CHAPTER 3

H_∞ control problem for linear time-varying system

In this chapter, we investigates the H_∞ control problem for certain dynamical systems with time-varying delay. We consider the H_∞ control problem for a class of linear uncertain time-varying systems with time-varying delay via the solution of certain Riccati differential equation.

3.1 H_∞ control problem for linear time-varying system with time-varying delay

In this section, all matrices are time-varying matrices except the identity matrix I. Also, for the sake of brevity, we will omit the variable t of matrix functions and from variables x, u, w and z whenever it does not cause any confusion. For the sake of technical simplification, without loss of generality, as in [28, 41, 45], we assume that

$$D^T[C\ D] = [0\ I], t \in \mathcal{R}^+. \tag{3.1}$$

For a given $\gamma > 0$, let us set

$$A_\gamma = A - BB^T + \tfrac{1}{\gamma}B_1B_1^T,\ B_\gamma = \sqrt{BB^T - \tfrac{1}{\gamma}B_1B_1^T}$$
$$B_{\gamma,h} = B(t-h(t))B(t-h(t))^T - \tfrac{1}{\gamma}B_1(t-h(t))B_{1(t-h(t))^T},\ t \in \mathcal{R}^+.$$

The following assumptions will be used in the proof of the main theorem:

A1. There exists $\gamma > 0$ such that $B_{\gamma,h} \geq 0$, $t \in \mathcal{R}^+$.

A2. There exist $\alpha, \gamma > 0$ such that

$$e^{-2\alpha h}(1-\delta)[B_{\gamma,h} + \tfrac{\lambda}{2}I] - C_1^TC_1 > 0,\ t \in \mathcal{R}^+.$$

3.1.1 Problem formulation

Consider the following uncertain linear time-varying system:

$$\dot{x}(t) = A(t)x(t) + A_1(t)x(t-h(t)) + B(t)u(t) + B_1(t)w(t),\ t \in \mathcal{R}^+,$$
$$z(t) = C(t)x(t) + C_1(t)x(t-h(t)) + D(t)u(t),\ x_0(t) = \phi(t), t \in [-h, 0], \tag{3.2}$$

where $x \in \mathcal{R}^n$ is the state; $x_0 = \phi \in \mathcal{C}$ is an initial state; $u \in \mathcal{R}^m$ is the control; $w \in \mathcal{R}^p$ is the uncertain input, $z \in \mathcal{R}^q$ is the observation output; $A(t), A_1(t), C(t), C_1(t), D(t)$ are given matrix functions continuous and bounded on $\mathcal{R}^+_{-h}; 0 \leq h(t) \leq h, \dot{h}(t) \leq \delta < 1$. In the sequel, we say that the control $u(t)$ is admissible if $u(t) \in \mathcal{L}_2([0,t), \mathcal{R}^m)$ for every $t \geq 0$, and the uncertainty $w(\cdot)$ is admissible if $w(\cdot) \in \mathcal{L}_2([0,+\infty), \mathcal{R}^p)$.

Theorem 3.1.1. *Assume that* ***A1.*** *and* ***A2.*** *hold. Then, the H_∞ control problem (3.2) has a solution provided that the RDE*

$$\begin{aligned}&\dot{P} + A_\gamma^T P + PA_\gamma - PB_\gamma B_\gamma^T P \\ &+ \tfrac{e^{2\alpha h}}{1-\delta} A_1^T (P+I) \left[B_{\gamma,h} + \tfrac{\lambda}{2} I\right]^{-1} (P+I) A_1 + Q = 0, \ t \in \mathcal{R}^+ \end{aligned} \tag{3.3}$$

has a solution $P \in BM^+(0,\infty)$, where $Q \in BM^+(0,\infty)$ is a matrix function satisfying

$$Q \geq A + A^T + C^T C + \lambda I, \ t \in \mathcal{R}^+ \tag{3.4}$$

such that

$$\tfrac{\lambda}{2} > \bar{\lambda}(VU^{-1}V^T) \geq 0 \tag{3.5}$$

where $V = A_1^T (P+I) C_1^T [C - DB^T (P+I)]$ and $U = e^{-2\alpha h}(1-\delta)\left[B_{\gamma,h} + \frac{\lambda}{2} I\right] - C_1^T C_1$. Moreover, the feedback stabilising control is

$$u = -B^T [P+I] x, \ t \in \mathcal{R}^+.$$

The proof of the theorem is based on the following lemma:

Lemma 3.1.2. *The H_∞ control problem for the system (3.2) has a solution if there exist $\alpha, \gamma, \lambda > 0$, matrix functions $X, R \in BMU^+(0,\infty)$, and matrix function $S \in BM^+(-h,\infty)$ such that $S > 0$ and for $t \in \mathcal{R}^+$, the following conditions hold:*

$$\begin{aligned}&(i)\dot{X} + A^T X + XA - X\left[BB^T - \tfrac{1}{\gamma} B_1 B_1^T\right] X + C^T C \\ &+ \tfrac{e^{-2\alpha h}}{1-\delta} A_1^T X S(t-h(t))^{-1} X A_1 + S \leq -R, \end{aligned} \tag{3.6}$$

$$(ii) U := e^{-2\alpha h}(1-\delta) S(t-h(t)) - C_1^T C_1 > 0, \tag{3.7}$$

$$(iii) \lambda - \bar{\lambda}(VU^{-1}V^T) > 0; where V = A_1^T X C_1^T [C - DB^T X], \tag{3.8}$$

$$(iv) R \geq \lambda I. \tag{3.9}$$

The feedback control is defined as

$$u = -B^T X x, \ t \in \mathcal{R}^+. \tag{3.10}$$

Proof : Using the feedback control (3.10), we consider the following Lyapunov function for the closed-loop system

$$\dot{x}(t) = A(t)x(t) + A_1(t)x(t-h(t)) + B(t)h(x(t)),\ x(t) = \phi(t),\ t \in [-h, 0] \tag{3.11}$$

where $w(t) = 0, h(x(t)) = -B^T X x(t)$:

$$V(t, x_t) = V_1(t, x_t) + V_2(t, x_t),$$

where $t \in \mathcal{R}^+,\ V_1(t, x_t) = \langle Xx, x\rangle$ and $V_2(t, x_t) = \int_{t-h(t)}^{t} e^{2\alpha}(s-t)x^T(s)Sx(s)ds$. Since $X \in BMU^+(0, \infty)$, we easily see that

$$\underline{\lambda}_0(X)\|x\|^2 \le V(t, x_t) \le \left(\bar{\lambda}_0(X) + \frac{\bar{\lambda}_{-h}(S)(1-e^{-2\alpha h})}{2\alpha}\right)\|x_t\|_h^2.$$

By taking the derivative of $V(t, \cdot)$ along the solution x of the closed-loop system, we have

$$\begin{aligned}
\dot{V}_1(t, x_t) &= \langle \dot{X}x, x\rangle + 2\langle X\dot{x}, x\rangle \\
&= \langle \dot{X}x, x\rangle + 2\langle X(Ax + A_1x(t-h(t)) + Bu), x\rangle \\
&= \langle \dot{X}x, x\rangle + 2\langle X(Ax + A_1x(t-h(t)) - BB^TXx), x\rangle \\
&= \langle (\dot{X} + A^TX + XA - 2XBB^TX)x, x\rangle + 2\langle XA_1x(t-h(t)), x\rangle.
\end{aligned}$$

It follows that

$$\dot{V}_1(t, x_t) = \langle (\dot{X} + A^TX + XA - 2XBB^TX)x, x\rangle + 2\langle A_1^TXx, x(t-h(t))\rangle. \tag{3.12}$$

Similarly, we obtain

$$\begin{aligned}
\dot{V}_2(t, x_t) &= -2\alpha V_2(t, x_t) + \langle Sx, x\rangle \\
&\quad - e^{-2\alpha h(t)}(1-\dot{h}(t))\langle S(t-h(t))x(t-h(t)), x(t-h(t))\rangle \\
&\le \langle Sx, x\rangle - e^{-2\alpha h}(1-\delta)\langle S(t-h(t))x(t-h(t)), x(t-h(t))\rangle.
\end{aligned} \tag{3.13}$$

Combining (3.12) and (3.13), we get

$$\begin{aligned}
\dot{V}(t, x_t) \le &\langle (\dot{X} + A^TX + XA - 2XBB^TX)x, x\rangle + 2\langle A_1^TXx, x(t-h(t))\rangle \\
&\langle Sx, x\rangle - e^{-2\alpha h}(1-\delta)\langle S(t-h(t))x(t-h(t)), x(t-h(t))\rangle.
\end{aligned} \tag{3.14}$$

From (2.6) and by Proposition 2.3.23, we have

$$\begin{aligned}
\dot{V}(t, x_t) \le &\langle \dot{X} + A^TX + XA - 2XBB^TX + S)x, x\rangle \\
&+ \tfrac{e^{2\alpha h}}{1-\delta}\langle A_1^TXS(t-h(t))^{-1}XA_1x, x\rangle \\
&= \left\langle \left(\dot{X} + A^TX + XA - 2XBB^TX + \tfrac{e^{2\alpha h}}{1-\delta}A_1^TXS(t-h(t))^{-1}XA_1 + S\right), x\right\rangle \\
&\le -\left\langle \tfrac{1}{\gamma}XB_1B_1^TXx, x\right\rangle - \langle XBB^TXx, x\rangle - \langle CC^Tx, x\rangle - \langle Rx, x\rangle.
\end{aligned} \tag{3.15}$$

Since $\langle CC^Tx,x\rangle \geq 0$,$\langle XBB^TXx,x\rangle \geq 0$, $\langle \frac{1}{\gamma}XB_1B_1^TXx,x\rangle \geq 0$, $t\geq 0$, it follows from (3.1.2)(iv) and (3.15) that

$$\dot{V}(t,x_t) - \langle Rx,x\rangle \leq -\lambda\|x\|^2.$$

By a similar argument as in the proof in [[12], Chapter 5, Proof of Theorem 2.1], the system without uncertainties is uniformly asymptotically stable. Therefore, the closed-loop system is uniformly asymptotically stable, i.e., the system is stabilisable. To complete the proof of the theorem, it remains to show the $\gamma-$ suboptimal condition (2.3). For this, we consider the relation

$$\int_0^t [\|z(s)\|^2 - \gamma\|w(s)\|^2]ds = \int_0^t [\|z(s)\|^2 - \gamma\|w(s)\|^2 + \dot{V}(s,x_s)]ds - \int_0^t \dot{V}(s,x_s)ds,$$

where (3.1.2)(i) and from positive definiteness of $\frac{e^{2\alpha h}}{1-\delta}A_1^TXS(t-h(t))^{-1}XA_1$,$\dot{V}(t,x(t))$ is estimated as

$$\begin{aligned}
\dot{V}(t,x_t) &\leq \langle \dot{X} + A^TX + XA - 2XBB^TX + S)x,x\rangle + 2\langle A_1^TXx, x(t-h(t))\rangle \\
&\quad - e^{-2\alpha h}(1-\delta)\langle Sx(t-h(t)), x(t-h(t))\rangle + 2\langle XB_1w,x\rangle \\
&\quad \leq \left\langle \left[-\tfrac{1}{\gamma}XB_1B_1^TX - XBB^TX - C^TC - R\right]x,x\right\rangle + 2\langle A_1^TXx, x(t-h(t))\rangle \\
&\quad - e^{-2\alpha h}(1-\delta)\langle Sx(t-h(t)), x(t-h(t))\rangle + 2\langle XB_1w,x\rangle \\
&\quad \leq -\lambda\|x\|^2 - \tfrac{1}{\gamma}XB_1B_1^TX - XBB^TX - C^TC + 2\langle A_1^TXx, x(t-h(t))\rangle \\
&\quad - e^{-2\alpha h}(1-\delta)\langle Sx(t-h(t)), x(t-h(t))\rangle + 2\langle XB_1w,x\rangle. \qquad (3.16)
\end{aligned}$$

Since $\dot{V}(t,x_t)\geq 0$, we have

$$\begin{aligned}
\int_0^t \dot{V}(s,x_s)ds &= V(t,x_t) - V(0,x_0) \\
&\geq -V(0,x_0) \\
&\geq -\left(\langle X(0)x_0,x_0\rangle + \int_{0-h(0)}^0 e^{2\alpha(s-t)}x^T(s)Sx(s)ds\right) \\
&\geq -\left(\lambda_{\max}(X(0))\|x_0\|^2 + \tfrac{1-e^{-2\alpha h}}{2\alpha}\bar{\lambda}_{-h}(S)\|x_0\|_h^2\right) \\
&\geq -\left(\lambda_{\max}(X(0)) + \tfrac{1-e^{-2\alpha h}}{2\alpha}\bar{\lambda}_{-h}(S)\right)\|x_0\|_h^2.
\end{aligned}$$

By denoting $\beta = \left(\lambda_{\max}(X(0)) + \frac{1-e^{-2\alpha h}}{2\alpha}\bar{\lambda}_{-h}(S)\right)\|x_0\|_h^2$, we get

$$\int_0^t [\|z(s)\|^2 - \gamma\|w(s)\|^2]ds = \int_0^t [\|z(s)\|^2 - \gamma\|w(s)\|^2 + \dot{V}(s,x_s)]ds + \beta. \qquad (3.17)$$

Taking the estimation of $\dot{V}(s,x_s)$ from (3.16) and putting

$$\begin{aligned}
\|z(t)\|^2 =& \langle [C^TC + XBB^TX]x,x\rangle + 2\langle C_1^T[C - DB^TX]x, x(t-h(t))\rangle \\
& \langle C_1^TC_1x(t-h(t)), x(t-h(t))\rangle,
\end{aligned}$$

in the inequality (5.25), we obtain

$$\int_0^t [\|z(s)\|^2 - \gamma\|w(s)\|^2]ds \le \int_0^t [-\lambda\|x(s)\|^2 + 2\langle A_1^T Xx, x(s-h(s))\rangle$$
$$- e^{-2\alpha h}(1-\delta)\langle S(s-h(s))x(s-h(s)), x(s-h(s))\rangle$$
$$+ 2\langle C_1^T[C - DB^T X]x, x(s-h(s))\rangle$$
$$+ \langle C_1^T C_1 x(s-h(s)), x(s-h(s))\rangle - \tfrac{1}{\gamma}\langle XB_1B_1^T Xx, x\rangle$$
$$+ 2\langle XB_1 w, x\rangle - \gamma\langle w, w\rangle]ds + \beta. \qquad (3.18)$$

Using Proposition 2.3.23 gives

$$2\langle XB_1 w, x\rangle - \tfrac{1}{\gamma}\langle XB_1B_1^T Xx, x\rangle. \qquad (3.19)$$

From (3.18) and (3.19), we obtain

$$\int_0^t [\|z(s)\|^2 - \gamma\|w(s)\|^2]ds$$
$$\le \int_0^t [-\lambda\|x(s)\|^2 + 2\langle A_1^T XC_1^T[C - DB^T X])x, x(s-h(s))\rangle$$
$$- \langle [e^{-2\alpha h}(1-\delta)S(s-h(s)) - C_1^T C_1]x(s-h(s)), x(s-h(s))\rangle]ds + \beta. \qquad (3.20)$$

Let $V = A_1^T XC_1^T[C - DB^T X]$ and $U = e^{-2\alpha h}(1-\delta)S(t-h(t)) - C_1^T C_1$. By (3.1.2)(ii) and Proposition 2.3.23, we get

$$\int_0^t [\|z(s)\|^2 - \gamma\|w(s)\|^2]ds \le \int_0^t [-\lambda\|x(s)\|^2 + \langle (VU^{-1}V^T x, x)\rangle]ds + \beta$$
$$\le -\int_0^t [\lambda - \bar{\lambda}(VU^1V^T]\|x(s)\|^2 ds + \beta. \qquad (3.21)$$

By (3.1.2)(iii) and (3.21), we obtain

$$\int_0^t [\|z(s)\|^2 - \gamma\|w(s)\|^2]ds \le \beta. \qquad (3.22)$$

By letting $t \to +\infty$ in (3.22), we finally obtain

$$\int_0^\infty [\|z(s)\|^2 - \gamma\|w(s)\|^2]ds \le \beta,$$

and hence

$$\int_0^\infty [\|z(t)\|^2 dt \le \gamma\left(\int_0^\infty \|w(t)\|^2 dt + \tfrac{\beta}{\gamma}\right).$$

Setting $c_0 = \frac{\beta}{\gamma}$, then from the last inequality, we have

$$\frac{\int_0^\infty \|z(t)\|^2 dt}{c_0\|x_0\|^2 + \int_0^\infty \|w(t)\|^2 dt} \le \gamma$$

for all x_0 and non-zero $w(t) \in \mathcal{L}_2([0,\infty), W)$. This completes the proof of the lemma. **Proof of Theorem 3.1.1:**. By assumptions, there exists $\lambda > 0$, $P, Q \in BM^+(0,\infty)$ such that

$$\tfrac{\lambda}{2} > \bar{\lambda}(VU^{-1}V^T) > 0,$$

where $V = A_1^T(P+I)C_1^T[C - DB^T(P+I)]$ and $U = e^{-2\alpha h}(1-\delta)\left[B_{\gamma,h} + \tfrac{\lambda}{2}I\right] - C_1^T C_1, Q$ satisfies

$$Q \geq A + A^T + C^T C + \lambda I, t \in \mathcal{R}^+ \tag{3.23}$$

(a matrix Q might be obtained by using Proposition 2.3.16), and P satisfies the RDE

$$\begin{aligned}&\dot{P} + A_\gamma^T P + PA_\gamma - PB_\gamma B_\gamma^T P \\ &+ \tfrac{e^{2\alpha h}}{1-\delta} A_1^T (P+I)\left[B_{\gamma,h} + \tfrac{\lambda}{2}I\right]^{-1}(P+I)A_1 + Q = 0.\end{aligned} \tag{3.24}$$

The RDE (3.24) can be reformulated as

$$\begin{aligned}&\dot{P} + A^T(P+I) + (P+I)A - (P+I)\left[BB^T - \tfrac{1}{\gamma}B_1B_1^T\right](P+I) \\ &+ \tfrac{e^{2\alpha h}}{1-\delta} A_1^T (P+I)\left[B_{\gamma,h} + \tfrac{\lambda}{2}I\right]^{-1}(P+I)A_1 + Q - (A^T + A) + \left[BB^T - \tfrac{1}{\gamma}B_1B_1^T\right] = 0.\end{aligned}$$

Therefore, by taking (3.23) into account, we have

$$\begin{aligned}&\dot{P} + A^T(P+I) + (P+I)A - (P+I)\left[BB^T - \tfrac{1}{\gamma}B_1B_1^T\right](P+I) + C^T C \\ &+ \tfrac{e^{2\alpha h}}{1-\delta} A_1^T (P+I)\left[B_{\gamma,h} + \tfrac{\lambda}{2}I\right]^{-1}(P+I)A_1 + Q + \left[BB^T - \tfrac{1}{\gamma}B_1B_1^T\right] + \lambda I \leq 0.\end{aligned} \tag{3.25}$$

By letting

$$X = P + I, \quad s = BB^T - \tfrac{1}{\gamma}B_1B_1^T + \tfrac{\lambda}{2}I \text{ and } R = \tfrac{\lambda}{2}I.$$

Then one might easily check that $X \gg 0$, $R(t) \geq \frac{\lambda}{2}I \gg 0$, and $S = BB^T - \frac{1}{\gamma}B_1B_1^T + \frac{\lambda}{2}I > 0$ by assumption A1. From (3.25), it follows that X, R and S satisfy the RDI

$$\begin{aligned}&\dot{X} + A^T X + XA - X\left[BB^T - \tfrac{1}{\gamma}B_1B_1^T\right]X + C^T C + \tfrac{e^{2\alpha h}}{1-\delta}A_1^T X S(t - h(t))^{-1} X A_1 \\ &+ S \leq -R,\end{aligned}$$

and the proof of the theorem is completed by using (3.1.2).

Remark 3.1.3. *Note that the problem of solving Riccati differential equation is in general still complicated, however, some various efficient approaches to solve this problem can be found, for instance, in [1, 25].*

The following simple procedure can be applied to solve the H_∞ control problem for the system (3.2):

Step 1. Given $\alpha, \gamma, \delta, \lambda, h > 0$ check assumptions A1 and A2.

Step 2. Check the condition (3.1).

Step 3. Find a matrix Q satisfying (3.4) and find a solution P of RDE (3.3).

Step 4. Check the condition (3.5).

Step 5. The feedback stabilising control is given by

$$u(t) = -B^T(t)[P + I]x(t), \quad t \in \mathcal{R}^+.$$

Remark 3.1.4. *For the system without time delay, namely, when* $A_1 \equiv C_1 \equiv 0$ *and* $h(t) \equiv 0$, *(3.2) becames*

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) + B_1(t)w(t), t \in \mathcal{R}^+ \\ z(t) &= C(t)x(t) + D(t)u(t), \quad x(0) = x_0. \end{aligned} \tag{3.26}$$

In this case, (3.26) and assumption A2 trivially hold, and we obtain the following result as a corollary of (3.1.1).

Proposition 3.1.5. *Assume that A1 holds and* $[A_\gamma, B_\gamma]$ *is GNC in finite time and let* λ *be a given positive real number. Then, the* H_∞ *control control problem for the system (3.2) has a solution and a feedback stabilising control is given by*

$$u = -B^T[P + I]x, \quad t \in \mathcal{R}^+,$$

where $P \in BM^+(0, \infty)$ *is a solution of RDE*

$$\dot{P} + A_\gamma^T P + P A_\gamma - P B_\gamma B_\gamma^T P + Q + 0, \quad t \in \mathcal{R}^+, \tag{3.27}$$

$$Q \geq A + A^T + C^T C + \lambda I, \quad t \in \mathcal{R}^+. \tag{3.28}$$

Proof : A matrix function Q satisfying (3.28) might be obtained by using Proposition 2.3.16 The existence of a solution P of the RDE (3.27) is guaranteed by Proposition 2.3.15. Proposition follows from Lemma 3.1.2 by observing that conditions (ii) and (iii) trivially hold and the RDI in Lemma 3.1.2 (i) becomes

$$\dot{X} + A^T X + XA - X\left[BB^T - \tfrac{1}{\gamma}B_1 B_1^T\right]X + C^T C + S \leq -R,$$

in which one might take $R = \lambda I$. The rest of the proof follows in the same manner as in Theorem 3.1.1.

Remark 3.1.6. *It is worth mentioning that for the system without time delay (3.26), the existence of solution of the H_∞ problem is guaranteed by the GNC infinite time of the system $[A_\gamma, B_\gamma]$ and one might use the rank condition in Proposition 2.3.14 to check the GNC condition of $[A_\gamma, B_\gamma]$.*

3.2 Numerical examples

In this section, we now provide an example to show the effectiveness of the result in Theorem

Example 3.2.1. Let

$$A = \begin{bmatrix} -\frac{e^{-2t}}{2} - 1 & 1 \\ -1 & -\frac{e^{-2t}}{2} - 1 \end{bmatrix}, A_1 = \begin{bmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{16} \end{bmatrix}, B = \begin{bmatrix} \sqrt{2}e^{-t} \\ 0 \end{bmatrix}, B_1 = \begin{bmatrix} \frac{e^{-2t}}{\sqrt{2}} \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} \frac{e^{-t}}{2} & -\frac{e^{-t}}{2} \\ -\frac{e^{-t}}{2} & \frac{e^{-t}}{2} \end{bmatrix}, C_1 = \begin{bmatrix} 0 & \frac{1}{8} \\ \frac{1}{8} & 0 \end{bmatrix}, D = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, P = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix},$$

$$Q = \begin{bmatrix} 4e^{-2t} - \frac{e^t}{4(4e^{e^{-t}}+e^t} + 2 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{e^{-2t}}{2} + \frac{55}{64} \end{bmatrix}, h(t) = e^{-t}, \quad h = 1$$

$\delta = 0, \lambda = \gamma = \frac{1}{2}, \alpha = ln2$. Then we have the following:

(i) $BB^T - \frac{1}{\gamma}B_1B_1^T = \begin{bmatrix} e^{-2t} & 0 \\ 0 & 0 \end{bmatrix}$.

(ii) $B_\gamma = \sqrt{BB^T - \frac{1}{\gamma}B_1B_1^T} = \begin{bmatrix} e^{-t} & 0 \\ 0 & 0 \end{bmatrix}, B_{\gamma,h} = \begin{bmatrix} e^{e^{-t}-t} & 0 \\ 0 & 0 \end{bmatrix}$.

(iii) $A_\gamma = A - BB^T + \frac{1}{\gamma}B_1B_1^T = \begin{bmatrix} -\frac{3e^{-2t}}{2} - \frac{3}{4} & 1 \\ -1 & -\frac{e^{-2t}}{2} - \frac{1}{4} \end{bmatrix}$.

(iv) By using Proposition 2.5, it is straightforward to show that$Q \geq 0$ and $Q \geq A + A^T + C^TC + \lambda I$.

(v) $D^TD = 1$ and $C^TD = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(vi) the following RDE holds for $t \in \mathcal{R}^+$:

$$\dot{P} + A_\gamma^T P + PA_\gamma - PB_\gamma B_\gamma^T P + \frac{e^{2\alpha h}}{1-\delta}A_1^T(P+I)\left[B_{\gamma,h} + \frac{\lambda}{2}I\right]^{-1}(P+I)A_1 + Q = 0.$$

(vii) $U := e^{-2\alpha h}(1-\delta)\left[B_{\gamma,h} + \frac{\lambda}{2}I\right] - C_1^T C_1 = \begin{bmatrix} \frac{e^{e^{-t}-t}}{4} + \frac{3}{64} & 0 \\ 0 & \frac{3}{64} \end{bmatrix} > 0,$

$$V := A_1^T(P+I)C_1^T[C - DB^T(P+I)] = \begin{bmatrix} -\frac{5e^{-t}}{128} & \frac{e^{-t}}{128} \\ -\frac{9e^{-t}}{512} & \frac{3e^{-t}}{512} \end{bmatrix} > 0,$$

and

$$U^{-1} = \begin{bmatrix} -\frac{64e^t}{3e^t + 16e^{e^{-t}}} & 0 \\ 0 & \frac{64}{3} \end{bmatrix}.$$

$(viii)$ By computation, we obtain $\bar{\lambda}(VU^{-1}V^T) \leq 0.0035$ which gives$\frac{\lambda}{2} = \frac{1}{4}$ $= 0.25 > \bar{\lambda}(VU^{-1}V^T)$.

From (i) to (viii), one might conclude from Theorem 3.1.1 that the H_∞ control problem has a solution and a feedback stabilising control is given by

$$\begin{aligned} u &= -B^T[P+I]x \\ &= -[2\sqrt{2}e^{-t} \quad 0]x \\ &= -2\sqrt{2}e^{-t}x_1. \end{aligned}$$

For simulation, we choose $h(t) = 0.5 + sin(t)$, $\phi(t) = [-5\cos(t), 3\cos(t)]$, $\forall t \in [0, 10]$. Figure 3.1 shows the trajectories of solutions $x_1(t)$ and $x_2(t)$ of the system without feedback control ($u(t) = 0$) and Figure 3.2 shows the trajectories of solutions $x_1(t)$, and $x_2(t)$ of the system with feedback control $u(t)$

Example 3.2.2. Consider system (3.26), where

$$A = \begin{bmatrix} -\frac{e^{-2t}}{2} - 1 & 1 \\ -1 & -\frac{e^{-2t}}{2} - 1 \end{bmatrix}, B(t) = B_1(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$C(t) = \begin{bmatrix} -\frac{e^{-2t}}{2} - 1 & 1 \\ -1 & -\frac{e^{-2t}}{2} - 1 \end{bmatrix}, D(t) = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

The assumption (3.1) hold, namely,

$$D^T D = I, \quad C^T D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

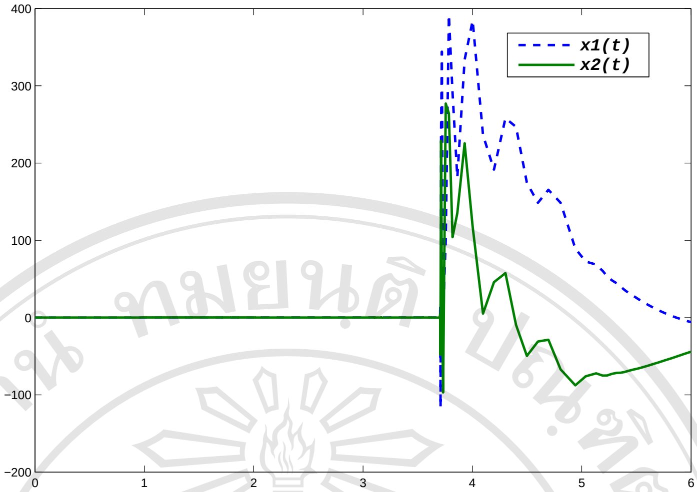

Figure 3.1: The trajectories of $x_1(t)$, and $x_2(t)$ of the system (3.2)and feedback control deactivated.

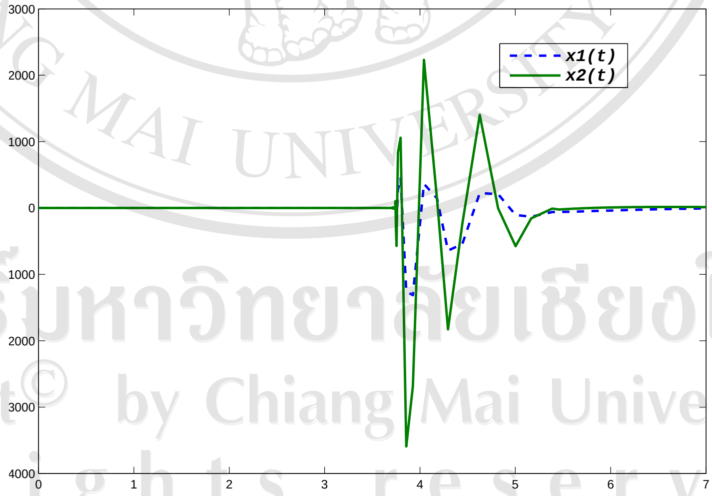

Figure 3.2: The trajectories of $x_1(t)$, and $x_2(t)$ of the system (3.2)and feedback control activated.

Taking $\gamma = \frac{3}{4}$, we have

$$(i)\ \ BB^T - \tfrac{3}{4}B_1B_1^T = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & 0 \end{bmatrix}.$$

$$(ii)\ \ B_\gamma = \sqrt{BB^T - \tfrac{3}{4}B_1B_1^T} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix}.$$

$$(iii)\ \ A_\gamma = A - BB^T - \tfrac{3}{4}B_1B_1^T = \begin{bmatrix} -\frac{e^{-2t}}{2} - 1 & 1 \\ -1 & -\frac{e^{-2t}}{2} - -\frac{1}{4} \end{bmatrix}.$$

(iv) It is clear that both matrix function A_γ $B_\gamma$are analytic. Moreover,

$M_1 = B_\gamma,\ \ M_2 = -A_\gamma,$

and

$$\text{rank}[M_1(t), M_2(t)] = \text{rank}\begin{bmatrix} -\frac{1}{4} & 0 & 1 + \frac{e^{-2t}}{2} & -1 \\ 0 & 0 & 1 & \frac{1}{4} + \frac{e^{-2t}}{2} \end{bmatrix}$$

and it is easy to verify the controllability condition (2.16): there exist $t_0 > 0$ so that $\text{rank}[B_\gamma(t_0 - A_\gamma(t_0] = 2$. Thus, by Proposition 2.14, the system $[A_\gamma, B_\gamma]$ is GNC in some finite time. Taking $\lambda = \frac{1}{2}$ and

$$Q = \begin{bmatrix} e^{-2t} - \frac{9}{4} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{e^{-2t}}{2} + \frac{1}{4} \end{bmatrix}.$$

It is easy to verify by Proposition 2.3.17 that $Q \geq 0$ and

$$Q - (A + A^T + C^TC + \lambda I) = \begin{bmatrix} \frac{9}{4} + \frac{3e^{-2t}}{2} & -\frac{1}{2} + \frac{e^{-2t}}{2} \\ -\frac{1}{2} + \frac{e^{-2t}}{2} & 1 + e^{-2t} \end{bmatrix} \geq 0$$

By Proposition 3.1.5 the H_∞ control problem has a solution. Finally, it is straightforward to show that the RDE

$$\dot{P} + A_\gamma^T P + PA_\gamma - PB_\gamma B_\gamma^T P + Q = 0,$$

has a solution

$$P = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} > 0.$$

Therefore, the H_∞ control problem has a solution and a feedback stabilising control is given by

$$U = -B^T(t)[P(t) + I]x = -[2\ \ 0]x,$$

where $x(t) = [x_1(t) \quad x_2(t)]$. For simulation, we choose $\phi(t) = [-5\cos(t), 3\cos(t)]$, $\forall t \in [0, 10]$. Figure 3.3 shows the trajectories of solutions $x_1(t)$ and $x_2(t)$ of the system without feedback control ($u(t) = 0$) and Figure 3.4 shows the trajectories of solutions $x_1(t)$, and $x_2(t)$ of the system with feedback control $u(t)$

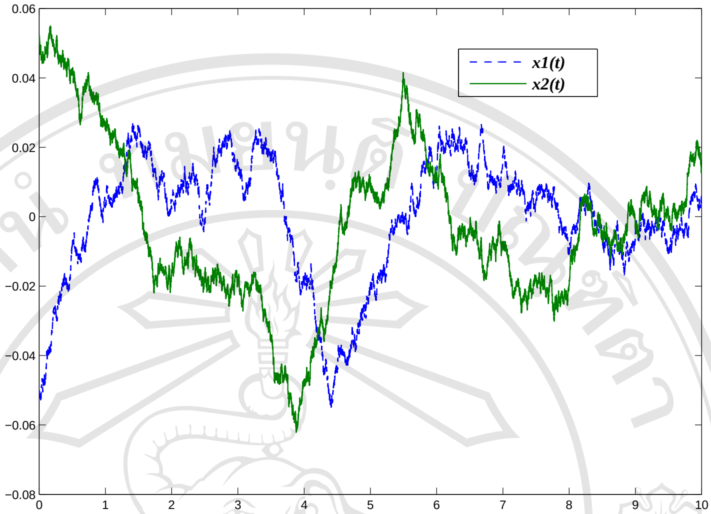

Figure 3.3: The trajectories of $x_1(t)$, and $x_2(t)$ of the system (3.26)and feedback control deactivated.

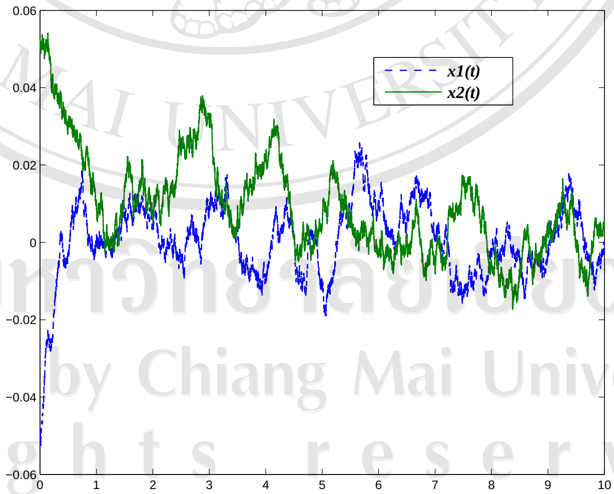

Figure 3.4: The trajectories of $x_1(t)$, and $x_2(t)$ of the system (3.26)and feedback control activated.