CHAPTER 5

Conclusion and Discussion

We have characterized all copulas satisfying Sklar's Theorem in the case of discrete marginal distribution functions which may be considered as an extension to higher dimensions of the result of Amo et al. [2] in the case of discrete random variables.

We also provided the lower and upper bounds of the set of all extensions of a discrete subcopula. However, these bounds are not copulas. The question whether there always exist the maximum and minimum extensions remains unanswered. If the answer is yes, finding the maximum extension may be easier than finding the minimum one. Since we know that M^n is the maximum copula, we can choose M^n instead all copulas $C_{\vec{k}}$. The remaining difficulty is to find the maximum of distribution functions $F_{i,\vec{k}}$ satisfying the condition

$$x = \frac{1}{a^+ - a}\sum_{\vec{b}} \beta_{\vec{b}} F_{i,\vec{b}}(x)$$

for all $x \in [0,1]$ and $a \in A_i$ where the sum is taken over all $\vec{b} \in \prod_{j=1}^{i-1} A_j \times \{a\} \times \prod_{j=i+1}^{n} A_j$. We hope that characterizing these distribution functions $F_{i,\vec{k}}$, will help to determine the maximum extension.

We have illustrated an application of the main result through the rate of convergence of copula approximations. Two examples of referred copula approximations are the checkmin and the checkerboard approximations. With our main result, we can provide the rate of convergence of the checkmin and the checkerboard approximations of a given copula.

The difference between two copulas extending the same subcopula is not greater than the difference between the lower bound and the upper bound of the set of all copulas extending that subcopula. By using this fact, we can prove that the difference between two copulas extending the same subcopula depends on only the grid size. Thus, as the grid size get smaller, the two copulas get closer. It can be concluded that if the grid size is approaching zero, then the two copulas extending the same subcopula are almost the same.

For a given copula, its restriction on a discrete set is a discrete subcopula. If we extend that subcopula to another copula, then we have an approximation of the given copula. Thus, the given copula and its approximation extend the same discrete subcopula. As an application of this work, we obtain the rate of convergence of copula approximation. We also give two examples of the rates of convergence of copula approximations: the checkmin and the checkerboard approximations.

As it remains unexplored, the characterization of all copulas satisfying Sklar's Theorem in the case of nondiscrete marginal distribution functions will be our future investigation. It should be possible to extend this work to nondiscrete case. The remaining difficulty to define the copula is defining values for $\overrightarrow{x} = (x_1, x_2, \ldots, x_n)$ where $x_i \in Ran\,(F_i)$ for some i but not for all i. We presume that we can define the function in analogous form of our main result but it still have not been proved.

In our main theorem, the condition needed of the distribution functions F_i is that $F_i\,(0) = 0$, $F_i\,(1) = 1$, and they satisfy

$$x = \sum \epsilon_i F_i\,(x)$$

for all $x \in [0, 1]$ and $0 < \epsilon_i \leq 1$ such that $\sum \epsilon_i = 1$ (see Equation (3.4)). As the readers may realize, it is still unclear to figure out the expositions of these distribution functions F_i. Although in Chapter 5 we show that these distribution functions F_i are $\frac{1}{\epsilon_i}$-Lipschitz functions (see Theorem 4.1) but the expositions of these distribution functions F_i are still mysterious. However, the easiest way to choose a collection of these distribution functions F_i is letting all of them be the identity function on $[0, 1]$ which is not hard to verify that they actually satisfy the condition. Thus it is interesting to characterize all functions satisfying the condition. This also will be our next investigation. If it is too hard to characterize all of them, at least, we hope we can give some various selections of these distribution functions F_i.