# CHAPTER 1

## Introduction

Graph theory is an old subject with many modern applications. Its basic ideas were introduced in the 18th century by the great Swiss mathematician Leonhard Euler. He used graphs to solve the famous Konigsberg bridge problem. After that, graph theory is used as a tool to solve problems in many fields. The graph theory is very useful in the theory of group and of semigroup. One of its branches studies Cayley digraphs that can be used to visualize the structure and problems in the semigroup theory.

The Cayley digraph was defined for the first time by Arthur Cayley in 1878 to study finite groups. The theory of Cayley digraphs has been developed to a rather big branch in algebraic theory of graphs. It has relations with many practical problems, and also with some classical problems in pure mathematics such as classification, isomorphism or enumeration problems. There are problems related to Cayley digraphs which are interesting to group and graph theorists such as hamiltonian or diameter problems, and to computer scientists and molecular biologists such as sorting by reversals. Cayley digraphs of semigroups have been studied extensively and many results have been very interesting as following.

In 2003, Sr. Arworn, U. Knauer and N. Na Chiangmai [1] characterized Cayley digraphs of semigroups which are the union of groups, so called right (left) groups. Cayley digraphs of Clifford semigroups were characterized by S. Panma, U. Knauer, N. Na Chiangmai and Sr. Arworn in 2006 [24]. Also in the same year, all of finite inverse semigroups and all of commutative inverse semigroups were studied in the sense of bipartite Cayley digraphs by A.V. Kelarev [14].

In 2010, S. Panma [22] characterized Cayley digraphs of some completely simple semigroups which are rectangular groups, right and left groups, and rectangular bands. Later, B. Khosravi and M. Mahmoudi [18] studied Cayley digraphs of rectangular groups. They described Cayley digraphs of finite rectangular groups. They gave necessary and sufficient conditions for Cayley digraphs of finite semigroups to be vertex-transitive. Y. Luo, Y. Hao and G.T. Clarke [20] studied the basic structure and properties of Cayley digraphs of a given completely simple semigroup. They gave necessary and sufficient conditions for such Cayley digraph to be the disjoint union of complete graphs. Moreover, they gave necessary and sufficient conditions for its strong subdigraph to be a strongly connected bipartite Cayley digraph. Y. Hao and Y. Luo [8] studied Cayley digraphs of left groups and right groups. They proved that Cayley digraphs of a left group is the union of Cayley digraphs of a group, and an undirected Cayley digraph of a right group is isomorphic to a Cayley digraph of a group with an appropriate connection set.

In 2011, Y. Hao, X. Gao and Y. Luo [7] studied Cayley digraphs of Brandt semigroups. They gave the basic structures and properties of this kind of Cayley digraphs. Moreover, they gave necessary and sufficient conditions for the components of a Cayley digraph of a Brandt semigroup to be strongly regular. L. John and A.N.P. Kumari [12] considered Cayley digraphs of rectangular bands relative to Green's equivalence classes. After that, they described Cayley digraphs of completely simple semigroups relative to Green's equivalence classes [13]. The results obtained from both studies are corresponding that a Cayley digraph which relative to an $\mathcal{L}$-class is isomorphic to a disjoint union of complete digraphs.

The isomorphism problem of Cayley digraphs of semigroups is of fundamental importance, and it has been studied extensively. We are interested in studying the isomorphism problem of Cayley digraphs of a completely simple semigroup. Because a completely simple semigroup is very important in semigroup theory, which plays an important role in automata theory. And as is known, a completely (0-, respectively) simple semigroup is isomorphic to a Rees matrix semigroup over a group (with zero, respectively). It is clear that a rectangular group and a Brandt semigroup are special kinds of a Rees matrix semigroup and Rees matrix semigroups over group with zero, respectively. So the isomorphism problem of Cayley digraphs of these semigroups are very interesting as well.

Inspired by these facts, we aim to characterize Cayley digraphs of completely simple semigroups, and find the isomorphism conditions for Cayley digraphs of rectangular groups. Finally, we describe Cayley digraphs of Brandt semigroups relative to Green's equivalence classes. Moreover, we give isomorphism conditions for Cayley digraphs of a Brandt semigroup relative to $\mathcal{H}$ classes.

The contents of each chapter in this thesis are as follows: In Chapter 2, we start with provide the basic definitions in semigroup theory and graph theory. We introduce a completely simple semigroup, a rectangular group and a Brandt semigroup. In the part of graph theory, we present definitions and examples of a digraph isomorphism, Cayley digraphs and operations on digraphs.

In Chapter 3, we characterize Cayley digraphs of completely simple semigroups. We also describe the structure of Cayley digraphs of a completely simple semigroup with a one-element connection set. Finally, we introduce the conditions for which they are isomorphic and connected.

In Chapter 4, we give the conditions for Cayley digraphs of a rectangular group to be isomorphic to each other. In particular, isomorphism conditions for Cayley digraphs of a right group and for Cayley digraphs of a rectangular band are given.

In Chapter 5, we describe the Cayley digraphs of a Brandt semigroup relative to the Green's relations which are equivalence classes of $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{H}$, respectively. We give isomorphism conditions for Cayley digraphs of a Brandt semigroup relative to $\mathcal{H}$-classes. Moreover, we give necessary conditions for two Cayley digraphs to be isomorphic to each other.

Chapter 6, we compile the all results obtained in this thesis. In addition, we present interesting open problems.