m}$ for all $\vec{x}\in T_{\frac{\vec{i}}{m}}$. Thus, the corollary follows. □

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