CHAPTER 4

Stability analysis and H_∞ control problem of linear systems with interval non-differentiable time-varying delays

In this chapter, we investigate the H_∞ control problem for linear systems with time-varying delay. We consider the H_∞ control problem for a class of linear uncertain time-varying systems with time-varying delay. New sufficient conditions for the existence of the H_∞ state-feedback for the system are given in terms of linear matrix inequalities (LMIs).

4.1 Stabilization and H_∞ control problem for linear system

Consider a linear system with interval time-varying delays of the form:

$$
\begin{aligned}
&\dot{x}(t) = (A + \Delta A(t))x(t) + (D + \Delta D(t))x(t - h(t)) + (B + \Delta B(t))u(t) + B_w w(t),\\
&z(t) = Cx(t) + C_d x(t - h(t)) + D_u u(t) + D_w w(t), \qquad (4.1)\\
&x(t) = \phi(t), t \in [-h_2, 0],
\end{aligned}
$$

where $x(t) \in \mathcal{R}^n$ is the state; $u(t) \in \mathcal{R}^m$ is the control input,$z(t) \in \mathcal{R}^q$ is the controlled output and $w(t) \in \mathcal{R}^p$ is a disturbance input belonging to $\mathcal{L}_2[0, \infty)$; $A, D, B, B_w, C, D_u, D_w$ are given matrices of appropriate dimensions and $\phi(t) \in C([-h_2, 0], \mathcal{R}^n)$ is the initial function with the norm $\| \phi \| = \sup_{s \in [-h_2, 0]} \| \phi(s) \|$; the uncertainties satisfy the following condition:

$$\Delta A(t) = E_1 F_1(t) H_1, \;\; \Delta D(t) = E_2 F_2(t) H_2, \;\; \Delta B(t) = E_3 F_3(t) H_3, \qquad (4.2)$$

where $E_i,\ H_i,\ i = 1, 2, 3$ are given constant matrices with appropriate dimensions, $F_i(t)$, $i = 1, 2, 3$ are unknown, real matrices with Lebesgue measurable elements satisfying

$$F_i^T(t) F_i(t) \le I, \quad i = 1, 2, 3, \quad \forall t \ge 0. \qquad (4.3)$$

The time-varying delay function $h(t)$ satisfies

$$0 \le h_1 \le h(t) \le h_2 \qquad (4.4)$$

It is worth noting that the time delay is assumed to be a continuous function belonging to a given interval, which means that the lower and upper bounds for the time-varying delay are available, but the delay function is bounded but not restricted to being zero. The objective of this study is to design a memoryless H_∞ state feedback controller

$$u(t) = Kx(t) \tag{4.5}$$

such that, for all admissible uncertainties satisfying (4.2) and any time-varying delay $h(t)$ satisfying(4.5) which the following criteria
The nominal system is given by

$$\begin{cases} \dot{x}(t) = Ax(t) + Dx(t-h(t)) + Bu(t) + B_w w(t), \\ z(t) = Cx(t) + C_d x(t-h(t)) + D_u u(t) + D_w w(t), \end{cases} \tag{4.6}$$
$$x(t) = \phi(t), t \in [-h_2, 0],$$

First, we present a delay-dependent exponential stabilization condition for the nominal system with interval time-varying delay (4.6). Let us set

$$\begin{aligned} \lambda_1 =& \lambda_{\min}(P^{-1}), \\ \lambda_2 =& \lambda_{\max}(P^{-1}) + 2h_2\lambda_{\max}[P^{-1}(Q)P^{-1}] \\ &+ 2h_2^2\lambda_{\max}[P^{-1}(R)P^{-1}] + (h_2-h_1)^2\lambda_{\max}[P^{-1}(U)P^{-1}]. \end{aligned}$$

Theorem 4.1.1. *Given $\alpha > 0, \gamma > 0$. The system (4.6) is $\alpha-$exponentially stabilizable and satisfies $\|z(t)\|_2 < \gamma\|w(t)\|_2$ for all nonzero $w \in \mathcal{L}_2[0,\infty)$ if there exist positive definite matrices P, Q, R, U such that the following LMI holds*

$$\begin{aligned} \mathcal{W}_1^* &= \mathcal{W}^* - \begin{bmatrix} 0 & 0 & 0 & -I & I & 0 & 0 & 0 \end{bmatrix}^T \times e^{-2\alpha h_2} U \begin{bmatrix} 0 & 0 & 0 & -I & I & 0 & 0 & 0 \end{bmatrix} \\ &< 0, \end{aligned} \tag{4.7}$$

$$\begin{aligned} \mathcal{W}_2^* &= \mathcal{W}^* - \begin{bmatrix} 0 & 0 & I & 0 & -I & 0 & 0 & 0 \end{bmatrix}^T \times e^{-2\alpha h_2} U \begin{bmatrix} 0 & 0 & I & 0 & -I & 0 & 0 & 0 \end{bmatrix} \\ &< 0, \end{aligned} \tag{4.8}$$

$$\mathcal{W}^* = \begin{bmatrix} W_{11}^* & W_{12} & W_{13} & W_{14} & W_{15} & W_{16}^* & P^TC^T & 0 \\ * & W_{22} & 0 & 0 & W_{25} & W_{26} & 0 & 0 \\ * & * & W_{33} & 0 & W_{35} & 0 & 0 & 0 \\ * & * & * & W_{44} & W_{45} & 0 & 0 & 0 \\ * & * & * & * & W_{55} & W_{56} & 0 & P^TCd^T \\ * & * & * & * & * & W_{66} & 0 & 0 \\ * & * & * & * & * & * & -I & 0 \\ * & * & * & * & * & * & 0 & -I \end{bmatrix},$$

where

$$W^*_{11} = [A+\alpha I]P + P[A+\alpha I]^T - BB^T - (e^{-2\alpha h_1} + e^{-2\alpha h_2})R + 2Q - 0.5P^TC^TD_uB^T$$
$$- 0.5BD_uCP - 0.25BB^TD_uD_u$$
$$W_{12} = PA^T - 0.5B^TB,$$
$$W_{13} = e^{-2\alpha h_1}R,$$
$$W_{14} = e^{-2\alpha h_2}R,$$
$$W_{15} = DP + C^TC_d - 0.5BD_uC_dP,$$
$$W^*_{16} = B_w + P^TC^TD_w - 0.5BD_uD_w,$$
$$W_{22} = (h_1^2 + h_2^2)R + (h_2 - h_1)^2U - 2P,$$
$$W_{25} = DP,$$
$$W_{26} = B_w,$$
$$W_{33} = -e^{-2\alpha h_1}Q - e^{-2\alpha h_1}R - e^{-2\alpha h_2}U,$$
$$W_{35} = e^{-2\alpha h_2}U,$$
$$W_{44} = -e^{-2\alpha h_2}Q - e^{-2\alpha h_2}R - e^{-2\alpha h_2}U,$$
$$W_{45} = e^{-2\alpha h_2}U,$$
$$W_{55} = -2e^{-2\alpha h_2}U,$$
$$W_{56} = P^TC_d^TD,$$
$$W_{66} = -\gamma^2 I + D_w^TD_w.$$

Moreover, the feedback control is given by

$$u(t) = -\frac{1}{2}B^TP^{-1}x(t), \quad t \geq 0, \tag{4.9}$$

and the solution $x(t,\phi)$ of the system satisfies

$$\| x(t,\phi) \| \leq \sqrt{\frac{\lambda_1}{\lambda_2}}e^{-\alpha t} \| \phi \|, \quad \forall t \geq 0.$$

Proof. Let $Y = P^{-1}, y(t) = Yx(t)$, we consider the following Lyapunov-Krasovskii functional

$$V(t, x_t) = \sum_{i=1}^{6} V_i, \tag{4.10}$$

where

$$V_1 = x^T(t)Yx(t),$$
$$V_2 = \int_{t-h_1}^{t} e^{2\alpha(s-t)}x^T(s)YQYx(s)\,ds,$$

$$V_3 = \int_{t-h_2}^{t} e^{2\alpha(s-t)} x^T(s) YQY x(s)\, ds,$$
$$V_4 = h_1 \int_{-h_1}^{0} \int_{t+s}^{t} e^{2\alpha(\tau-t)} \dot{x}^T(\tau) YRY \dot{x}(\tau)\, d\tau\, ds,$$
$$V_5 = h_2 \int_{-h_2}^{0} \int_{t+s}^{t} e^{2\alpha(\tau-t)} \dot{x}^T(\tau) YRY \dot{x}(\tau)\, d\tau\, ds,$$
$$V_6 = (h_2 - h_1) \int_{-h_2}^{-h_1} \int_{t+s}^{t} e^{2\alpha(\tau-t} \dot{x}^T(\tau) YUY \dot{x}(\tau)\, d\tau\, ds.$$

It easy to check that

$$\lambda_1 \parallel x(t) \parallel^2 \leq V(t, x_t) \leq \lambda_2 \parallel x_t \parallel^2, \quad \forall t \geq 0. \tag{4.11}$$

Using the feedback control (4.9), take the derivative of V_1 along the solution of system and set $w(t) = 0$, we have

$$\begin{aligned}
\dot{V}_1 =& 2x^T(t) Y \dot{x}(t) \\
=& y^T(t)[PA^T + AP]y(t) - y^T(t)BB^T y(t) + 2y^T(t)DPy(t - h(t)), \\
\dot{V}_2 =& y^T(t)Qy(t) - e^{-2\alpha h_1} y^T(t - h_1)Qy(t - h_1) - 2\alpha V_2, \\
\dot{V}_3 =& y^T(t)Qy(t) - e^{-2\alpha h_2} y^T(t - h_2)Qy(t - h_2) - 2\alpha V_3, \\
\dot{V}_4 =& h_1^2 \dot{y}^T(t)R\dot{y}(t) - h_1 e^{-2\alpha h_1} \int_{t-h_1}^{t} \dot{y}^T(s)R\dot{y}(s)\, ds - 2\alpha V_4, \\
\dot{V}_5 =& h_2^2 \dot{y}^T(t)R\dot{y}(t) - h_2 e^{-2\alpha h_2} \int_{t-h_2}^{t} \dot{y}^T(s)R\dot{y}(s)\, ds - 2\alpha V_5, \\
\dot{V}_6 =& (h_2 - h_1)^2 \dot{y}^T(t)U\dot{y}(t) - (h_2 - h_1)e^{-2\alpha h_2} \int_{t-h_2}^{t-h_1} \dot{y}^T(s)U\dot{y}(s)\, ds - 2\alpha V_6.
\end{aligned}$$

Applying Proposition 2.3.19 and the Leibniz - Newton formula, we have

$$\begin{aligned}
-h_1 \int_{t-h_1}^{t} \dot{y}^T(s)R\dot{y}(s)\, ds &\leq -\left[\int_{t-h_1}^{t} \dot{y}(s)\, ds\right]^T R \left[\int_{t-h_1}^{t} \dot{y}(s)\, ds\right] \\
&\leq -[y(t) - y(t - h_1)]^T R[y(t) - y(t - h_1)] \\
&= -y^T(t)Ry(t) + 2y^T(t)Ry(t - h_1)) - y^T(t - h_1)Ry(t - h_1),
\end{aligned} \tag{4.12}$$

and

$$\begin{aligned}
-h_2 \int_{t-h_2}^{t} \dot{y}^T(s)R\dot{y}(s)\, ds &\leq -\left[\int_{t-h_2}^{t} \dot{y}(s)\, ds\right]^T R \left[\int_{t-h_2}^{t} \dot{y}(s)\, ds\right] \\
&\leq -[y(t) - y(t - h_2)]^T R[y(t) - y(t - h_2)] \\
&= -y^T(t)Ry(t) + 2y^T(t)Ry(t - h_2)) - y^T(t - h_2)Ry(t - h_2).
\end{aligned} \tag{4.13}$$

Note that

$$
\begin{aligned}
-(h_2-h_1)\int_{t-h_2}^{t-h_1}\dot{y}^T(s)U\dot{y}(s)\,ds &= -(h_2-h_1)\int_{t-h_2}^{t-h(t)}\dot{y}^T(s)U\dot{y}(s)\,ds \\
&\quad -(h_2-h_1)\int_{t-h(t)}^{t-h_1}\dot{y}^T(s)U\dot{y}(s)\,ds \\
&= -(h_2-h(t))\int_{t-h_2}^{t-h(t)}\dot{y}^T(s)U\dot{y}(s)\,ds \\
&\quad -(h(t)-h_1)\int_{t-h_2}^{t-h(t)}\dot{y}^T(s)U\dot{y}(s)\,ds \\
&\quad -(h(t)-h_1)\int_{t-h(t)}^{t-h_1}\dot{y}^T(s)U\dot{y}(s)\,ds \\
&\quad -(h_2-h(t))\int_{t-h(t)}^{t-h_1}\dot{y}^T(s)U\dot{y}(s)\,ds.
\end{aligned}
$$

Using Proposition 2.3.19, this gives

$$
\begin{aligned}
-(h_2-h(t))\int_{t-h_2}^{t-h(t)}\dot{y}^T(s)U\dot{y}(s)\,ds &\le -\Big[\int_{t-h_2}^{t-h(t)}\dot{y}(s)ds\Big]^T U\Big[\int_{t-h_2}^{t-h(t)}\dot{y}(s)ds\Big] \qquad (4.14)\\
&\le -[y(t-h(t)-y(t-h_2)]^T U[y(t-h(t)-y(t-h_2)],
\end{aligned}
$$

and

$$
\begin{aligned}
-(h(t)-h_1)\int_{t-h(t)}^{t-h_1}\dot{y}^T(s)U\dot{y}(s)\,ds &\le -\Big[\int_{t-h(t)}^{t-h_1}\dot{y}(s)ds\Big]^T U\Big[\int_{t-h(t)}^{t-h_1}\dot{y}(s)ds\Big] \qquad (4.15)\\
&\le -[y(t-h_1)-y(t-h(t))]^T U[y(t-h_1)-y(t-h(t))].
\end{aligned}
$$

Let $\beta=\frac{h_2-h(t)}{h_2-h_1}\le 1$. Then

$$
\begin{aligned}
-(h_2-h(t))\int_{t-h(t)}^{t-h_1}\dot{y}^T(s)U\dot{y}(s)\,ds &= -\beta\int_{t-h(t)}^{t-h_1}(h_2-h_1)\dot{y}^T(s)U\dot{y}(s)\,ds \\
&\le -\beta\int_{t-h(t)}^{t-h_1}(h(t)-h_1)\dot{y}^T(s)U\dot{y}(s)\,ds \qquad (4.16)\\
&\le -\beta[y(t-h_1)-y(t-h(t))]^T U[y(t-h_1)-y(t-h(t))],
\end{aligned}
$$

and

$$
\begin{aligned}
-(h(t)-h_1)\int_{t-h_2}^{t-h(t)}\dot{y}^T(s)U\dot{y}(s)\,ds &= -(1-\beta)\int_{t-h_2}^{t-h(t)}(h_2-h_1)\dot{y}^T(s)U\dot{y}(s)\,ds \\
&\le -(1-\beta)\int_{t-h_2}^{t-h(t)}(h_2-h(t))\dot{y}^T(s)U\dot{y}(s)\,ds \\
&\le -(1-\beta)[y(t-h(t))-y(t-h_2)]^T \qquad (4.17)\\
&\quad \times U[y(t-h(t))-y(t-h_2)].
\end{aligned}
$$

Therefore, from (4.14)-(4.17), we obtain

$$\begin{aligned}
-(h_2-h_1)\int_{t-h_2}^{t-h_1} \dot{y}^T(s)U\dot{y}(s)\,ds \quad \leq \quad & -[y(t-h(t)-y(t-h_2)]^T U[y(t-h(t)-y(t-h_2)] \\
& -[y(t-h_1)-y(t-h_(t))]^T U[y(t-h_1)-y(t-h(t))] \\
& -\beta[y(t-h_1)-y(t-h(t))]^T U[y(t-h_1)-y(t-h(t)) \\
& -(1-\beta)[y(t-h(t))-y(t-h_2)]^T \\
& \times U[y(t-h(t))-y(t-h_2)]. \qquad (4.18)
\end{aligned}$$

By using the following identity relation

$$P\dot{y}(t)-APy(t)-DPy(t-h(t))+0.5BB^Ty(t)=0,$$

we obtain

$$\begin{aligned}
&-2\dot{y}^T(t)P\dot{y}(t)+2\dot{y}^T(t)APy(t)+2\dot{y}^T(t)DPy(t-h(t)) \\
&\qquad -\dot{y}^T(t)BB^Ty(t)=0. \qquad (4.19)
\end{aligned}$$

According to (4.12)-(4.13), (4.18) and by adding the zero term of (5.11), we have

$$\begin{aligned}
\dot{V}(t,x_t)+2\alpha V(t,x_t) \quad \leq \quad & y^T(t)[PA^T+AP+2\alpha P]y(t)+2y^T(t)DPy(t-h(t)) \\
& -y^T(t)BB^Ty(t)+2y^T(t)Qy(t)-e^{-2\alpha h_1}y^T(t-h_1)Qy(t-h_1) \\
& -e^{-2\alpha h_1}y^T(t-h_1)Qy(t-h_1)-e^{-2\alpha h_2}y^T(t-h_2)Qy(t-h_2) \\
& +\dot{y}^T(t)[(h_1^2+h_2^2)R+(h_2-h_1)^2U]\dot{y}(t) \\
& -e^{-2\alpha h_1}[y(t)-y(t-h_1)]^T R[y(t)-y(t-h_1)] \\
& -e^{-2\alpha h_2}[y(t)-y(t-h_2)]^T R[y(t)-y(t-h_2)] \\
& -e^{-2\alpha h_2}[y(t-h(t))-y(t-h_2)]^T U[y(t-h(t))-y(t-h_2)] \\
& -e^{-2\alpha h_2}[y(t-h_1)-y(t-h(t))]^T U[y(t-h_1)-y(t-h(t))]
\end{aligned}$$

$$\begin{aligned}
& -\beta[y(t-h_1)-y(t-h(t))]^T e^{-2\alpha h_2}U[y(t-h_1)-y(t-h(t)) \\
& -(1-\beta)[y(t-h(t))-y(t-h_2)]^T e^{-2\alpha h_2}U[y(t-h(t))-y(t-h_2)] \\
& -2\dot{y}^T(t)P\dot{y}(t)+2\dot{y}^T(t)APy(t)+2\dot{y}^T(t)DPy(t-h(t)) \\
& -\dot{y}^T(t)BB^Ty(t) \\
= \quad & \zeta^T(t)\mathcal{W}\zeta(t)-\beta[y(t-h_1)-y(t-h(t))]^T e^{-2\alpha h_2}U[y(t-h_1)-y(t-h(t)) \\
& -(1-\beta)[y(t-h(t))-y(t-h_2)]^T e^{-2\alpha h_2}U[y(t-h(t))-y(t-h_2)] \\
= \quad & \zeta^T(t)[(1-\beta)\mathcal{W}_1+\beta\mathcal{W}_2]\zeta(t), \qquad (4.20)
\end{aligned}$$

where

$$\mathcal{W}_1 = \mathcal{W} - \begin{bmatrix} 0 & 0 & 0 & -I & I \end{bmatrix}^T \times e^{-2\alpha h_2} U \begin{bmatrix} 0 & 0 & 0 & -I & I \end{bmatrix} < 0, \quad (4.21)$$

$$\mathcal{W}_2 = \mathcal{W} - \begin{bmatrix} 0 & 0 & I & 0 & -I \end{bmatrix}^T \times e^{-2\alpha h_2} U \begin{bmatrix} 0 & 0 & I & 0 & -I \end{bmatrix} < 0, \quad (4.22)$$

$$\mathcal{W} = \begin{bmatrix} W_{11} & W_{12} & W_{13} & W_{14} & W_{15} \\ * & W_{22} & 0 & 0 & W_{25} \\ * & * & W_{33} & 0 & W_{35} \\ * & * & * & W_{44} & W_{45} \\ * & * & * & * & W_{55} \end{bmatrix},$$

and

$$\zeta(t) = [y(t), \dot{y}(t), y(t-h_1), y(t-h_2), y(t-h(t))].$$

Since $0 \le \beta \le 1$, $(1-\beta)\mathcal{W}_1 + \beta\mathcal{W}_2$ is a convex combination of $\mathcal{W}_1$ and $\mathcal{W}_2$. Therefore, $(1-\beta)\mathcal{W}_1 + \beta\mathcal{W}_2 < 0$ is equivalent to $\mathcal{W}_1 < 0$ and $\mathcal{W}_2 < 0$. Thus, it follows from (**??**), (4.21) and (4.22), that

$$\dot{V}(t, x_t) \le -2\alpha V(t, x_t), \quad \forall t \ge 0. \quad (4.23)$$

Integrating both sides of (4.23) from 0 to t, we obtain

$$V(t, x_t) \le V(\phi)e^{-2\alpha t}, \quad \forall t \ge 0.$$

Furthermore, taking condition (4.11) into account, we have

$$\lambda_1 \parallel x(t,\phi) \parallel^2 \le V(x_t) \le V(\phi)e^{-2\alpha t} \le \lambda_2 e^{-2\alpha t} \parallel \phi \parallel^2 .$$

This implies

$$\parallel x(t,\phi) \parallel \le \sqrt{\frac{\lambda_2}{\lambda_1}} e^{-\alpha t} \parallel \phi \parallel, \quad t \ge 0.$$

Therefore, the nominal system (4.6) is $\alpha-$exponentially stabilizable. To complete the proof of theorem, we need to find the upper bound $\gamma\|w(t)\|_2$, for the $L_2[0,\infty)$ norm of $z(t)$, assume that $x(t) = 0$, $t \in [-h(t), 0]$, we introduce

$$J = \int_0^\infty \Big[z^T(t)z(t) - \gamma^2 w^T(t)w(t)\Big]dt. \quad (4.24)$$

The facts that $V(t,x_t)|_{t=0} = 0$ under a zero initial condition, $V(t,x_t)|_{t=\infty} \ge 0$ and $\int_0^\infty 2\alpha V(t)dt \ge 0$ lead to

$$J_\tau \le \int_0^\tau \Big[z^T(t)z(t) - \gamma^2 w^T(t)w(t) + \dot{V}(t,x) + 2\alpha V(t)\Big]dt \quad (4.25)$$

$$= \int_0^\tau \Psi^T(t)\Theta\Psi(t)dt, \quad (4.26)$$

where

$$
\begin{aligned}
\Psi &= [y(t), \dot{y}(t), y(t-h_1), y(t-h_2), y(t-h(t)), w(t)] && (4.27)\\
\Theta &= [(1-\beta)\mathcal{W}_1^* + \beta \mathcal{W}_2^*], && (4.28)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{W}_1^* &= \mathcal{W}^* - \begin{bmatrix} 0 & 0 & 0 & -I & I & 0 & 0 & 0 \end{bmatrix}^T \times e^{-2\alpha h_2} U \begin{bmatrix} 0 & 0 & 0 & -I & I & 0 & 0 & 0 \end{bmatrix}\\
&< 0, && (4.29)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{W}_2^* &= \mathcal{W}^* - \begin{bmatrix} 0 & 0 & I & 0 & -I & 0 & 0 & 0 \end{bmatrix}^T \times e^{-2\alpha h_2} U \begin{bmatrix} 0 & 0 & I & 0 & -I & 0 & 0 & 0 \end{bmatrix}\\
&< 0, && (4.30)
\end{aligned}
$$

$$
\mathcal{W}^* = \begin{bmatrix}
W_{11}^* & W_{12} & W_{13} & W_{14} & W_{15} & W_{16}^* & P^T C^T & 0\\
* & W_{22} & 0 & 0 & W_{25} & W_{26} & 0 & 0\\
* & * & W_{33} & 0 & W_{35} & 0 & 0 & 0\\
* & * & * & W_{44} & W_{45} & 0 & 0 & 0\\
* & * & * & * & W_{55} & 0 & 0 & P^T C_d^T\\
* & * & * & * & * & W_{66} & 0 & 0\\
* & * & * & * & * & * & -I & 0\\
* & * & * & * & * & * & 0 & -I
\end{bmatrix}.
$$

We have $\Theta < 0$, then $J_\tau \le 0$. Thus, we have

$$
J_\tau = \int_0^\tau \Big[z^T(t) z(t) - \gamma^2 w^T(t) w(t) \Big] dt \le 0,
$$

that is

$$
\int_0^\tau z^T(t) z(t) dt \le \int_0^\tau \gamma^2 w^T(t) w(t) dt \le \int_0^\infty \gamma^2 w^T(t) w(t) dt
$$

for all $\tau > 0$. Hence, the control output $z(T) \in L_2[0, \infty)$ satisfies

$$
\|z(t)\|_2 \le \gamma \|w(t)\|_2, \forall w(t) \in L_2[0, \infty)
$$

The proof is complete.

4.2 Robustly H_∞ control for uncertain linear systems

Based on Theorem 4.1.1, we derive robustly H_∞ control conditions of uncertain linear control systems with interval time-varying delay (4.1) in terms of LMIs.

Theorem 4.2.1. *Given $\alpha > 0, \gamma > 0$. The system (4.1) is $\alpha-$exponentially stabilizable and satisfies $\|z(t)\|_2 < \gamma \|w(t)\|_2$ for all nonzero $w \in L_2[0, \infty)$ if there exist positive definite matrices P, Q, R, U and $\epsilon_i > 0, i = 1, 2, \ldots, 6$ such that the following LMI holds*

$$\begin{aligned} \mathcal{M}_1 &= \mathcal{M} - \begin{bmatrix} 0 & 0 & 0 & -I & I & 0 & 0 & 0 \end{bmatrix}^T \times e^{-2\alpha h_2} U \begin{bmatrix} 0 & 0 & 0 & -I & I & 0 & 0 & 0 \end{bmatrix} \\ &< 0, \end{aligned} \tag{4.31}$$

$$\begin{aligned} \mathcal{M}_2 &= \mathcal{M} - \begin{bmatrix} 0 & 0 & I & 0 & -I & 0 & 0 & 0 \end{bmatrix}^T \times e^{-2\alpha h_2} U \begin{bmatrix} 0 & 0 & I & 0 & -I & 0 & 0 & 0 \end{bmatrix} \\ &< 0, \end{aligned} \tag{4.32}$$

$$\mathcal{M}_3 = \begin{bmatrix} M_{311} & PH_1^T & PH_1^T & \frac{BH_3^T}{2} & \frac{BH_3^T}{2} \\ * & -\epsilon_1 I & 0 & 0 & 0 \\ * & * & -\epsilon_2 I & 0 & 0 \\ * & * & * & \frac{-\epsilon_5 I}{2} & 0 \\ * & * & * & * & \frac{-\epsilon_6 I}{2} \end{bmatrix} < 0, \tag{4.33}$$

$$\mathcal{M}_4 = \begin{bmatrix} M_{411} & PH_2^T & PH_2^T \\ * & -\epsilon_3 I & 0 \\ * & * & -\epsilon_4 I \end{bmatrix} < 0, \tag{4.34}$$

where

$$\mathcal{M} = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} & M_{15} & M_{16} & M_{17} & 0 \\ * & M_{22} & 0 & 0 & M_{25} & M_{26} & 0 & 0 \\ * & * & M_{33} & 0 & M_{35} & 0 & 0 & 0 \\ * & * & * & M_{44} & M_{45} & 0 & 0 & 0 \\ * & * & * & * & M_{55} & M_{56} & 0 & M_{58} \\ * & * & * & * & * & M_{66} & 0 & 0 \\ * & * & * & * & * & * & -I & 0 \\ * & * & * & * & * & * & 0 & -I \end{bmatrix},$$

$$
\begin{aligned}
M_{11} &= [A+\alpha I]P + P[A+\alpha I]^T - 0.9BB^T - (e^{-2\alpha h_1} + e^{-2\alpha h_2})R + 2Q \\
&\quad -0.5P^TC^TD_uB^T - 0.5BD_uCP - 0.25BB^TD_uD_u + \epsilon_1 E_1^TE_1 \\
&\quad +\epsilon_3 E_2^TE_2 + 0.5\epsilon_5 E_3^TE_3, \\
M_{12} &= PA^T - 0.5B^TB, \\
M_{13} &= e^{-2\alpha h_1}R, \\
M_{14} &= e^{-2\alpha h_2}R, \\
M_{15} &= DP + C^TC_d - 0.5BD_uC_dP, \\
M_{16} &= B_w + P^TC^TD_w - 0.5BD_uD_w, \\
M_{17} &= P^TC^T, \\
M_{22} &= (h_1^2 + h_2^2)R + (h_2 - h_1)^2U - 2P + \epsilon_2 E_1^TE_1 + \epsilon_4 E_2^TE_2 + 0.5\epsilon_6 E_3^TE_3 \\
M_{25} &= DP, \\
M_{26} &= B_w, \\
M_{33} &= -e^{-2\alpha h_1}Q - e^{-2\alpha h_1}R - e^{-2\alpha h_2}U, \\
M_{35} &= e^{-2\alpha h_2}U, \\
M_{44} &= -e^{-2\alpha h_2}Q - e^{-2\alpha h_2}R - e^{-2\alpha h_2}U, \\
M_{45} &= e^{-2\alpha h_2}U, \\
M_{55} &= -1.9e^{-2\alpha h_2}U, \\
M_{56} &= P^TCd^TD, \\
M_{58} &= P^TCd^T, \\
M_{66} &= -\gamma^2 I + D_w^TD_w \\
M_{311} &= -0.1BB^T - 0.1(e^{-2\alpha h_1} + e^{-2\alpha h_2})R, \\
M_{411} &= -0.1e^{-2\alpha h_2}U.
\end{aligned}
$$

Moreover, the feedback control is given by

$$u(t) = -\frac{1}{2}B^TP^{-1}x(t), \quad t \geq 0, \tag{4.35}$$

and the solution $x(t,\phi)$ of the system satisfies

$$\| x(t,\phi) \| \leq \sqrt{\frac{\lambda_1}{\lambda_2}} e^{-\alpha t} \| \phi \|, \quad \forall t \geq 0.$$

Proof. Choose Lyapunov-Krasovskii functional as in (4.10), we may proof the Theorem by using a similar argument as in the proof of Theorem 4.1.1. By replacing A, D and B in (4.21) with $A + E_1F_1(t)H_1$, $D + E_2F_2(t)H_2$ and $B + E_3F_3(t)H_3$, respectively.

We have the following

$$
\begin{aligned}
\dot{V}(t,x_t)+2\alpha V(t,x_t) \ \leq\ & y^T(t)[P(A+E_1F_1(t)H_1)^T+(A+E_1F_1(t)H_1)P+2\alpha P]y(t) \\
&+2y^T(t)(D+E_2F_2(t)H_2)Py(t-h(t))-y^T(t)BB^Ty(t)+2y^TB_ww(t) \\
&-y^T(t)E_3F_3(t)H_3B^Ty(t)+2y^T(t)Qy(t)-e^{-2\alpha h_1}y^T(t-h_1)Qy(t-h_1) \\
&-e^{-2\alpha h_1}y^T(t-h_1)Qy(t-h_1)-e^{-2\alpha h_2}y^T(t-h_2)Qy(t-h_2) \\
&+\dot{y}^T(t)[(h_1^2+h_2^2)R+(h_2-h_1)^2U]\dot{y}(t) \\
&-e^{-2\alpha h_1}[y(t)-y(t-h_1)]^TR[y(t)-y(t-h_1)] \\
&-e^{-2\alpha h_2}[y(t)-y(t-h_2)]^TR[y(t)-y(t-h_2)] \\
&-e^{-2\alpha h_2}[y(t-h(t))-y(t-h_2)]^TU[y(t-h(t))-y(t-h_2)] \\
&-e^{-2\alpha h_2}[y(t-h_1)-y(t-h(t))]^TU[y(t-h_1)-y(t-h(t))] \\
&-\beta[y(t-h_1)-y(t-h(t))]^Te^{-2\alpha h_2}U[y(t-h_1)-y(t-h(t)) \\
&-(1-\beta)[y(t-h(t))-y(t-h_2)]^Te^{-2\alpha h_2}U[y(t-h(t))-y(t-h_2)] \\
&-2\dot{y}^T(t)P\dot{y}(t)+2\dot{y}^T(t)(A+E_1F_1(t)H_1)Py(t) \\
&+2\dot{y}^T(t)(D+E_2F_2(t)H_2)Py(t-h(t))-\dot{y}^T(t)BB^Ty(t) \\
&-\dot{y}^T(t)E_3F_3(t)H_3B^Ty(t)+2\dot{y}^TB_ww(t). \qquad (4.36)
\end{aligned}
$$

Applying Lemma 2.3.20 and Lemma 2.3.21, the following estimations hold

$$
\begin{aligned}
& y^T(t)[P(A+E_1F_1(t)H_1)^T+(A+E_1F_1(t)H_1)Py(t) \\
\leq\ & y(t)^T[PA^T+PA]y(t)+\epsilon_1 y(t)^TE_1^TE_1y(t)\epsilon_1^{-1}y(t)^TPH_1^TH_1Py(t), \qquad (4.37)
\end{aligned}
$$

$$
\begin{aligned}
2\dot{y}^T(t)(A+E_1F_1(t)H_1)Py(t) \ =\ & 2\dot{y}^T(t)APy(t)+2\dot{y}^T(t)E_1F_1(t)H_1Py(t) \\
\leq\ & 2\dot{y}^T(t)APy(t)+\epsilon_2\dot{y}^T(t)E_1^TE_1\dot{y}(t) \\
&+\epsilon_2^{-1}y^T(t)PH_1^TH_1Py(t), \qquad (4.38)
\end{aligned}
$$

$$
\begin{aligned}
2y^T(t)(D+E_2F_2(t)H_2)Py(t-h(t)) \ =\ & 2y^T(t)DPy(t-h(t))+2y^T(t)E_2F_2(t)H_2Py(t-h(t)) \\
\leq\ & 2y^T(t)DPy(t-h(t))+\epsilon_3y^T(t)E_2^TE_2y(t) \\
&+\epsilon_3^{-1}y^T(t-h(t))PH_2^TH_2Py(t-h(t)), \qquad (4.39)
\end{aligned}
$$

$$
\begin{aligned}
2\dot{y}^T(t)(D+E_2F_2(t)H_2)Py(t-h(t)) \ =\ & 2\dot{y}^T(t)DPy(t-h(t))+2\dot{y}^T(t)E_2F_2(t)H_2Py(t-h(t)) \\
\leq\ & 2\dot{y}^T(t)DPy(t-h(t))+\epsilon_4\dot{y}^T(t)E_2^TE_2\dot{y}(t) \\
&+\epsilon_4^{-1}y^T(t-h(t))PH_2^TH_2Py(t-h(t)), \qquad (4.40)
\end{aligned}
$$

$$-y^T(t)E_3F_3(t)H_3B^Ty(t) \leq \frac{\epsilon_5}{2}y^T(t)E_3^TE_3y(t) + \frac{\epsilon_5^{-1}}{2}y^T(t)BH_3^TH_3B^Ty(t), \quad (4.41)$$

and

$$-\dot{y}^T(t)E_3F_3(t)H_3B^Ty(t) \leq \frac{\epsilon_6}{2}\dot{y}^T(t)E_3^TE_3\dot{y}(t) + \frac{\epsilon_6^{-1}}{2}y^T(t)BH_3^TH_3B^Ty(t). \quad (4.42)$$

Therefore, from (4.37)-(4.42), it follows that

$$\begin{aligned}\dot{V}(t,x_t) + 2\alpha V(t,x_t) &\leq \zeta^T(t)[(1-\beta)\mathcal{M}_1 + \beta\mathcal{M}_2]\zeta(t) + y^T(t)M_3y(t) \\ &\quad + y^T(t-h(t))M_4y(t-h(t)),\end{aligned}$$

where

$$\begin{aligned}M_3 &= M_{311} + \epsilon_1^{-1}PH_1^TH_1P + \epsilon_2^{-1}PH_1^TH_1P + \frac{\epsilon_5^{-1}}{2}BH_3^TH_3B^T + \frac{\epsilon_6^{-1}}{2}BH_3^TH_3B^T, \\ M_4 &= M_{411} + \epsilon_3^{-1}PH_2^TH_2P + \epsilon_4^{-1}PH_2^TH_2P.\end{aligned}$$

From Lemma 2.3.20, the inequalities $M_3 < 0$ and $M_4 < 0$ are equivalent to $\mathcal{M}_3 < 0$ and $\mathcal{M}_4 < 0$, respectively. Therefore, system (4.1) if robustly $\alpha-$exponential stabilizability and satisfies $\|z(t)\|_2 < \gamma\|w(t)\|_2$ for all nonzero $w \in L_2[0,\infty)$ if the conditions (4.31)-(4.34) holds. The proof is thus completed.

Remark 4.2.2. *The proposed method has no restrictions on the derivatives of the time-varying delays, while traditional design methods require the derivatives to be less than 1. So the proposed method can deal with fast time-varying delays.*

4.3 Numerical example

In this section, we now provide an example to show the effectiveness of the result in Theorem 4.1.1 and Theorem 4.2.1.

Example 4.3.1. Consider the uncertain linear system with interval time-varying delay with the following parameters :

$$\begin{aligned}&\dot{x}(t) = (A + \Delta A(t))x(t) + (D + \Delta D(t))x(t-h(t)) + (B + \Delta B(t))u(t) + B_ww(t), \\ &z(t) = Cx(t) + C_dx(t-h(t)) + D_uu(t) + D_ww(t),\end{aligned}$$

where

$$A = \begin{bmatrix} -2 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad D = \begin{bmatrix} -0.1 & 0.1 \\ 0.1 & -0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$E_1 = E_2 = E_3 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad H_1 = H_2 \begin{bmatrix} 0.02 & 0 \\ 0 & 0.01 \end{bmatrix},$$

$$H_3 = \begin{bmatrix} 0.02 \\ 0 \end{bmatrix}, \quad C = C_d = \begin{bmatrix} 0.1 & 0 \\ 0 & 0 \end{bmatrix}, \quad D_u = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$D_w = \begin{bmatrix} 0.1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_w = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Solution: From the conditions (4.31)-(4.34) of Theorem 4.2.1, we let $\alpha = 0.6$, $\gamma = 6$, $h_1 = 0.5$ and $h_2 = 1.5$. By using the LMI Toolbox in MATLAB, we obtain

$$P = \begin{bmatrix} 0.5687 & 0.0209 \\ 0.0209 & 0.3532 \end{bmatrix}, \quad Q = \begin{bmatrix} 0.0157 & 0.0016 \\ 0016 & 0.0029 \end{bmatrix},$$

$$R = \begin{bmatrix} 0.1369 & 0.0223 \\ 0.0223 & 0.0398 \end{bmatrix}, \quad U = \begin{bmatrix} 0.3584 & 0.0307 \\ 0.0307 & 0.1276 \end{bmatrix},$$

$$\epsilon_1 = 2.1929, \quad \epsilon_2 = 2.4869 \quad \epsilon_3 = 2.7168, \quad \epsilon_4 = 0.2437, \quad \epsilon_5 = 1.1614, \quad \epsilon_6 = 1.4615,$$

with a stabilizing controller

$$u(t) = \begin{bmatrix} 0.5180 & -1.4108 \end{bmatrix} x(t). \tag{4.43}$$

Thus, the system (4.1) is 0.6−exponentially stabilizable and the value $\sqrt{\frac{\lambda_2}{\lambda_1}} = 0.8391$, so the solution of the closed-loop system satisfies

$$\|x(t, \phi)\| \leq 0.8391 e^{-0.6t} \|\phi\|, \quad \forall t \in \mathbb{R}^+.$$

We let $h(t) = 0.5 + |\sin(t)|$, $\phi(t) = [e^{\cos^2 t}, e^{\sin^2 t}]$, $\forall t \in [-1.5, 0]$. Since the delay function $h(t)$ is non-differentiable, the stability conditions derived in [2, 16, 32, 42] are not applicable to this system. Figure 4.1. shows the trajectories of solutions $x_1(t)$, and $x_2(t)$ of the uncertain linear system with interval time-varying delay (4.1) and feedback control. Figure 4.2. shows The trajectories of feedback control $u(t)$ depend on $x_1(t)$, and $x_2(t)$ of the uncertain linear system with interval time-varying delay (4.1) and feedback control $u(t) = \begin{bmatrix} 0.5180 & -1.4108 \end{bmatrix} x(t)$.
In Table 4.1, we shows the maximum allowable upper bounds h_2 of the uncertain linear system with interval time-varying delay (4.1) for different values of the lower bounds h_1 and decay rate.

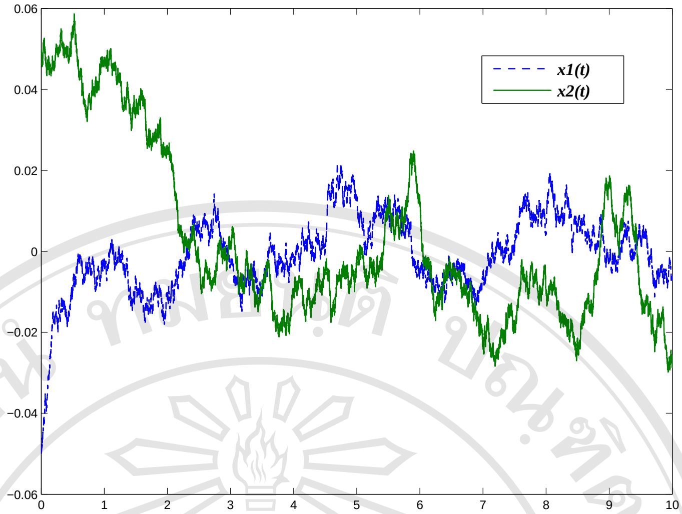

Figure 4.1: The trajectories of $x_1(t)$, and $x_2(t)$ of the uncertain linear system with interval time-varying delay (4.1) and feedback control deactivated.

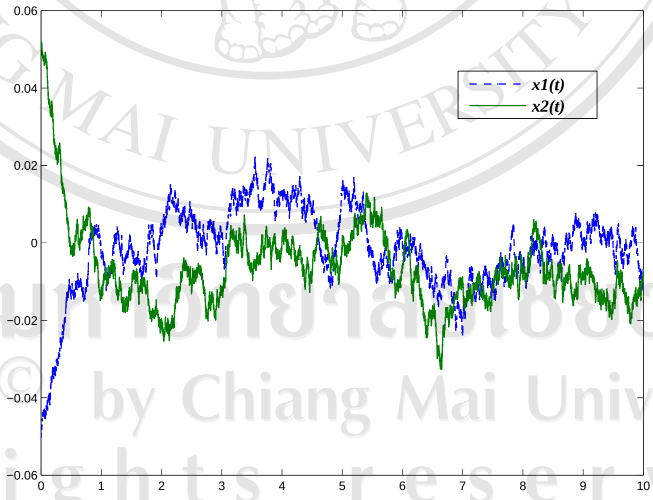

Figure 4.2: The trajectories of $x_1(t)$, and $x_2(t)$ of the uncertain linear system with interval time-varying delay (4.1) and feedback control activated..

α				
0.2	h_1	0	1	2
	max h_2	2.4700	2.6550	2.6760
0.4	h_1	0	1	
	max h_2	1.7200	1.8410	
0.6	h_1	0	1	
	max h_2	1.2030	1.2860	

Table 4.1: Maximum allowable upper bounds h_2 of the uncertain linear system with interval time-varying delay (4.1) for different values of the lower bounds h_1 and decay rate.