CHAPTER 2

Preliminaries

In this chapter, we will introduce the basic concepts, notations and mathematical definitions of uncertain linear systems, linear parameter dependent systems, linear systems with nonlinear perturbation. Finally, we give some general concepts of stability, important definitions, lemma, propositions and results which will be used in later chapters..

2.1 Notations

The following notations will be used throughout this thesis:

$\mathcal{R}^+$ denotes the set of all non-negative real numbers;

$\mathcal{R}^n$ denotes the n-dimensional Euclidean space;

Matrix M is positive definite ($M > 0$) if $x^T Mx > 0$ for all $x \in \mathcal{R}^n$, $x \neq 0$;

Matrix M is semi-positive definite ($M \geq 0$) if $x^T Mx \geq 0$ for all $x \in \mathcal{R}^n$;

Matrix M is negative definite ($M < 0$) if $x^T Mx < 0$ for all $x \in \mathcal{R}^n$, $x \neq 0$;

Matrix M is semi-negative definite ($M \leq 0$) if $x^T Mx \leq 0$ for all $x \in \mathcal{R}^n$;

$M > 0$ ($M \geq 0$) denotes the square symmetric positive (semi-) definite matrix;

$M < 0$ ($M \leq 0$) denotes the square symmetric negative (semi-) definite matrix;

$M > N$ ($M \geq N$) means that the matrix $M - N$ is square symmetric positive (semi-) definite matrix;

$M < N$ ($M \leq N$) means that the matrix $M - N$ is square symmetric negative (semi-) definite matrix;

$M^{n\times m}$ denotes the space of all $(n \times m)$ matrices;

A^T denotes the transpose of the vector/matrix A;

A^{-1} denotes the inverse of a non-singular matrix A;

A is symmetric if $A = A^T$;

I denotes the identity matrix;

$\lambda(A)$ denotes the set of all eigenvalues of A;

$\lambda_{max}(A) = \max \{\text{Re } \lambda : \lambda \in \lambda(A)\}$,

$\lambda_{min}(A) = \min \{\text{Re } \lambda : \lambda \in \lambda(A)\}$;

$\langle x, y\rangle$ or $x^\top y$ represent the scalar product of two vector x, y ;

$\|x\|$ denotes the Euclidean vector norm of x;

$\|\cdot\|_{cl} = \sup_{-h\le\theta\le 0} \{\|x(t+\theta)\|, \|\dot{x}(t+\theta)\|\}$;

$\mathcal{L}_2([0,t],\mathcal{R}^m)$ denotes the set of all $\mathcal{R}^m$-valued square integrable functions on $[0,t]$;

$C([0,t],\mathcal{R}^n)$ denotes the set of all $\mathcal{R}^n$-valued continuous functions on $[0,t]$;

$BM^+(-h,\infty)$ denotes the set of all symmetric semi-positive definite matrix functions bounded on $[0,\infty)$;

$BMU^+(a,\infty)$ denotes the set of all symmetric uniformly positive definite matrix functions bounded on $[a,\infty)$;

$\|x\|_c$ denotes the continuous norm $\max_{a\le\xi\le b} \|\phi(\xi)\|$ for $\phi \in C([0,t],\mathcal{R}^n)$;

$x_t = \{x(t+s) : s \in [-h,0]\}, \|x_t\| = \sup_{s\in[-h,0]} \|x(t+s)\|$.

2.2 Examples of dynamical systems with time-varying delay

2.2.1 Uncertain linear systems with time-varying delay

We consider a linear system with interval time-varying delays of the form

$$\begin{aligned}
&\dot{x}(t) = (A + \Delta A(t))x(t) + (D + \Delta D(t))x(t-h(t)) + (B + \Delta B(t))u(t),\\
&x(t) = \phi(t), t \in [-h_2, 0],
\end{aligned}$$

where $x(t) \in \mathcal{R}^n$ is the state; $u(t) \in \mathcal{R}^m$ is the control; A, D, B are given matrices of appropriate dimensions and $\phi(t) \in C([-h_2,0],\mathcal{R}^n)$ is the initial function with the norm $\| \phi \| = \sup_{s\in[-h_2,0]} \| \phi(s) \|$; the uncertainties satisfy the following condition:

$$\Delta A(t) = E_1F_1(t)H_1, \quad \Delta D(t) = E_2F_2(t)H_2, \quad \Delta B(t) = E_3F_3(t)H_3,$$

where $E_i,\ H_i,\ i = 1,2,3$ are given constant matrices with appropriate dimensions, $F_i(t)$, $i = 1,2,3$ are unknown real matrices with Lebesgue measurable elements satisfying

$$F_i^T(t)F_i(t) \le I, \quad i = 1,2,3, \quad \forall t \ge 0.$$

The time-varying delay functions $h(t)$ satisfies

$$0 \le h_1 \le h(t) \le h_2.$$

2.2.2 Linear uncertain time-varying system with time-varying delay

We propose to study the linear uncertain time-varying systems with time-varying delays system of the form

$$\dot{x}(t) = A(t)x(t) + A_1(t)x(t-h(t)) + B(t)u(t) + B_1(t)w(t),\ t \in \mathcal{R}^+,$$
$$z(t) = C(t)x(t) + C_1(t)x(t-h(t)) + D(t)u(t),\ x_0(t) = \phi(t), t \in [-h, 0]$$

where $x \in \mathcal{R}^n$ is the state; $x_0 = \phi \in \mathcal{C}$ is an initial state; $u \in \mathcal{R}^m$ is the control; $w \in \mathcal{R}^p$ is the uncertain input, $z \in \mathcal{R}^q$ is the observation output; $A(t), A_1(t), C(t), C_1(t), D(t)$ are given continuous and bounded matrix functions on $\mathcal{R}^+_{-h}; 0 \leq h(t) \leq h, \dot{h}(t) \leq \delta < 1$. In the sequel, we say that the control $u(t)$ is admissible if $u(t) \in \mathcal{L}_2([0,t), \mathcal{R}^m)$ for every $t \geq 0$, and the uncertainty $w(\cdot)$ is admissible if $w(\cdot) \in \mathcal{L}_2([0,+\infty), \mathcal{R}^p)$.

2.2.3 Linear systems with time-varying delay and nonlinear perturbations

We consider the following linear system with constant and interval time-varying delays and nonlinear perturbations

$$\dot{x}(t) = (A + \Delta A(t))x(t) + (D + \Delta D(t))x(t-h(t)) + (B + \Delta B(t))u(t) + B_w w(t),$$
$$z(t) = Cx(t) + C_d x(t-h(t)) + D_u u(t) + D_w w(t),$$
$$x(t) = \phi(t), t \in [-h_2, 0],$$

where $x(t) \in \mathcal{R}^n$ is the state; $u(t) \in \mathcal{R}^m$ is the control input, $z(t) \in \mathcal{R}^q$ is the controlled output and $w(t) \in \mathcal{R}^p$ is a disturbance input belonging to $\mathcal{L}_2[0, \infty)$; $A, D, B, B_w, C, D_u, D_w$ are given matrices of appropriate dimensions and $\phi(t) \in C([-h_2, 0], \mathcal{R}^n)$ is the initial function with the norm $\| \phi \| = \sup_{s \in [-h_2, 0]} \| \phi(s) \|$; the uncertainties satisfy the following condition:

$$\Delta A(t) = E_1 F_1(t) H_1,\ \ \Delta D(t) = E_2 F_2(t) H_2,\ \ \Delta B(t) = E_3 F_3(t) H_3, \tag{2.1}$$

where $E_i,\ H_i,\ i = 1,2,3$ are given constant matrices with appropriate dimensions, $F_i(t)$, $i = 1,2,3$ are unknown real matrices with Lebesgue measurable elements satisfying

$$F_i^T(t) F_i(t) \leq I, \quad i = 1,2,3, \quad \forall t \geq 0.$$

The time-varying delay function $h(t)$ satisfies

$$0 \leq h_1 \leq h(t) \leq h_2 \tag{2.2}$$

The objective of this study is to design a memoryless H_∞ state feedback controller

$$u(t) \quad = \quad Kx(t) \tag{2.3}$$

such that, for all admissible uncertainties satisfying (2.3) and any time-varying delay $h(t)$ satisfying (2.4)

2.2.4 Nonlinear systems with time-varying delays

Consider the following system with time-varying delays and control input:

$$\begin{aligned}\dot{x}(t) =& Ax(t) + Dx(t-\tau(t)) + Bu(t) + Cw(t)\\ &+ f(t, x(t), x(t-\tau(t)), u(t), w(t)),\\ z(t) =& Ex(t) + Gx(t-\tau(t)) + Fu(t)\\ &+ g(t, x(t), x(t-\tau(t)), u(t)),\end{aligned}$$

$$x(t_0+\theta) = \phi(\theta), \theta \in [-\tau, 0], (t_0, \phi) \in \mathcal{R}^+ \times \mathcal{C}([-\tau, 0], \mathcal{R}^n),$$

where $x(t) \in \mathcal{R}^n$ is the state; $u(t) \in \mathcal{R}^m$ is the control input, $w(t) \in \mathcal{L}_2([0, \infty], \mathcal{R}^r)$ is a disturbance input and $z(t) \in \mathcal{R}^s$ is the observation output.

The delay, $\tau(t)$ is time-varying continuous function that satisfies

$$0 \le \tau_1 \le \tau(t) \le \tau_2,\ \mu_1 \le \dot{\tau}(t) \le \mu_2.$$

Let $x^\tau = x(t-\tau(t))$, the nonlinear functions $f(t, x, x^\tau, u, w) : \mathcal{R}^+ \times \mathcal{R}^n \times \mathcal{R}^n \times \mathcal{R}^m \times \mathcal{R}^r \to \mathcal{R}^n$, $g(t, x, x^\tau, u) : \mathcal{R}^+ \times \mathcal{R}^n \times \mathcal{R}^n \times \mathcal{R}^m \to \mathcal{R}^s$ satisfy the following growth condition:

$$\begin{aligned}\exists a, b, c, d > 0 :& \|f(t, x, x^\tau, u, w)\|\\ &\le a\|x\| + b\|x^\tau\| + c\|u\| + d\|w\|, \forall (x, x^\tau, u, w)\\ \exists a_1, b_1, c_1 > 0 :& \|g(t, x, x^\tau, u)\|^2\\ &\le a_1\|x\|^2 + b_1\|x^\tau\|^2 + c_1\|u\|^2, \forall (x, x^\tau, u).\end{aligned}$$

2.3 Preliminary results

Consider a dynamical system described by

$$\dot{x}(t) = f(t, x(t)) \tag{2.4}$$

where $x \in \mathcal{R}^n$ and f is a vector having components $f_i(t, x_1, ..., x_n)$, $i = 1, 2, ..., n$. We shall assume that the f_i are continuous and satisfy standard conditions, such as having continuous first partial derivatives so that the solution of (2.4) exists and is unique for the given initial conditions. If f_i do not depend explicitly on t, (2.4) is called autonomous (otherwise, nonautonomous). If $f(t, c) = 0$ for all t, where c is some constant vector, then it follows at once from (2.4) that if $x(t_0) = c$ then $x(t) = c$ for all $t \geq t_0$. Thus solutions starting at c remain there, and c is said to be an *equilibrium* or *critical point*. Clearly, by introducing new variables $\acute{x_i} = x_i - c_i$ we can arrange for the equilibrium point to be transferred to the origin; we shall assume that this has been done for any equilibrium point under consideration (there may well be several for a given system (2.4) so that we then have $f(t, 0) = 0$, $t \geq t_0$.

2.3.1 Autonomous systems

Consider the autonomous system

$$\dot{x} = f(x) \tag{2.5}$$

where $f : D \to \mathcal{R}^n$ is locally Lipschitz map from a domain $D \subset \mathcal{R}^n$ into $\mathcal{R}^n$. We shall always assume that $f(x)$ satisfies $f(0) = 0$, and study stability of the origin $x = 0$.

Definition 2.3.1. [21] The equilibrium point $x = 0$ of (2.5) is

(i) **stable** if for each $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$ such that

$$\|x(0)\| < \delta \Rightarrow \|x\| < \epsilon, \quad \forall t \geq 0,$$

(ii) **unstable** if it is not stable, that is, there exists $\varepsilon > 0$ such that for every $\delta > 0$ there exist an $x(0)$ with $\|x(0)\| < \delta$ so that $\|x(t_1)\| \geq \varepsilon$ for some $t_1 > 0$. If this holds for every $x(0)$ in $\|x(0)\| < \delta$ the equilibrium is completely unstable.

(iii) **asymptotically stable** if it is stable and δ can be chosen such that

$$\|x(0)\| < \delta \Rightarrow \lim_{t \to \infty} x(t) = 0.$$

Definition 2.3.2. [21] A function $V(\cdot) : \mathcal{R}^n \to \mathcal{R}$ is said to be Lyapunov function if it satisfies the following:

1. $V(x)$ and all its partial derivatives $\dfrac{\partial V}{\partial x_i}$ are continuous.

2. $V(x)$ is positive definite, i.e. $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$ in some neighbourhood $\| x \| \leq k$ of the origin.

3. The derivative of V with respect to (2.5), namely

$$\begin{aligned}\dot{V} &= \frac{\partial V}{\partial x_1}\dot{x_1} + \frac{\partial V}{\partial x_2}\dot{x_2} + ... + \frac{\partial V}{\partial x_n}\dot{x_n} \\ &= \frac{\partial V}{\partial x_1}f_1 + \frac{\partial V}{\partial x_2}f_2 + ... + \frac{\partial V}{\partial x_n}f_n \end{aligned} \quad (2.6)$$

is negative semidefinite i.e. $\dot{V}(0) = 0$, and for all x satisfy $\|x\| \leq k$, $V\dot{(x)} \leq 0$.

Theorem 2.3.1. *[21] Let $x = 0$ be an equilibrium point for (2.5) and $D \subset \mathcal{R}^n$ be a domain containing $x = 0$. Let $V(x) : D \rightarrow \mathcal{R}$ be a continuously differentiable function, such that*

$$V(0) = 0 \quad \text{and} \quad V(x) > 0 \quad \text{in} \quad D - \{0\},$$
$$\dot{V}(x) \leq 0 \quad \text{in} \quad D.$$

Then, $x = 0$ is stable. Moreover, if

$$\dot{V}(x) < 0 \quad \text{in} \quad D - \{0\},$$

then $x = 0$ is asymptotically stable.

Theorem 2.3.2. *[21] Let $x = 0$ be an equilibrium point for (2.5). Let $V(x) : \mathcal{R}^n \rightarrow \mathcal{R}$ be a continuously differentiable function, such that*

$$V(0) = 0 \quad \text{and} \quad V(x) > 0, \quad \forall x \neq 0,$$
$$\|x\| \rightarrow \infty \Rightarrow V(x) \rightarrow \infty,$$
$$\dot{V}(x) < 0, \quad \forall x \neq 0,$$

then $x = 0$ is globally asymptotically stable.

Theorem 2.3.3. *[21] Let $x = 0$ be an equilibrium point for (2.5) and $f : D \rightarrow \mathcal{R}^n$ is continuously differentiable and D is a neighborhood of the origin. Let*

$$A = \frac{\partial f}{\partial x}(x)\mid_{x=0}.$$

Then,

1. *The origin is asymptotically stable if $Re(\lambda_i) < 0$ for all eigenvalues of A.*
2. *The origin is unstable if $Re(\lambda_i) > 0$ for one or more of the eigenvalues of A.*

2.3.2 Nonautonomous systems

Consider the nonautonomous system

$$\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0, \quad x \in \mathcal{R}^n, \quad t \in \mathcal{R}^+, \tag{2.7}$$

where $f : \mathcal{R}^+ \times D \to \mathcal{R}^n$ is piecewise continuous in t and locally Lipschitz in x on $\mathcal{R}^+ \times \mathcal{R}^n$ and $D \subset \mathcal{R}^n$ is domain that contains the origin $x = 0$. The origin is an equilibrium point for (2.7) if

$$f(t, 0) = 0, \quad \forall t \geq t_0.$$

Definition 2.3.3. [21] The equilibrium point $x = 0$ of the system (2.7) is

(i) **stable** if, for each $\epsilon > 0$, there is $\delta = \delta(\epsilon, t_0) > 0$ such that

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \quad \forall t \geq t_0 \geq 0,$$

(ii) **uniformly stable** if, for each $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$, independent of t_0, such that (2.14) is satisfied,

(iii) **unstable** if not stable,

(iv) **asymptotically stable** if it is stable and there is $c = c(t_0) > 0$ such that $x(t) \to 0$ as $t \to \infty$, for all $\|x(t_0)\| < c$,

(v) **uniformly asymptotically stable** if it is uniformly stable and there is $c > 0$, independent of t_0, such that for all $\|x(t_0)\| < c$, $x(t) \to 0$ as $t \to \infty$, uniformly in t_0, for each $\epsilon > 0$, there is $T = T(\epsilon) > 0$ such that

$$\|x(t)\| < \epsilon, \quad \forall t \geq t_0 + T(\epsilon), \quad \forall \|x(t_0)| < c,$$

(vi) **globally uniformly asymptotically stable** if it is uniformly stable and, for each pair of positive numbers ϵ and c, there is $T = T(\epsilon, c) > 0$ such that

$$\|x(t)\| < \epsilon, \quad \forall t \geq t_0 + T(\epsilon, c), \quad \forall \|x(t_0)| < c.$$

Definition 2.3.4. [31] The equilibrium point $x = 0$ of the system (2.7) is *exponentially stable* if there exist three positive real constants ϵ, K and λ such that

$$\|x(t)\| \leq K\|x_0\|e^{-\lambda(t-t_0)}, \quad \forall \|x_0\| < \epsilon, \quad t \geq t_0;$$

The largest constant λ which may be utilized in above inequality is called the rate of convergence.

Definition 2.3.5. [21] A continuous function $\alpha : [0, a) \to [0, \infty]$ is said to belong to class K if it is strictly increasing and $\alpha(0) = 0$. It is said to belong class K_∞ if $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$.

Example 2.3.4. [21] We give some example for class K and class K_∞ :

1. $\alpha(r) = \tan^{-1} r$ is strictly increasing since $\alpha'(r) = \frac{1}{1+r^2} > 0$. It belong to class K, but not to class K_∞ since $\lim\limits_{r\to\infty} \alpha(r) = \dfrac{\pi}{2} < \infty$.
2. $\alpha(r) = r^c$, for any positive real number c, is strictly increasing since $\alpha'(r) = cr^{c-1} > 0$. Moreover, $\lim\limits_{r\to\infty} \alpha(r) = \infty$, thus, it belong to class K_∞.

Definition 2.3.6. [21] A continuous function $\beta : [0, a) \times [0, \infty] \to [0, \infty]$ is said to belong to class KL if, for each fixed s, the mapping $\beta(r, s)$ belong to class K with respect to r and, for each fixed r, the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$.

Example 2.3.5. [21] We give some example for class KL :

1. $\beta(r, s) = \frac{r}{ksr+1}$, for any positive real number k, is strictly increasing in r since
$$\frac{\partial \beta}{\partial r} = \frac{1}{(ks+1)^2} > 0$$
and strictly decreasing in s since
$$\frac{\partial \beta}{\partial s} = \frac{-kr^2}{(ks+1)^2} < 0.$$
Moreover, $\beta(r, s) \to 0$ as $s \to \infty$. Hence, it belong to class KL.

Lemma 2.3.6. *[21] The equilibrium point $x = 0$ of (2.7) is*

(1) ***uniformly stable*** *if and only if there exist a class K function $\alpha(.)$ and a positive constant c, independent of t_0, such that*
$$\|x(t)\| \le \alpha(\|x(t_0)\|), \quad \forall t \ge t_0 \ge 0, \quad \forall \|x(t_0)\| < c, \tag{2.8}$$

(2) ***uniformly asymptotically stable*** *if and only if there exist a class KL function $\beta(.,.)$ and a positive constant c, independent of t_0, such that*
$$\|x(t)\| \le \beta(\|x(t_0)\|, t - t_0), \quad \forall t \ge t_0 \ge 0, \quad \forall \|x(t_0)\| < c, \tag{2.9}$$

(3) ***globally uniformly asymptotically stable*** *if and only if inequality (2.9) is satisfied for any initial state $x(t_0)$.*

Definition 2.3.7. [21] The equilibrium point $x = 0$ of (2.7) is exponentially stable if inequality (2.9) is satisfied with

$$\beta(r, s) = kre^{-\gamma s}, \quad k > 0, \quad \gamma > 0$$

and is globally exponentially stable if this condition is satisfied for any initial state.

Definition 2.3.8. [27] The function $W(x)$ is said to be positive (negative) definite if $W(x) > 0(-W(x) > 0)$ and $W(x) = 0$ if and only if $x = 0$. The function $W(x)$ is said to be positive (negative) semi-definite if $W(x) \geq 0(-W(x) \geq 0)$.

Definition 2.3.9. [27] The function $W(x)$ is said to be radially unbounded, positive definite if $W(x)$ is positive definite and $W(x) \to \infty$ as $\|x\| \to \infty$.

Let B_ϵ be a ball of size ϵ around the origin,

$$B_\epsilon = \{x \in \mathcal{R}^n : \|x\| < \epsilon\}.$$

Definition 2.3.10. [29] A function $V(\cdot) : \mathcal{R}^+ \times \mathcal{R}^n \to \mathcal{R}$ is said to be *Lyapunov function* if it satisfies the following:

(i) $V(t, x)$ and all its partial derivatives $\dfrac{\partial V}{\partial t}, \dfrac{\partial V}{\partial x_i}$ are continuous for all $i = 1, 2, 3, ..., n$.

(ii) $V(t, x)$ is positive definite function, i.e., $V(0) = 0$ and $V(t, x) > 0$, $x \neq 0$, $\forall x \in B_\epsilon$.

(iii) The derivative of $V(t, x)$ with respect to system (2.7), namely

$$\begin{aligned} \dot{V}(t, x) &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_1}\dot{x_1} + \frac{\partial V}{\partial x_2}\dot{x_2} + ... + \frac{\partial V}{\partial x_n}\dot{x_n} \\ &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_1}f_1 + \frac{\partial V}{\partial x_2}f_2 + ... + \frac{\partial V}{\partial x_n}f_n. \end{aligned} \tag{2.10}$$

$\dot{V}(t, x)$ is negative semi-definite i.e., $\dot{V}(t, 0) = 0$ and $\forall x \in B_\epsilon$, $\dot{V}(t, x) \leq 0$.

Theorem 2.3.7. *[21] Let $x = 0$ be an equilibrium point for (2.7) and $D \subset \mathcal{R}^n$ be a domain containing $x = 0$. Let $V : \mathcal{R}^+ \times D \to \mathcal{R}$ be a continuously differentiable function, such that*

$$W_1(x) \leq V(t, x) \leq W_2(x), \tag{2.11}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}f(t, x) \leq -W_3(x) \tag{2.12}$$

$\forall t \geq t_0 \geq 0$, $\forall x \in D$ where $W_1(x)$, $W_2(x)$ and $W_3(x)$ are continuous positive definite functions on D. Then, $x = 0$ is uniformly asymptotically stable.

Corollary 2.3.8. *[21] Suppose that all the assumptions of Theorem 2.3.7 are satisfied globally (for all $x \in \mathcal{R}^n$) and $W_1(x)$ is radially unbounded. Then, $x = 0$ is globally uniformly asymptotically stable.*

Corollary 2.3.9. *[21] Suppose all the assumptions of Theorem 2.3.7 are satisfied with*

$$W_1(x) \geq k_1\|x\|^c, \quad W_2(x) \leq k_2\|x\|^c, \quad W_3(x) \geq k_3\|x\|^c$$

for some positive constants k_1, k_2, k_3 and c. Then, $x = 0$ is exponentially stable. Moreover, if the assumptions hold globally, then, $x = 0$ is globally exponentially stable.

Theorem 2.3.10. *[21] Let $x = 0$ be an equilibrium point for the nonlinear system*

$$\dot{x}(t) = f(t, x)$$

where $f : [0, \infty) \times D \to \mathcal{R}^n$ is continuously differentiable, $D = \{x \in \mathcal{R}^n | \|x\|_2 < r\}$, and Jacobian matrix $[\frac{\partial f}{\partial x}]$ is bounded and Lipschitz on D, uniformly in t. Let

$$A(t) = \frac{\partial f}{\partial x}(t, x)|_{x=0}.$$

Then, the origin is an exponentially stable equilibrium point for nonlinear system if and only if it is an exponentially stable equilibrium point for linear system

$$\dot{x}(t) = A(t).$$

Theorem 2.3.11. *[21] Let $x = 0$ be an equilibrium point for the nonlinear system*

$$\dot{x}(t) = f(t, x)$$

where $f : [0, \infty) \times D \to \mathcal{R}^n$ is continuously differentiable, $D = \{x \in \mathcal{R}^n | \|x\| < r\}$, and Jacobian matrix $[\frac{\partial f}{\partial x}]$ is bounded and Lipschitz on D, uniformly in t. Let $\beta(.,.)$ be a class KL function and r_0 be a positive constant such that $\beta(r_0, 0) < r$. Let $D_0 = \{x \in \mathcal{R}^n | \|x\| < r_0\}$. Assume that the trajectory of the system satisfied

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \quad \forall x(t_0) \in D_0, \quad \forall t \geq t_0 \geq 0.$$

Then, there is a continuously differentiable function $V : [0, \infty) \times D_0 \to \mathcal{R}$ that satisfies the inequalities

$$\begin{aligned} \alpha_1(\|x\|) \leq V(t, x) &\leq \alpha_2(\|x\|), \\ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) &\leq -\alpha_3(\|x\|), \\ \|\frac{\partial V}{\partial x}\| &\leq \alpha_4(\|x\|), \end{aligned}$$

where $\alpha_1(.)$, $\alpha_2(.)$, $\alpha_3(.)$ and $\alpha_4(.)$ are class K function defined on $[0, r_0]$. If the system is autonomous, V can be chosen independent of t.

2.3.3 Nonautonomous systems with time delay

We consider the nonautonomous system with time-delay of the form [10]

$$\begin{aligned} \dot{x}(t) &= f(t, x(t-h)), \qquad \forall t \geq 0, \\ x(t_0+\theta) &= \phi(\theta), \qquad \forall \theta \in [-h, 0], \end{aligned} \tag{2.13}$$

where $x(t) \in \mathcal{R}^n$ is the state variable, $h \in \mathcal{R}^+$ is the delay and $f : \mathcal{R}^+ \times C(C([-h,0], \mathcal{R}^n)) \to \mathcal{R}^n$. $\phi(t)$ is a continuous vector-valued initial condition. We assume $f(t,0) = 0$ so that system (2.13) admits the trivial solution. We also assume that system (2.13) has an existence and uniqueness solution.

Definition 2.3.11. For the system described by (2.13), the trivial solution $x(t) = 0$ is said to be

(i) **stable** if for any $t_0 \in \mathcal{R}$ and any $\epsilon > 0$, there exists a $\delta = \delta(t_0, \epsilon) > 0$ such that

$$\|x_{t_0}\|_c < \delta \Rightarrow \|x(t)\| < \epsilon, \quad \forall t \geq t_0,$$

(ii) **uniformly stable** if it is stable and $\delta(t_0, \epsilon)$ can be chosen independently of t_0,

(iii) **asymptotically stable** if it is stable, and for any $t_0 \in \mathcal{R}$ and any $\epsilon > 0$ there exists a $\delta_a = \delta_a(t_0, \epsilon) > 0$ such that

$$\|x_{t_0}\|_c < \delta_a \Rightarrow \lim_{t \to \infty} = 0,$$

(iv) **uniformly asymptotically stable** if it is uniformly stable and there exists a $\delta_a > 0$ such that for any $\eta > 0$, there exists a $T = T(\delta_a, \eta)$, such that

$$\|x_{t_0}\|_c < \delta_a \Rightarrow \|x(t)\| < \eta, \quad \forall t \geq t_0 + T.$$

Suppose that $u, v, w : \mathcal{R}^+ \to \mathcal{R}^+$ are continuous nondecreasing functions, where additionally $u(s)$ and $v(s)$ are positive for $s > 0$, and $u(0) = v(0) = 0$. If there exist a continuous differentiable functional $V : \mathcal{R}^+ \times C \to \mathcal{R}$ such that

$$u(\|\phi(0)\|) \leq V(t, x(t)) \leq v(\|\phi\|),$$

the equilibrium point $x^* = 0$ of system (2.13) is

(i) uniformly stable if

$$\dot{V}(t, x(t)) \leq -w(\|\phi(0)\|),$$

(ii) uniformly asymptotically stable if

$$\dot{V}(t, x(t)) \leq -w(\|\phi(0)\|),$$

where $w(s) > 0$ for $s > 0$,

(iii) globally uniformly asymptotically stable if

$$\dot{V}(t, x(t)) \leq -w(\|\phi(0)\|),$$

and $u(s)$ is radially unbounded.

Definition 2.3.12. [22] A functional $V : \mathcal{R}^+ \times C \to \mathcal{R}^+$ is called a Lyapunov-Krasovskii functional for the system (2.13) if it has the following properties. There exist $\lambda_1, \lambda_2, \lambda_3 > 0$ such that

(i) $\lambda_1 \|x(t)\|^2 \leq V(t, x_t) \leq \lambda_2 \|x_t\|^2$,

(ii) $\dot{V}(t, x_t) \leq -\lambda_3 \|x(t)\|^2$.

Definition 2.3.13. [5] System $[A(t), B(t)]$ is globally null controllable (GNC) in a finite time $T > 0$ if for every initial state x_0, there is an admissible control $u(t)$ such that

$$U(T, 0)x_0 + \int_0^T U(T, \tau)B(\tau)u(\tau)d\tau = 0.$$

Lemma 2.3.12. *[10] Consider the non autonomous time-delay system* (2.13). *If there exist a Lyapunov function $V(t, x_t)$ and $\lambda_1, \lambda_2 > 0$ such that for every solution $x(t)$ of the system, the following conditions hold,*

(i) $\lambda_1 \|x(t)\|^2 \leq V(t, x_t) \leq \lambda_2 \|x_t\|^2$,

(ii) $\dot{V}(t, x_t) \leq 0$,

then the solution of the system is bounded, i.e., there exists $N > 0$ such that $\|x(t, \phi)\| \leq N\|\phi\|, \forall t \geq 0$.

Lemma 2.3.13. *[22] Consider the autonomous time-delay system* (2.13). *If there exist a Lyapunov-Krasovskii function $V(x_t)$ and $\lambda_1, \lambda_2, \lambda_3 > 0$ such that for every solution $x(t)$ of the system, the following conditions hold,*

(i) $\lambda_1 \|x(t)\|^2 \leq V(x_t) \leq \lambda_2 \|x_t\|^2$,

(ii) $\dot{V}(x_t) \leq -\lambda_3 \|x(t)\|^2$,

then the solution of the system (2.13) *is exponentially stable.*

Proposition 2.3.14. *Assume that the matrix function* $A(t), B(t)$ *are analytic on* $\mathcal{R}^+$. *The system* $[A(t), B(t)]$ *is GNC in some finite time if there exists* $t_0 > 0$ *such that*

$$rank[M_1(t_0), M_2(t_0), \ldots, M_n(t_0)] = n, \tag{2.14}$$

where

$$M_1(t) = B(t), M_k(t) = -A(t) + \frac{d}{dt}M_{k-1}(t), \quad k = 2, \ldots, n-1.$$

Consider the following linear time-varying control system, briefly denoted by $[A(t), B(t)]$

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad t \in \mathcal{R}^+. \tag{2.15}$$

Associated with the control system 2.15, we consider the following RDE:

$$\dot{P}(t) + A^T(t)P(t) + P(t)A(t) - P(t)B(t)B^T(t)P(t) + Q(t) = 0. \tag{2.16}$$

Proposition 2.3.15. *[5] If system* $[A(t), B(t)]$ *is globally null controllable in some finite time, then for any matrix* $Q \in BM^+(0, \infty)$, *the RDE (2.16) has a solution* $P \in BM^+(0, \infty)$

Proposition 2.3.16. *[5] For any matrix function* $A(t)$ *bounded on* $\mathcal{R}^+$, *there exists* $Q \in BM^+(0, \infty)$ *such that* $Q(t) - A(t) \geq 0$

Proposition 2.3.17. *[5] The matrix* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, *where* $a > 0$ *or* $c > 0$, *is positive definite if* $b^2 < ac$

Proposition 2.3.18. *(Cauchy inequality) For any symmetric positive definite matrix* $N \in M^{n\times n}$ *and* $x, y \in \mathcal{R}^n$ *we have*

$$\pm 2x^T y \leq x^T N x + y^T N^{-1} y.$$

Proposition 2.3.19. *[10] For any symmetric positive definite matrix* $M > 0$, *scalar* $\gamma > 0$ *and vector function* $\omega : [0, \gamma] \to \mathcal{R}^n$ *such that the integrations concerned are well defined, the following inequality holds*

$$\Big(\int_0^\gamma \omega(s)\, ds\Big)^T M \Big(\int_0^\gamma \omega(s)\, ds\Big) \leq \gamma \Big(\int_0^\gamma \omega^T(s) M \omega(s)\, ds\Big).$$

Lemma 2.3.20. *[10] (Schur complement lemma). Given constant symmetric matrices X, Y, Z with appropriate dimensions satisfying $X = X^T, Y = Y^T > 0$. Then $X + Z^T Y^{-1} Z < 0$ if and only if*

$$\begin{pmatrix} X & Z^T \\ Z & -Y \end{pmatrix} < 0 \quad or \quad \begin{pmatrix} -Y & Z \\ Z^T & X \end{pmatrix} < 0.$$

Lemma 2.3.21. *[59] Given matrices $Q = Q^T$, H, E and $R = R^T > 0$ with appropriate dimensions. Then*

$$Q + HFE + E^T F^T H^T < 0,$$

for all F satisfying $F^T F \leq R$, if and only if there exists an $\epsilon > 0$ such that

$$Q + \epsilon H H^T + \epsilon^{-1} E^T R E < 0.$$

Lemma 2.3.22. *[62] For any constant symmetric matrix $M \in \mathcal{R}^{n\times n}$, $M = M^T > 0$, $0 \leq h_m \leq h(t) \leq h_M$, $t \geq 0$, and any differentiable vector function $x(t) \in \mathcal{R}^n$, we have*

$$\begin{aligned}
(a)\ \Big[\int_{t-h_m}^{t} \dot{x}(s)ds\Big]^T M \Big[\int_{t-h_m}^{t} \dot{x}(s)ds\Big] &\leq h_m \int_{t-h_m}^{t} \dot{x}^T(s) M \dot{x}(s)ds, \\
(b)\Big[\int_{t-h(t)}^{t-h_m} \dot{x}(s)ds\Big]^T M \Big[\int_{t-h(t)}^{t-h_m} \dot{x}(s)ds\Big] &\leq (h(t) - h_m) \int_{t-h(t)}^{t-h_m} \dot{x}^T(s) M \dot{x}(s)ds \\
&\leq (h_M - h_m) \int_{t-h(t)}^{t-h_m} \dot{x}^T(s) M \dot{x}(s)ds.
\end{aligned}$$

Lemma 2.3.23. *[Completing the Square, [34]] Let Q, S be matrices of appropriate dimensions, and $S > 0$ be symmetric. Then*

$$2\langle Qy, x\rangle - \langle Sy, y\rangle \leq \langle QS^{-1}Q^T x, x\rangle, \qquad \forall (x, y).$$

Lemma 2.3.24. *[46] For a given matrix $R > 0$, the following inequality holds for any continuously differentiable function $w : [a, b] \to \mathcal{R}^n$*

$$\int_a^b \dot{\omega}^T(u) R \dot{\omega} du \geq \tfrac{1}{b-a}(\Gamma_1^T R \Gamma_1 + 3\Gamma_2^T R \Gamma_2) \tag{2.17}$$

where

$$\begin{aligned}
\Gamma_1 &:= \omega(b) - \omega(a) \\
\Gamma_2 &:= \omega(b) + \omega(a) - \tfrac{2}{b-a} \int_a^b \omega(u)du.
\end{aligned}$$

Before the novel integral inequalities are established, we denote

$$\begin{aligned}
\nu_1(t) &:= \tfrac{1}{\tau_2-\tau(t)} \int_{t-\tau_2}^{t-\tau(t)} y(s)ds \\
\nu_2(t) &:= \tfrac{1}{\tau(t)-\tau_1} \int_{t-\tau(t)}^{t-\tau_1} y(s)ds \\
\nu_3(t) &:= \tfrac{1}{\tau_1} \int_{t-\tau_1}^{t} y(s)ds
\end{aligned} \tag{2.18}$$

Lemma 2.3.25. *[53] For a given scalar $\tau_1 \geq 0$ and any $n \times n$ real matrices $Y_1 > 0$ and $Y_2 > 0$ and a vector $\dot{y} : [-\tau_1, 0] \to \mathcal{R}^n$ such that the integration concerned below is well defined, the following inequality holds for any vector-valued function $\pi_1(t) : [0, \infty) \to \mathcal{R}^k$ and matrices $M_1 \in \mathcal{R}^{k\times k}$ and $N_1 \in \mathcal{R}^{k\times n}$ satisfying $\begin{bmatrix} M_1 & N_1 \\ N_1^T & Y_1 \end{bmatrix} \geq 0$,*

$$\begin{aligned}
\wp_1 &:= \int_{t-\tau_1}^{t} (\tau_1 - t + s)\dot{y}^T(s)Y_1\dot{y}(s)ds \\
&\geq -\tfrac{\tau_1^2}{2}\pi_1^T(t)M_1\pi_1(t) - 2\tau_1\pi_1^T N_1[y(t) - \nu_3(t)] \\
\wp_2 &:= \int_{t-\tau_1}^{t} (\tau_1 - t + s)^2\dot{y}^T(s)Y_2\dot{y}(s)ds \\
&\geq \tau[y(t) - \nu_3(t)]^T Y_2[y(t) - \nu_3(t)]
\end{aligned}$$

where $\nu_3(t)$ being defined in (2.18)

Lemma 2.3.26. *[54] Let $\tau(t)$ be a continuous function satisfying $0 \leq \tau_1 \leq \tau(t) \leq \tau_2$. For any $n \times n$ real matrix $R_2 > 0$ and a vector $\dot{y} : [-\tau_2, 0] \to \mathcal{R}^n$ such that the integration concerned below is well defined, the following inequality holds for any $\phi_{i1} \in \mathcal{R}^q$ and real matrices $Z_i \in \mathcal{R}^{q\times q}$, $F_i \in \mathcal{R}^{q\times n}$ satisfying $\begin{bmatrix} Z_i & F_i \\ F_i^T & R_2 \end{bmatrix} \geq 0, (i = 1, 2)$,*

$$\begin{aligned}
&-\int_{t-\tau_2}^{t-\tau_1} (\tau_2 - t + s)\dot{y}^T(s)R_2\dot{y}(s)ds \\
\leq &\frac{1}{2}(\tau_2 - \tau(t))^2\phi_{11}^T Z_1\phi_{11} + 2(\tau_2 - \tau(t))\phi_{11}^T F_1\phi_{12} \\
&+ \frac{1}{2}[(\tau_2 - \tau_1)^2 - (\tau_2 - \tau(t))^2]\phi_{21}^T Z_2\phi_{21} \\
&+ 2\phi_{21}^T F_2[(\tau_2 - \tau(t))\phi_{22} + (\tau(t) - \tau_1)\phi_{23}]
\end{aligned}$$

where

$$\phi_{12} := y(t-\tau(t)) - \nu_1(t)$$
$$\phi_{22} := y(t-\tau_1) - x(t-\tau(t))$$
$$\phi_{23} := y(t-\tau_1) - \nu_2(t)$$

with $\nu_i(t)(i=2,3)$ being defined in (2.18).

Lemma 2.3.27. *[53] Let $\tau(t)$ be a continuous function satisfying $0 \le \tau_1 \le \tau(t) \le \tau_2$. For any $n \times n$ real matrix $R_1 > 0$ and a vector $\dot{y} : [-\tau_2, 0] \to \mathcal{R}^n$ such that the integration concerned below is well defined, the following inequality holds for any $2n \times 2n$ real matrix S_1 satisfying $\begin{bmatrix} \tilde{R}_1 & S_1 \\ S_1^T & \tilde{R}_1 \end{bmatrix} \ge 0$,*

$$-(\tau_2-\tau_1)\int_{t-\tau_2}^{t-\tau_1} \dot{y}^T(s)R_1\dot{y}(s)ds$$
$$\le 2\psi_{11}^T S_1 \psi_{21} - \psi_{11}^T \tilde{R}_1 \psi_{11} - \psi_{21}^T \tilde{R}_1 \psi_{21}$$

where $\tilde{R_1} := \operatorname{diag}\{R_1, 3R_1\}$; and

$$\psi_{11} := \begin{bmatrix} y(t-\tau(t)) - y(t-\tau_2) \\ y(t-\tau(t)) + y(t-\tau_2) - 2\nu_1(t) \end{bmatrix}$$
$$\psi_{21} := \begin{bmatrix} y(t-\tau_1) - y(t-\tau(t)) \\ y(t-\tau_1) + y(t-\tau(t)) - 2\nu_2(t) \end{bmatrix}$$

Remark 2.3.28. *:In Lemma(2.3.27), when $\tau_1 = 0$, the inequality reduces to similar one in [46]. It contains slack matrix variable S_1 with dimension $2n \times 2n$ comparing to slack matric variable with dimension $2n \times 5n$ introduced in [24].*

Lemma 2.3.29. *[53] Let χ_0, χ_1 and χ_2 be $m \times m$ real symmetric matrices and a continuous function τ satisfy $\tau_1 \le \tau \le \tau_2$, where τ_1 and τ_2 are constants satisfying $0 \le \tau_1 \le \tau_2$. If $\chi_0 \ge 0$, then*

$$\tau^2\chi_0 + \tau\chi_1 + \chi_2 < 0(\le 0), \forall \tau \in [\tau_1, \tau_2]$$
$$\Leftrightarrow \tau_i^2\chi_0 + \tau_i\chi_1 + \chi_2 < 0(\le 0), (i=1,2)$$

or

$$\tau^2\chi_0 + \tau\chi_1 + \chi_2 > 0(\ge 0), \forall \tau \in [\tau_1, \tau_2]$$
$$\Leftrightarrow \tau_i^2\chi_0 + \tau_i\chi_1 + \chi_2 > 0(\ge 0), (i=1,2)$$

2.3.4 H_∞ control problem

Consider a linear system observation control system described by

$$\dot{x}(t) = Ax(t) + Bu(t) + B_1 w(t)$$
$$z(t) = Cx(t).$$

where $w(t) \in W$ is uncertain/perturbation variable and $z(t)$ is observation output. H_∞ control problem for the system has s solution if for given $\gamma > 0$, find feedback control $u(t)$ such that

i) the system is stabilizability for all perturbation $w(t) \in W$

ii) $\sup_{w(t)\neq 0} \parallel T \parallel_{z,w} \leq \gamma$,

where$\|T\|_z, w$ in $\mathcal{L}_2$-reduced normed of transformation $w \to z$:

$$\|T\|_{z,w} = \frac{\|z\|_{\mathcal{L}_2}}{\|w\|_{\mathcal{L}_2}}, \|z\|_{\mathcal{L}_2} = \int_0^\infty \|z(t)\|^2 dt < +\infty$$

Definition 2.3.14. [23] A system is said to be exponentially stable with an H_∞ norm bound γ if the following conditions hold:

(1) For the system with $w(t) = 0$, the trivial solution (equilibrium point) is globally exponentially stable if $\lim_{t\to\infty} x(t) = 0$; and

(2) Under the assumption of zero initial condition, the controlled output $z(t)$ satisfies

$$\parallel z(t) \parallel_2 \leq \gamma \parallel w(t) \parallel_2$$

for any nonzero $w(t) \in \mathcal{L}_2[0, \infty)$.

2.3.5 Numerical analysis

Fourth-Order Runge-Kutta Method

In order to solve an initial-value problem

$$\frac{dx}{dt} = f(t, x), \quad x(t_0) = x_0,$$

where $x = [x_1, x_2, \ldots, x_n]^T$ and $f = [f_1, f_2, \ldots, f_n]^T$.

The best known Runge-Kutta method of the first stage and fourth order is given by

$$X_{i+1} = X_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where

$$\begin{aligned}
k_1 &= hf(t_i, X_i),\\
k_2 &= hf(t_i + \frac{h}{2}, X_i + \frac{k_1}{2}),\\
k_3 &= hf(t_i + \frac{h}{2}, X_i + \frac{k_2}{2}),\\
k_4 &= hf(t_i + h, X_i + k_3),
\end{aligned}$$

where X_i is an approximation of $x(t_i)$ when $X_i = [X_{i1}, X_{i2}, \ldots, X_{in}]^T$, $t_i = t_0 + ih$, h is step size and $k_i = [k_{i1}, k_{i2}, \ldots, k_{in}]^T$, $\forall i = 1, \ldots, 4$.