CHAPTER 5

Stabilization and H_∞ control problem for nonlinear system with time-varying delay

In this paper, the problem of asymptotical stabilization and H_∞ control of nonlinear system with time-varying delay is considered. By constructing a set of improved Lyapunov-Krasovskii functional includes some integral terms in the form $\int_{t-h}^{t}(h-t-s)^j \dot{x}^T(s)R_j\dot{x}(s)ds$ $(j=1,2)$ a matrix-based on quadratic convex, combined with Wirtinger inequalities and some useful integral inequality. H_∞ controller is designed via memoryless state feedback control and new sufficient conditions for the existence of the H_∞ state-feedback for the system are given in terms of linear matrix inequalities (LMIs). Numerical examples are given to illustrate the effectiveness of the obtained result.

5.1 Problem Formulation

Consider the following system with time-varying delays and control input:

$$\begin{cases} \dot{x}(t) = Ax(t) + Dx(t-\tau(t)) + Bu(t) + Cw(t) + f(t, x(t), x(t-\tau(t)), u(t), w(t)), \\ z(t) = Ex(t) + Gx(t-\tau(t)) + Fu(t) + g(t, x(t), x(t-\tau(t)), u(t)), \end{cases} \tag{5.1}$$

$$x(t_0+\theta) = \phi(\theta), \theta \in [-\tau_2, 0], (t_0, \phi) \in \mathcal{R}^+ \times \mathcal{C}([-\tau_2, t_0], \mathcal{R}^n),$$

where $x(t) \in \mathcal{R}^n$ is the state; $u(t) \in \mathcal{R}^m$ is the control input and $w(t) \in \mathcal{L}_2([0,\infty], \mathcal{R}^r)$ is a disturbance input; $z(t) \in \mathcal{R}^s$ is the observation output. The delay, $\tau(t)$ is time-varying continuous function that satisfies

$$0 \le \tau_1 \le \tau(t) \le \tau_2, \ \mu_1 \le \dot{\tau}(t) \le \mu_2.$$

Let $x^\tau = x(t-\tau(t))$, the nonlinear functions $f(t, x, x^\tau, u, w) : \mathcal{R}^+ \times \mathcal{R}^n \times \mathcal{R}^n \times \mathcal{R}^m \times \mathcal{R}^r \to \mathcal{R}^n$, $g(t, x, x^\tau, u) : \mathcal{R}^+ \times \mathcal{R}^n \times \mathcal{R}^n \times \mathcal{R}^m \to \mathcal{R}^s$ satisfy the following growth condition:

$$\begin{cases} \exists a, b, c, d > 0 : \|f(t, x, x^\tau, u, w)\| \le a\|x\| + b\|x^\tau\| + c\|u\| + d\|w\|, \forall (x, x^\tau, u, w) \\ \exists a_1, b_1, c_1 > 0 : \|g(t, x, x^\tau, u)\|^2 \le a_1\|x\|^2 + b_1\|x^\tau\|^2 + c_1\|u\|^2, \forall (x, x^\tau, u). \end{cases} \tag{5.2}$$

In this section, we give a design of memoryless H_∞ feedback control for the system (5.1). First, we present a delay-dependent asymptotical stabilizability analysis conditions for the nonlinear system with time-varying delay (5.1). Now, we operate the matrix-based quadratic convex approach with the integral inequalities in [53] to formulate a new stability criterion for the system (5.1). For our goal, we choose the following Lyapunov-Krasovskii functional

$$V(t, x_t, \dot{x}_t) = V_1(t) + V_2(t) + V_3(t) \tag{5.3}$$

where x_t denotes the function $x(s)$ defined on the interval $[t-\tau_2, t]$. Setting $P_1 = P^{-1}$, $y(t) = P_1 x(t)$ or $x(t) = Py(t)$ $\tau_{21} := \tau_2 - \tau_1$ and

$$\begin{aligned}
V_1(t) &:= y^T(t)Py(t) + \int_{t-\tau_1}^{t} \dot{y}^T(s)Q_0\dot{y}(s)ds \\
V_2(t) &:= \int_{t-\tau_1}^{t} [y^T(t) \;\; y^T(s)]Q_1[y^T(t) \;\; y^T(s)]^T ds \\
&\quad + \int_{t-\tau(t)}^{t-\tau_1} [y^T(t) \;\; y^T(s)]Q_2[y^T(t) \;\; y^T(s)]^T ds \\
&\quad + \int_{t-\tau_2}^{t-\tau(t)} [y^T(t) \;\; y^T(s)]Q_3[y^T(t) \;\; y^T(s)]^T ds \\
V_3(t) &:= \int_{t-\tau_1}^{t} [\tau_1(\tau_1 - t + s)\dot{y}^T(s)W_1\dot{y}(s) + (\tau_1 - t + s)^2\dot{y}^T(s)W_2\dot{y}(s)]ds \\
&\quad + \int_{t-\tau_2}^{t-\tau_1} [\tau_{21}(\tau_2 - t + s)\dot{y}^T(s)R_1\dot{y}(s) + (\tau_2 - t + s)^2\dot{y}^T(s)R_2\dot{y}(s)]ds
\end{aligned}$$

where $Q_j > 0, (j = 0, 1, 2, 3), W_1 > 0, W_2 > 0, R_1 > 0, R_2 > 0$ and P are real matrices to be determined. Before introducing the main theorem, for simplicity we set

$$\varepsilon = a + b + c + \tfrac{4d^2}{\gamma}.$$

Theorem 5.1.1. *Given $\gamma > 0$. Then system (5.1) is asymptotically stabilizable and satisfies $\|z(t)\|_2 < \gamma\|w(t)\|_2$ for all nonzero $w \in \mathcal{L}_2[0,\infty)$ if there exist positive definite matrices $P, Q_j > 0, (j = 0, 1, 2, 3), W_1, W_2, R_1, R_2, S_1, Z_1, Z_2, Z_3, N_1, N_2, N_3$ and Y such that the following LMIs holds*

$$\begin{bmatrix} \tilde{R}_1 & S_1 \\ S_1^T & \tilde{R}_1 \end{bmatrix} \geq 0, \begin{bmatrix} Z_i & N_i \\ N_i^T & R_2 \end{bmatrix} \geq 0, (i = 1, 2)$$

$$\begin{bmatrix} Z_3 & N_3 \\ N_3^T & W_2 \end{bmatrix} \geq 0, \qquad Z_1 \geq Z_2,$$

$$\begin{cases} \Xi_2(\tau_1,\mu_1)+\Xi_3(\tau_1)+\hat{\Xi}_4<0 \\ \Xi_2(\tau_1,\mu_2)+\Xi_3(\tau_1)+\hat{\Xi}_4<0 \\ \Xi_2(\tau_2,\mu_1)+\Xi_3(\tau_2)+\hat{\Xi}_4<0 \\ \Xi_2(\tau_2,\mu_2)+\Xi_3(\tau_2)+\hat{\Xi}_4<0 \end{cases} \tag{5.4}$$

where $\tilde{R}_1 = \mathrm{diag}\{R_1, 3R_1\}$;*and*

$$\Xi_2(\tau(t),\dot{\tau}(t)) := \Xi_{20} + [\tau(t)-\tau_1]\Xi_{21} + [\tau_2-\tau(t)]\Xi_{22} \tag{5.5}$$

$$\begin{aligned} \Xi_3(\tau(t)) :=& \tilde{\varphi}_1^T S_1\tilde{\varphi}_2 + \tilde{\varphi}_2^T S_1^T\tilde{\varphi}_1 - \tilde{\varphi}_1^T\tilde{R}_1^T\tilde{\varphi}_1 \\ &+ (\tau_2-\tau(t))^2(Z_1-Z_2) + (\tau_2-\tau(t))\Xi_{31} \\ &+ (\tau(t)-\tau_1)\Xi_{32} + \tau_{21}^2 Z_2 - \tilde{\varphi}_2^T\tilde{R}_1\tilde{\varphi}_2 \end{aligned} \tag{5.6}$$

$$\begin{aligned} \hat{\Xi}_4 :=& \tau_1^2 Z_3 - \tilde{\varphi}_3^T\tilde{W}_1\tilde{\varphi}_3 + e_9^T(\tau_1^2W_1+\tau_1^2W_2)e_9 + 2\tau_1N_3(e_1-e_7) + e_8^T(\tau_{21}^TR_1+\tau_{21}^TR_2)e_8 \\ &+ 2\tau_1(e_1-e_7)^TN_3^T + e_1^T(AP+PA^T+(BY+Y^TB^T)+\tfrac{4}{\gamma}CC^T+\varepsilon I)e_1 \\ &+ e_1^T(PD)e_2 + e_2^T(D^TP)e_1 + e_1^T(AP+YB^T)e_9 + e_9^T(PA^T+Y^TB)e_1 \\ &+ e_2^T(DP)e_9 + e_9^T(PD^T)e_2 + e_8^T(Q_0)e_8 + e_9^T(-2P+\tfrac{4}{\gamma}CC^T+\varepsilon I+Q_0)e_9 \\ &+ e_1^T(P)e_{10} + e_{10}^T(P)e_1 + e_1^T(Y^T)e_{11} + e_{11}^T(Y)e_1 + e_1^T(PE^T)e_{12} + e_{12}^T(EP)e_1 \\ &+ e_2^T(PG^T)e_{13} + e_{13}^T(GP)e_2 + e_2^T(P)e_{14} + e_{14}^T(P)e_2 + e_{10}^T(\tfrac{-1}{2a+4a_1}I)e_{10} \\ &+ e_{11}^T(\tfrac{-1}{2+2c+4c_1}I)e_{11} + e_{12}^T(-\tfrac{I}{3})e_{12} + e_{13}^T(-\tfrac{I}{3})e_{13} + e_{14}^T(\tfrac{-1}{2b+4b_1}I)e_{14} \end{aligned} \tag{5.7}$$

with $e_i (i = 1, 2, \ldots, 14)$ *denoting the i-th row-block vector of the* $14n \times 14n$ *identity matrix* $\tilde{W}_1 = \mathrm{diag}\{W_1, 3W_1\}$; *and*

$$\begin{aligned} \Xi_{20} :=& [e_1^T\ e_3^T](Q_2-Q_1)[e_1^T\ e_3^T]^T \\ &+ \tau_1[e_9^T\ 0]Q_1[e_1^T\ e_7^T]^T + \tau_1[e_1^T\ e_7^T]Q_1[e_9^T\ 0]^T \\ &- (1-\dot{\tau}(t))[e_1^T\ e_2^T](Q_2-Q_3)[e_1^T\ e_2^T]^T \\ &- [e_1^T\ e_4^T]Q_3[e_1^T\ e_4^T]^T + [e_1^T\ e_1^T]Q_1[e_1^T\ e_1^T]^T \\ \Xi_{21} :=& [e_1^T\ e_6^T]Q_2[e_9^T\ 0]^T + [e_9^T\ 0]Q_2[e_1^T\ e_6^T]^T \\ \Xi_{22} :=& [e_1^T\ e_5^T]Q_3[e_9^T\ 0]^T + [e_9^T\ 0]Q_3[e_1^T\ e_5^T]^T \\ \Xi_{31} :=& 2N_1(e_2-e_5) + 2N_2(e_3-e_2) \\ &+ 2(e_3-e_2)^TN_2^T + 2(e_2-e_5)^TN_1^T \\ \Xi_{32} :=& 2N_1(e_3-e_6) + 2(e_3-e_6)^TN_1^T \end{aligned}$$

$$\tilde{\varphi}_1 := \mathrm{col}\{e_2 - e_4, e_2 + e_4 - 2e_5\}$$
$$\tilde{\varphi}_2 := \mathrm{col}\{e_3 - e_2, e_3 + e_2 - 2e_6\}$$
$$\tilde{\varphi}_3 := \mathrm{col}\{e_1 - e_3, e_1 + e_3 - 2e_7\}$$

Moreover, the feedback control is given by

$$u(t) = YP^{-1}x(t), t \geq 0 \tag{5.8}$$

Proof. Taking the derivative of V along the solution of system (5.1), we obtain

$$\dot{V}_1 = 2y^T(t)P\dot{y}(t) + \dot{y}^T(t)Q_0\dot{y}(t) - \dot{y}^T(t-\tau_1)Q_0\dot{y}(t-\tau_1) \tag{5.9}$$

$$\begin{aligned}
\dot{V}_2 =& [y^T(t) \;\; y^T(t)]Q_1[y^T(t) \;\; y^T(t)]^T - [y^T(t) \;\; y^T(t-\tau_1)]Q_1[y^T(t) \;\; y^T(t-\tau_1)]^T \\
&+ \int_{t-\tau_1}^{t} 2[y^T(t) \;\; y^T(s)Q_1[\dot{y}(t)^T \;\; 0]^T ds + [y^T(t) \;\; y^T(t-\tau_1)]Q_2[y^T(t) \;\; y^T(t-\tau_1)]^T \\
&- (1-\dot{\tau(t)})[y^T(t) \;\; y^T(t-\tau(t))]Q_2[y^T(t) \;\; y^T(t-\tau(t))]^T \\
&+ 2\int_{t-\tau(t)}^{t-\tau_1} [y^T(t) \;\; y^T(s)]Q_2[\dot{y}^T(t) \;\; 0]^T ds - [y^T(t) \;\; y^T(t-\tau_2)]Q_3[y^T(t) \;\; y^T(t-\tau_2)]^T \\
&+ (1-\dot{\tau(t)})[y^T(t) \;\; y^T(t-\tau(t))]Q_3[y^T(t) \;\; y^T(t-\tau(t))]^T \\
&+ \int_{t-\tau_2}^{t-\tau(t)} 2[y^T(t) \;\; y^T(s)]Q_2[\dot{y}^T(t) \;\; 0]^T ds
\end{aligned}$$

$$\begin{aligned}
\dot{V}_3 =& \tau_1\dot{y}^T(t)\tau_1 W_1\dot{y}(t) + \tau_1^2\dot{y}^T(t)W_2\dot{y}(t) - \int_{t-\tau_1}^{t} \dot{y}^T(s)\tau_1 W_1\dot{y}(s)ds \\
&- 2\int_{t-\tau_1}^{t} (\tau_1 - t + s)\dot{y}^T(s)W_2\dot{y}(s)ds + (\tau_{21})\dot{y}^T(t-\tau_1)R_1\dot{y}(t-\tau_1) \\
&+ (\tau_{21})^2\dot{y}^T(t-\tau_1)R_2\dot{y}(t-\tau_1) - \int_{t-\tau_2}^{t-\tau_1} \tau_{21}\dot{y}^T(s)R_1\dot{y}(s)ds \\
&- 2\int_{t-\tau_2}^{t-\tau_1} (\tau_2 - t + s)\dot{y}^T(s)R_2\dot{y}(s)ds.
\end{aligned}$$

From (5.2) and Cauchy inequality, we get the following inequalities:

$$\begin{aligned}
&2x^T P_1 f(t, x, x_h, u, w) \leq 2\|P_1 x\|\|f(t, x, x_h, u, w)\| \\
&\leq 2\|P_1 x\|(a\|x\| + b\|x_h\| + c\|u\| + d\|w\|) \\
&\leq a\|P_1 x\|^2 + a\|x\|^2 + b\|P_1 x\|^2 + b\|P_1 x_h\|^2 + c\|P_1 x\|^2 \\
&+ c\|u\|^2 + \tfrac{4d^2}{\gamma}\|P_1 x\|^2 + \tfrac{\gamma}{4}\|w\|^2 \\
&= a\|x\|^2 + b\|x_h\|^2 + c\|u\|^2 + \tfrac{\gamma}{4}\|w\|^2 + \varepsilon\|P_1 x\|^2
\end{aligned} \tag{5.10}$$

Similarly,

$$2\dot{x}^T P_1 f(t, x, x_\tau, u, w) \leq a\|x\|^2 + b\|x_\tau\|^2 + c\|u\|^2 + \tfrac{\gamma}{4}\|w\|^2 + \varepsilon\|P_1\dot{x}\|^2,$$

$$2x(t)^T P_1 Cw \leq \frac{\gamma}{4}\|w\|^2 + \frac{4}{\gamma}x(t)^T P_1 CC^T P_1 x(t),$$

and

$$2\dot{x}(t)^T P_1 Cw \leq \frac{\gamma}{4}\|w\|^2 + \frac{4}{\gamma}\dot{x}(t)^T P_1 CC^T P_1 \dot{x}(t),$$

By using the following identity relation:

$$-2\dot{x}^T(t)P_1[\dot{x}(t) - Ax(t) - Dx(t-\tau(t)) - Bu(t) - Cw(t) - f(\cdot)] = 0$$

we obtain the following,

$$\begin{aligned}
0 = &- 2\dot{x}^T(t)P_1[\dot{x}(t) - Ax(t) - Dx(t-\tau(t)) - Bu(t)\\
&- Cw(t) - f(\cdot)]\\
\leq &- 2\dot{x}^T(t)P_1[\dot{x}(t) - Ax(t) - Dx(t-\tau(t))\\
&- BYP_1x(t))] + 2\dot{x}^T(t)P_1Cw(t) + 2\dot{x}^T(t)P_1f(\cdot)\\
\leq &- 2\dot{y}^T(t)P\dot{y}(t) + 2\dot{y}^T(t)APy(t)\\
&+ 2\dot{y}^T(t)DPy(t-\tau(t)) + 2\dot{y}^T(t)BYy(t))\\
&+ \frac{4}{\gamma}\dot{y}(t)^T CC^T\dot{y}(t) + ay(t)^T P^2 y(t)\\
&+ by^T(t-\tau(t))P^2y(t-\tau(t)) + cy^T(t)Y^TYy(t)\\
&+ \frac{\gamma}{2}w(t)^T w(t) + \varepsilon\dot{y}^T(t)\dot{y}(t).
\end{aligned} \tag{5.11}$$

From (5.9) and (5.10)-(5.11), $\dot{V}_1$ is estimated as

$$\begin{aligned}
\dot{V}_1 \leq &y^T(t)[AP + PA^T + (BY + Y^TB^T)\\
&+ \frac{4}{\gamma}CC^T + aP^2 + cY^TY + \varepsilon]y(t)\\
&+ \frac{\gamma}{2}\|w(t)^2\| + 2y^T(t)[DP]y^T(t-\tau(t))\\
&+ by^T(t-\tau(t))P^2y(t-\tau(t)\\
&- 2\dot{y}^T(t)P\dot{y}(t) + 2\dot{y}^T(t)APy(t)\\
&+ 2\dot{y}^T(t)DPy(t-\tau(t)) + 2\dot{y}^T(t)BYy(t))\\
&+ \frac{4}{\gamma}\dot{y}(t)^T CC^T\dot{y}(t) + ay(t)^T P^2 y(t)\\
&+ by^T(t-\tau(t))P^2y(t-\tau(t)) + cy^T(t)Y^TYy(t)\\
&+ \frac{\gamma}{2}w(t)^T w(t) + \varepsilon\dot{y}^T(t)\dot{y}(t)\\
&+ \dot{y}^T(t)Q_0\dot{y}(t) - \dot{y}^T(t-\tau_1)Q_0\dot{y}(t-\tau_1)\\
&= \xi^T(t)\Xi_1\xi(t)
\end{aligned} \tag{5.12}$$

where

$$\xi(t) := \text{col}\{y(t), y(t-\tau(t)), y(t-\tau_1), y(t-\tau_2), \nu_1(t), \nu_2(t), \nu_3(t), \dot{y}(t-\tau_1), \dot{y}(t)\}, \quad (5.13)$$

$$\begin{aligned}\Xi_1 :=& e_1^T(AP + PA^T + (BY + Y^TB^T) + \tfrac{4}{\gamma}CC^T + 2aP^2 + 2cY^TY + \varepsilon I)e_1 \\ &+ e_1^T(PD)e_2 + e_2^T(D^TP)e_1 + e_1^T(AP + YB^T)e_9 \\ &+ e_9^T(PA^T + Y^TB)e_1 + e_2^T(DP)e_9 + e_9^T(PD^T)e_2 \\ &+ e_9^T(-2P + \tfrac{4}{\gamma}CC^T + \varepsilon I + Q_0)e_9 + e_2^T(2bP^2)e_2 + e_8^T(-Q_0)e_8. \quad (5.14)\end{aligned}$$

With the consideration of the three terms of $\dot{V}_2(t)$, we obtained the following inequality:

$$\begin{aligned}&\int_{t-\tau_1}^{t} 2[y^T(t) \;\; y^T(s)]Q_1[\dot{y}(t)^T \;\; 0]^T ds \\ &\leq 2[\int_{t-\tau_1}^{t} y^T(t)ds \;\; \int_{t-\tau_1}^{t} y^T(s)ds]Q_1[\dot{y}^T(t) \;\; 0]^T \\ &= 2\tau_1[y^T(t) \;\; \nu_3^T]Q_1[\dot{y}^T(t) \;\; 0]^T,\end{aligned}$$

$$\begin{aligned}&\int_{t-\tau(t)}^{t-\tau_1} 2[y^T(t) \;\; y^T(s)]Q_2[\dot{y}(t)^T \;\; 0]^T ds \\ &\leq 2[\int_{t-\tau(t)}^{t-\tau_1} y^T(t)ds \;\; \int_{t-\tau(t)}^{t-\tau_1} y^T(s)ds]Q_2[\dot{y}^T(t) \;\; 0]^T \\ &= 2(\tau(t)-\tau_1)[y^T(t) \;\; \nu_2^T]Q_2[\dot{y}^T(t) \;\; 0]^T\end{aligned}$$

and

$$\begin{aligned}&\int_{t-\tau_2}^{t-\tau(t)} 2[y^T(t) \;\; y^T(s)]Q_3[\dot{y}(t)^T \;\; 0]^T ds \\ &\leq 2[\int_{t-\tau_2}^{t-\tau(t)} y^T(t)ds \;\; \int_{t-\tau_2}^{t-d(t)} y^T(s)ds]Q_3[\dot{y}^T(t) \;\; 0]^T \\ &= 2(\tau_2-\tau(t))[y^T(t) \;\; \nu_1^T]Q_3[\dot{y}^T(t) \;\; 0].\end{aligned}$$

Therefore, the estimation of $\dot{V}_2(t)$ is estimated as

$$\begin{aligned}\dot{V}_2(t) &\leq \Xi_{20} + (\tau(t)-\tau_1)\Xi_{21} + (\tau_2-\tau(t))\Xi_{22} \\ &= \xi^T(t)\Xi_2(\tau(t), \dot{\tau}(t))\xi^T(t). \quad (5.15)\end{aligned}$$

where Ξ_2 is defined in (5.5). Similarly, $\dot{V}_3(t)$ is estimate as

$$\dot{V}_3(t) = \xi^T(t)\Xi_{30}\xi(t) + \delta_1(t) + \delta_2(t)$$

where

$$\Xi_{30} := e_9^T(\tau_1^2W_1 + \tau_1^2W_2)e_9 + e_8^T(\tau_{21}^2R_1 + \tau_{21}^2R_2)e_8,$$

$$\delta_1(t) = -\int_{t-\tau_2}^{t-\tau_1} \tau_{21}\dot{y}^T(s)R_1\dot{y}(s)ds - 2\int_{t-\tau_2}^{t-\tau_1}(\tau_2 - t + s)\dot{y}^T(s)R_2\dot{y}(s)ds,$$

$$\delta_2(t) = -\int_{t-\tau_1}^{t} \tau_1\dot{y}^T(s)W_1\dot{y}(s)ds - 2\int_{t-\tau_1}^{t}(\tau_1 - t + s)\dot{y}^T(s)W_2\dot{y}(s)ds.$$

By Lemma 2.3.26 and Lemma 2.3.27, we obtain the following

$$-(\tau_2 - \tau_1)\int_{t-\tau_2}^{t-\tau_1} \dot{y}^T(s)R_1\dot{y}(s)ds \leq 2\psi_{11}^T S_1\psi_{21} - \psi_{11}^T\tilde{R}_1\psi_{11} - \psi_{21}^T\tilde{R}_1\psi_{21},$$

where $\tilde{R}_1 := \mathrm{diag}\{R_1, 3R_1\}$; and

$$\psi_{11} := \begin{bmatrix} y(t-\tau(t)) - y(t-\tau_2) \\ y(t-\tau(t)) + y(t-\tau_2) - 2\nu_1(t) \end{bmatrix},$$
$$\psi_{21} := \begin{bmatrix} y(t-\tau_1) - y(t-\tau(t)) \\ y(t-\tau_1) + y(t-\tau(t)) - 2\nu_2(t). \end{bmatrix},$$

and

$$\begin{aligned}-2\int_{t-\tau_2}^{t-\tau_1}(\tau_2 - t + s)\dot{y}^T(s)R_2\dot{y}(s)ds \leq & -2\{\frac{1}{2}(\tau_2 - \tau(t))^2\xi^T(t)Z_1\xi(t) \\ & + 2(\tau_2 - \tau(t))\xi^T(t)N_1[y(t-\tau(t)) - \nu_1] \\ & + \tfrac{1}{2}[(\tau_{21})^2(\tau_2 - \tau(t))^2]\xi^T(t)Z_2\xi(t) \\ & + 2\xi^T(t)N_2[(\tau_2 - \tau(t))[y(t-\tau_1) - y(t-\tau(t))] \\ & + (\tau(t) - \tau_1)[y(t-\tau_1) - \nu_2]]\}.\end{aligned}$$

Thus,

$$\begin{aligned}\delta_1(t) \leq & [2\psi_{11}^T S_1\psi_{21} - \psi_{11}^T\tilde{R}_1\psi_{11} \\ & - \psi_{21}^T\tilde{R}_1\psi_{21} - 2\{\frac{1}{2}(\tau_2 - \tau(t))^2\xi^T(t)Z_1\xi(t) \\ & + 2(\tau_2 - \tau(t))\xi^T(t)N_1[y(t-\tau(t)) - \nu_1] \\ & + \tfrac{1}{2}[(\tau_2 - \tau_1)^2(\tau_2 - \tau(t))^2]\xi^T(t)Z_2\xi(t) \\ & + 2\xi^T(t)N_2[(\tau_2 - \tau(t))[y(t-\tau_1) - y(t-\tau(t))] \\ & + (\tau(t) - \tau_1)[y(t-\tau_1) - \nu_2]]\}. \\ = & \xi^T(t)\Xi_3(\tau(t))\xi(t) \end{aligned} \tag{5.16}$$

where $\Xi_3(\tau(t))$ is given in (5.6). From Lemma 2.3.24 and Lemma 2.3.25, we obtain

$$-\int_{t-\tau_1}^{t} \dot{y}^T(s)\tau_1 W_1\dot{y}(s)ds \leq [y(t) - y(t-\tau_1)]^T W_1[y(t) - y(t-\tau_1)] + 3\tilde{\Omega_1}^T W_1\tilde{\Omega_1}$$

and

$$-2\int_{t-\tau_1}^{t}(\tau_1 - t + s)\dot{y}^T(s)W_1\dot{y}(s)ds \le -\tau_1^2\xi^T(t)Z_3\xi(t) - 2\tau_1\xi^T(t)N_3[y(t) - \nu_3].$$

From which it follows that

$$\begin{aligned}\delta_2(t) \le&[y(t) - y(t-\tau_1)]^T W_1[y(t) - y(t-\tau_1)]\\ &+ 3\tilde{\Omega_1}^T W_1\tilde{\Omega_1} - \tau_1^2\xi^T(t)M_1\xi(t) - 2\tau_1\xi^T(t)N_1[y(t) - \nu_3],\\ =&\xi^T(t)\Xi_{33}\xi(t) \end{aligned}\tag{5.17}$$

where

$$\begin{aligned}\tilde{\Omega_1} =&y(t) + y(t-\tau_1) - 2\nu_3\\ \tilde{\Omega_2} =&y(t-\tau(t)) + y(t+\tau_2) - \nu_1\\ \tilde{\Omega_3} =&y(t-\tau_1) + y(t-\tau(t)) - \nu_2,\\ \Xi_{33} :=& -\tilde{\varphi}_3^T\text{diag}\{W_1, 3W_1\}\tilde{\varphi}_3\\ &+ \tau_1^2 Z_3 + 2\tau_1 N_3(e_1 - e_7) + 2\tau_1(e_1 - e_7)^T N_3^T\end{aligned}$$

Hence, from (5.16) and (5.17), we obtain

$$\dot{V}_3 \le \xi^T(t)[\Xi_3(\tau(t)) + \Xi_4]\xi(t) \tag{5.18}$$

where $\Xi_4 := \Xi_{30} + \Xi_{33}$. From (5.10),(5.15) and (5.18), we obtain $\dot{V}(t, y_t, \dot{y}_t)$ along the solution of the system (5.1) as

$$\dot{V}(t, x_t, \dot{x}_t) \le \xi^T\Delta(\tau(t), \dot{\tau}(t))\xi(t) \tag{5.19}$$

where

$$\begin{aligned}\Delta(\tau(t), \dot{\tau}(t)) =&\hat{\Xi}_1 + \Xi_2(\tau(t), \dot{\tau}(t)) + \Xi_3(\tau(t)) + \Xi_4\\ =&\Xi_2(\tau(t), \dot{\tau}(t)) + \Xi_3(\tau(t)) + \hat{\Xi}_4\end{aligned}$$

Therefore, we have

$$\begin{aligned}\dot{V}(t, x_t) \le&\xi^T(t)\Delta(\tau(t), \dot{\tau}(t))\xi(t) + \gamma\|w(t)\|^2\\ &+ y^T(t)[3PE^TEP + 4a_1P^2 + (2+4c_1)Y^TY]y(t)\\ &- y^T(t)[3PE^TEP + 4a_1P^2 + (2+4c_1)Y^TY]y(t)\\ &+ y^T(t-\tau(t))[3PG^TGP + 4b_1P^2]y(t-\tau(t))\\ &- y^T(t-\tau(t))[3PG^TGP + 4b_1P^2]y(t-\tau(t))\end{aligned}$$

$$
\begin{aligned}
=&\xi^T(t)\hat{\Delta}(\tau(t),\dot{\tau}(t))\xi(t)+\gamma\|w(t)\|^2\\
&-y^T(t)[3PE^TEP+4a_1P^2+(2+4c_1)Y^TY]y(t)\\
&-y^T(t-\tau(t))[3PG^TGP+4b_1P^2]y(t-\tau(t)),
\end{aligned}
\tag{5.20}
$$

where

$$
\begin{aligned}
\hat{\Delta}(\tau(t),\dot{\tau}(t)) &= \Xi_2(\tau(t),\dot{\tau}(t))+\Xi_3(\tau(t))+(\hat{\Xi}_1+\Xi_4)\\
&= \Xi_2(\tau(t),\dot{\tau}(t))+\Xi_3(\tau(t))+\hat{\Xi}_4,
\end{aligned}
$$

$$
\hat{\Xi}_1=\Xi_1+e_1^T((2+4c_1)Y^TY+3PE^TEP+4a_1)e_1+e_2^T(3PE^TEP+4b_1P^2)e_2
$$

and $\hat{\Xi}_4=\hat{\Xi}_1+\Xi_4$ is defined in (5.7). Observe that $\hat{\Delta}(\tau(t),\dot{\tau}(t))$ may be rewritten as

$$
\hat{\Delta}(\tau(t),\dot{\tau}(t))=\tau^2(t)\Delta_0+\tau(t)\Delta_1+\Delta_2 \tag{5.21}
$$

where $\Delta_0=Z_1-Z_2$ and Δ_1,Δ_2 are $\tau(t)$-independent real matrices. By Lemma (2.3.29), if $Z_1-Z_2\geq 0$ and the inequalities in (5.4) hold, then $\hat{\Delta}(\tau(t),\dot{\tau}(t))<0$, $\forall\tau(t)\in[\tau_1,\tau_2]$, $\forall\dot{\tau}(t)\in[\mu_1,\mu_2]$. Moreover, $\hat{\Delta}(\tau(t),\dot{\tau}(t))$ may be rewritten as a convex combination of $\dot{\tau}(t)$ as following,

$$
\hat{\Delta}(\tau(t),\dot{\tau}(t))=(1-\dot{\tau}(t))\Box_0+\dot{\tau}(t)\Box_1+\Box_2 \tag{5.22}
$$

where $\Box_0=Q_2-Q_3$ and $\Box_1,\Box_2$ are $\dot{\tau}(t)$-independent real matrices. By utilizing the Schur complement lemma, it follows from (5.21) and (5.22) and (6) that $\hat{\Delta}(\tau(t),\dot{\tau}(t))<0$ holds. From which it follows from inequality (5.20) that

$$
\begin{aligned}
\dot{V}(t,x_t)\leq&\gamma w(t)^Tw(t)-y^T(t)[3PE^TEP+4a_1P^2+(2+4c_1)Y^TY]y(t)\\
&-y^T(t-\tau(t))[3PG^TGP+4b_1P^2]y(t-\tau(t)).
\end{aligned}
\tag{5.23}
$$

Letting $w(t)=0$ and from

$$
\begin{aligned}
-y^T(t)[3PE^TEP+4a_1P^2+(2+4c_1)Y^TY]y(t)\leq 0\\
-y^T(t-\tau(t))[3PG^TGP+4b_1P^2]y(t-\tau(t))\leq 0
\end{aligned}
$$

there exists a scalar $\varepsilon_3>0$ such that

$$
\dot{V}(t,x_t)\leq-\varepsilon_3\|x(t)\|^2<0,\quad \forall t\geq 0.
$$

Therefore, the system (5.1) with $w(t)\equiv 0$ is asymptotically stable. To complete the proof of theorem, next we consider the H_∞ performance $\|z\|_2<\gamma\|w\|_2$. By assuming that

$x(t) = 0$, $t \in [-\tau_2, t_0]$, it follows from definition of $z(t)$ that

$$\begin{aligned}
\|z\|^2 \leq & \|E(x)\|^2 + \|Gx(t-\tau(t))\|^2 + \|u(t)\|^T \\
& + 2x^T(t)E^TGx(t-\tau(t) + 2x^T(t)E^Tg(\cdot) \\
& + 2x^T(t-\tau(t))E^Tg(\cdot) + 2u^T(t)F^Tg(\cdot) + \|g(\cdot)\|^2 \\
\leq & 3\|Ex(t)\|^2 + 3\|Gx(t-\tau(t))\|^2 + 2\|u(t)\|^2 + 4\|g(\cdot)\|^2 \\
\leq & x^T(t)[3E^TE + 4a_1]x(t) + x^T(t-\tau(t)) \\
& [3G^TG + 4b_1]x(t-\tau(t)) + [2+4c_1]\|u(t)\|^2 \\
= & y^T(t)[3PE^TEP + 4a_1P^2 + (2+4c_1)Y^TY]y(t) \\
& + y^T(t-\tau(t))[3PG^TGP + 4b_1P^2]y(t-\tau(t)).
\end{aligned} \tag{5.24}$$

From (5.23), we obtain

$$\begin{aligned}
\dot{V}(t, x_t) \leq & \gamma w(t)^T w(t) - y^T(t)[3PE^TEP + 4a_1P^2 + (2+4c_1)Y^TY]y(t) \\
& - y^T(t-\tau(t))[3PG^TGP + 4b_1P^2]y(t-\tau(t)).
\end{aligned}$$

From estimations of $\dot{V}(t, x_t)$ and $\|z(t)\|^2$ in (5.20) and (5.24), we obtain

$$\dot{V}(t, x_t) + z^T(t)z(t) - \gamma^2 w^T(t)w(t) < 0.$$

Integrating both sides of above equation from t_0 to t, we get

$$\int_{t_0}^{t} \Big[\dot{V}(t, x_t) + z^T(t)z(t) - \gamma^2 w^T(t)w(t)\Big] dt < 0.$$

It follows that

$$\begin{aligned}
\int_{t_0}^{t} \Big[z^T(t)z(t) - \gamma^2 w^T(t)w(t)\Big] dt & \leq V(t_0, x_{t_0}) - V(t, x_t) \\
& \leq 0.
\end{aligned} \tag{5.25}$$

Therefore, under zero initial condition $x(t) = 0$, $t \in [-\tau_2, t_0]$, by letting $t \to +\infty$ in (5.25), we get

$$\int_{t_0}^{\infty} z^T(t)z(t)dt < \gamma^2 \int_{t_0}^{\infty} w^T(t)w(t)dt$$

which gives $\|z\|_2 < \gamma\|w\|_2$. This completes the proof.

When $\tau_1 = 0, B = 0, C = 0$, the nonlinear function is $f(t, x, x^\tau)$, and (5.1) reduces to

$$\dot{x}(t) = Ax(t) + Dx(t-\tau(t)) + f(t, x(t), x(t-\tau(t))), \tag{5.26}$$

where $0 \leq \tau(t) \leq \tau_2,\ \mu_1 \leq \dot{\tau}(t) \leq \mu_2$. In the following, we present a stability criterion for the case when $\tau_1 = 0$. We consider the following LFK candidate

$$\begin{aligned}\hat{V}(t, x_t, \dot{x}_t) &= x^T(t)Px(t)\\ &+ \int_{t-\tau(t)}^{t} [x^T(t) \;\; x^T(s)]Q_2[x^T(t) \;\; x^T(s)]^T ds\\ &+ \int_{t-\tau_2}^{t-\tau(t)} [x^T(t) \;\; x^T(s)]Q_3[x^T(t) \;\; x^T(s)]^T ds\\ &+ \int_{t-\tau_2}^{t} \Big\{\tau_2(\tau_2 - t + s)\dot{x}^T(s)R_1\dot{x}(s)\\ &+ (\tau_2 - t + s)^2\dot{x}^T(s)R_2\dot{x}(s)\Big\}\, ds. \end{aligned} \tag{5.27}$$

Corollary 5.1.2. *For given scalars τ_2, μ_1, μ_2, a and b (5.26) is asymptotically stable if there exist symmetric positive definite matrices $P, R_1, R_2, N_1, N_2, Z_1, Z_2, Q_2, Q_3, S_1$ such that the following LMIs hold:*

$$\tau_2\Upsilon_1 + (1-\mu_1)\Upsilon_3 + \Upsilon_4 < 0$$
$$\tau_2\Upsilon_1 + (1-\mu_2)\Upsilon_3 + \Upsilon_4 < 0$$
$$\tau_2\Upsilon_2 + (1-\mu_1)\Upsilon_3 + \Upsilon_4 < 0$$
$$\tau_2\Upsilon_2 + (1-\mu_2)\Upsilon_3 + \Upsilon_4 < 0.$$

$$\begin{bmatrix} \tilde{R}_1 & S_1 \\ S_1^T & \tilde{R}_1 \end{bmatrix} \geq 0, Z_1 \geq Z_2,$$

$$\begin{bmatrix} Z_i & N_i \\ N_i^T & R_2 \end{bmatrix} \geq 0, (i = 1, 2),$$

$$Q_j = \begin{bmatrix} Q_{j1} & Q_{j2} \\ * & Q_{j3} \end{bmatrix} \geq 0, (i = 2, 3),$$

where $\tilde{R}_1 \triangleq diag\{R_1, 3R_1\}$

$$\begin{aligned}\Upsilon_1 :=& [\tilde{e}_1^T \; \tilde{e}_4^T]Q_3[\tilde{e}_6^T \; 0]^T + [\tilde{e}_6^T \; 0]Q_3[\tilde{e}_1^T \; \tilde{e}_4^T]^T\\ &+ h_2(Z_1 - Z_2) + 2N_1[\tilde{e}_2 - \tilde{e}_4]\\ &+ 2[\tilde{e}_2 - \tilde{e}_4]^T N_1^T + 2N_2[\tilde{e}_1 - \tilde{e}_2]\\ &+ 2[\tilde{e}_1 - \tilde{e}_2]^T N_2^T\end{aligned}$$

$$
\begin{aligned}
\Upsilon_2 :=& [\tilde{e}_1^T \ \tilde{e}_5^T] Q_2 [\tilde{e}_6^T \ 0]^T + [\tilde{e}_6^T \ 0] Q_2 [\tilde{e}_1^T \ \tilde{e}_5^T]^T \\
& + 2N_2[\tilde{e}_1 - \tilde{e}_5] + 2[\tilde{e}_1 - \tilde{e}_5]^T N_2^T \\
\Upsilon_3 :=& [\tilde{e}_1 \ \tilde{e}_2](Q_3 - Q_2)[\tilde{e}_1 \ \tilde{e}_2]^T \\
\Upsilon_4 :=& \tilde{e}_1^T(A^TP + PA)\tilde{e}_1 + \tilde{e}_1^T(a+b)I\tilde{e}_1 \\
& + \tilde{e}_1^T(PD)\tilde{e}_2 + \tilde{e}_2^T(D^TP)\tilde{e}_1 \\
& + \tilde{e}_1^T(PA)\tilde{e}_6 + \tilde{e}_6^T(A^TP)\tilde{e}_1 \\
& + \tilde{e}_1^T(P^T)\tilde{e}_7 + \tilde{e}_7^T(P)\tilde{e}_1 \\
& + \tilde{e}_2^T(PD)\tilde{e}_6 + \tilde{e}_6^T(D^TP)\tilde{e}_2 \\
& + \tilde{e}_2^T(P^T)\tilde{e}_8 + \tilde{e}_8^T(P)\tilde{e}_2 \\
& + \tilde{e}_6^T(-2P)\tilde{e}_6 + \tilde{e}_6^T(a+b)I\tilde{e}_6 \\
& + \tilde{e}_7^T(\frac{-1}{2a})I\tilde{e}_7 + \tilde{e}_8^T(\frac{-1}{2b})I\tilde{e}_8 \\
& + [\tilde{e}_1^T \ \tilde{e}_1^T] Q_2 [\tilde{e}_1^T \ \tilde{e}_1^T]^T \\
& - [\tilde{e}_1^T \ \tilde{e}_3^T] Q_3 [\tilde{e}_1^T \ \tilde{e}_3^T]^T + h_2^2 \tilde{e}_6^T(R_1 + R_2)\hat{e}_6 \\
& + \Theta_1^T S_1 \Theta_2 + \Theta_2^T S_1^T \Theta_1 - \Theta_1^T \tilde{R}_1 \Theta_1 - \Theta_2^T \tilde{R}_1 \Theta_2 \\
& + h_2^2 Z_2, \\
\Theta_1 :=& \operatorname{col}\{(\tilde{e}_2 - \tilde{e}_3), (\tilde{e}_2 + \tilde{e}_3 - 2\tilde{e}_4)\} \\
\Theta_2 :=& \operatorname{col}\{(\tilde{e}_1 - \tilde{e}_2), (\tilde{e}_1 + \tilde{e}_2 - 2\tilde{e}_5)\}
\end{aligned}
$$

with $\tilde{e}_1 = [I\ 0\ 0\ 0\ 0\ 0\ 0\ 0],_2 = [0\ I\ 0\ 0\ 0\ 0\ 0\ 0], ..., \tilde{e}_8 = [0\ 0\ 0\ 0\ 0\ 0\ 0\ I]$.

Proof. The proof is the same as in Theorem 5.1.1 by using LKF (5.27). The proof is omitted. □

5.2 Numerical examples

In this section, we provide examples to show the effectiveness of the result in Theorem 5.1.1 and Corollary 5.1.2.

Example 5.2.1. Consider the nonlinear system with interval time-varying delays (5.1) which was considered in [50] where

$$A = \begin{bmatrix} -1.3 & 0.3 \\ 0.5 & 0.1 \end{bmatrix}, \qquad D = \begin{bmatrix} -0.01 & 0.02 \\ 0.03 & -0.04 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.2 & 0 \\ 0.3 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} -0.02 & 0.01 \\ 0.02 & -0.03 \end{bmatrix},$$

$$E = G = \begin{bmatrix} 0.06 & -0.06 \\ -0.08 & 0.08 \end{bmatrix}, \qquad F = \begin{bmatrix} 0.8 & 0 \\ 0.6 & 0 \end{bmatrix},$$

$$f(\cdot) = g(\cdot) = 0.01 \begin{bmatrix} \sqrt{x_1^2(t) + x_2^2(t - \tau(t))} \\ \sqrt{x_2^2(t) + x_1^2(t - \tau(t))} \end{bmatrix},$$

$a = b = c = d = a_1 = b_1 = c_1 = 0.01$, $\tau_1 = 0.3, \tau_2 = 0.5, \mu_1 = -0.1, \mu_2 = 0.1, \gamma = 4$. By using LMI Toolbox in Matlab, the LMI in Theorem (5.1.1) is feasible. From (5.8) the H_∞ controller feedback gain can be computed as

$$K = YP^{-1} = \begin{bmatrix} -3.4638 & -6.8069 \\ -4.3243 & -4.1846 \end{bmatrix}.$$

For simulation, we choose $\tau(t) = 0.4 + 0.1\cos(t)$, $\phi(t) = [-5\cos(t), 3\cos(t)]$, $\forall t \in [0, 10]$ and $w(t)$ is the Gaussian noise which set of random numbers with fluctuation range between -1 and 1. Figure 1 shows the trajectories of solutions $x_1(t)$ and $x_2(t)$ of the system (5.1) without feedback control ($u(t) = 0$) and Figure 2 shows the trajectories of solutions $x_1(t)$, and $x_2(t)$ of the system with feedback control $u(t)$

In Table I, we shows the value of minimum γ with $\mu_1 = -0.1$ and $\mu_2 = 0.1$ by using Theorem (5.1.1).Table II, we shows the value of minimum γ with $\mu_1 = 0.05$ and $\mu_2 = 0.1$ by using Theorem (5.1.1)

Example 5.2.2. Consider the following nonlinear system with interval time-varying delays which was considered in [57]:

$$\dot{x}(t) = Ax(t) + Dx(t - \tau(t)) + f(t, x(t), x(t - \tau(t))),$$

where

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, D = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix},$$

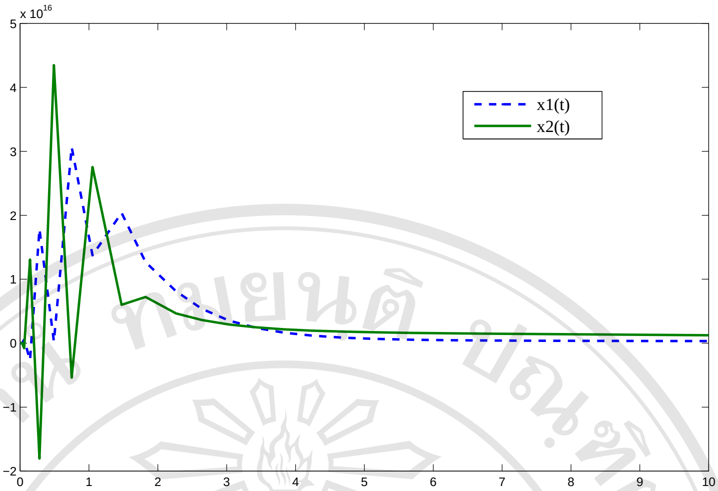

Figure 5.1: The trajectories of $x_1(t)$, and $x_2(t)$ of 5.1 in Example 5.2.1 without feedback control.

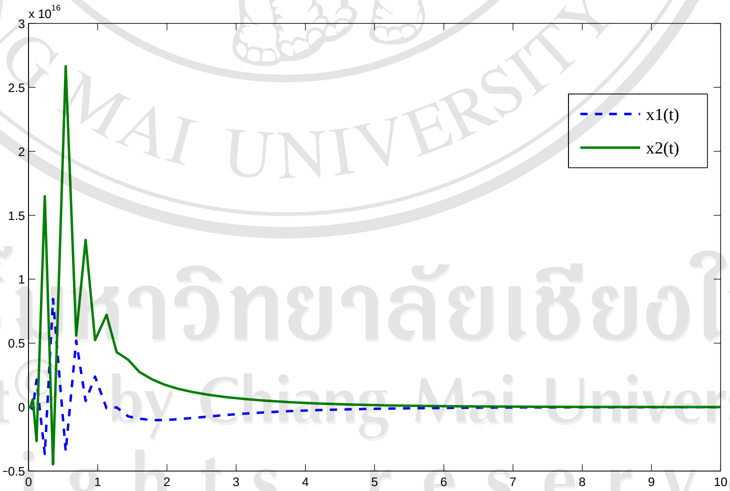

Figure 5.2: The trajectories of $x_1(t)$, and $x_2(t)$ of 5.1 in Example 5.2.1 with feedback control.

$a = b = c = d = a_1 = b_1 = c_1 = 0.01$			
Method	τ_1	τ_2	γ_{min}
By theorem 5.1.1	0.1	0.3	0.2377
	0.1	0.5	0.2474

$a = b = c = d = a_1 = b_1 = c_1 = 0.05$			
Method	τ_1	τ_2	γ_{min}
By theorem 5.1.1	0.1	0.3	0.8991
	0.1	0.5	0.9643

Table 5.1: The value of the minimum allowable disturbance attenuation γ with $\mu_1 = -0.1$ and $\mu_2 = 0.1$.

$a = b = c = d = a_1 = b_1 = c_1 = 0.01$			
Method	τ_1	τ_2	γ_{min}
By theorem 5.1.1	0.1	0.5	0.2487

Table 5.2: The value of the minimum allowable disturbance attenuation γ with $\mu_1 = 0.05$ and $\mu_2 = 0.1$.

	a=0.1	b=0.1	
Method	μ_1	μ_2	$max\ \tau_2$
By corollary 5.1.2	-0.1	0.1	2.1443
	-0.3	0.3	2.1126
	-0.5	0.5	2.0833
	-0.8	0.8	2.0473
	-1	1	2.0332

Table 5.3: The value of maximum allowable delay τ_2 in Example 5.2.2 with $\tau_1 = 0$.

$a = b = 0.1$ and $0 \leq \tau(t) \leq \tau_2,\ \mu_1 \leq \dot{\tau}(t) \leq \mu_2$. By using LMI Toolbox in Matlab, the LMI in corollary (5.1.2) is feasible. Table 5.3 shows the maximum allowable upper bound of τ_2 with different values of μ_1 and μ_2.

Example 5.2.3. Consider the following nonlinear system with interval time-varying delays which was considered in [43]:

$$\dot{x}(t) = Ax(t) + Dx(t - \tau(t)) + f(t, x(t), x(t - \tau(t))),$$

	a=0.1	b=0.1
μ	0.5	0.9
[65]	1.009	0.714
[60]	1.284	1.209
[43]	1.287	1.279
By corollary 5.1.2	1.7706	1.7355

Table 5.4: Comparison of maximum allowable delay τ_2 in Example 5.2.3 with $\tau_1 = 0$.

where

$$A = \begin{bmatrix} -1.2 & 0.1 \\ -0.1 & -1 \end{bmatrix}, D = \begin{bmatrix} -0.6 & 0.7 \\ -1 & -0.8 \end{bmatrix},$$

$a = b = 0.1$ and $0 \leq \tau(t) \leq \tau_2,\ \mu_1 \leq \dot{\tau}(t) \leq \mu_2$. By using LMI Toolbox in Matlab, the LMI in corollary (5.1.2) is feasible. For comparison, we now calculate the admissible maximum upper bounds of τ_2 and set $\mu_2 = -\mu_1 = \mu$. Table IV shows the comparison of maximum allowable upper bound of τ_2 with given $\tau_1 = 0$.