# CHAPTER 2

# Preliminaries

In this chapter some of the basic of semigroup theory and graph theory needed to understand the results of the thesis will be presented.

## 2.1 Semigroups

A set $S$ together with a binary operation $\cdot$ is a *groupoid*. A groupoid $(S, \cdot)$ satisfying the *associative law*

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

for all $a, b, c \in S$ is a *semigroup*. A semigroup having only one element is *trivial*. It is customary to omit the symbol for multiplication by writing simply $S$ for $(S, \cdot)$ and denote the product of $a$ and $b$ by juxtaposition $ab$ rather than $a \cdot b$.

An element $e$ of $S$ is the *identity* of $S$ if $s = es = se$ for all $s \in S$. A semigroup with an identity is a *monoid*.

One can always *adjoin the identity* to $S$ by taking $e \notin S$ and declaring a multiplication on $S \cup \{e\}$ such that $es = se = s$ for all $s \in S \cup \{e\}$. We write $S = S^1$ if $S$ is a monoid; otherwise, $S^1$ stands for the semigroup $S$ with the identity adjoined. The usual notation for the identity is 1. A *group* is a semigroup with the identity (denoted by 1 or $e$ as convenient) and in which for every element $a$ there axists an element $a^{-1}$ with $aa^{-1} = a^{-1}a = 1$.

An element $z$ of $S$ is a *left* (respectively, *right*) *zero* of $S$ if $zs = z$ (respectively, $sz = z$) for all $s \in S$. If $z$ is both a left and right zero then we call the *zero* of $S$. The usual symbol for the zero is 0. A semigroup can have at most one zero element. A semigroup all of whose elements are left (respectively, right) zeros is a *left* (respectively, *right*) *zero semigroup*.

The zero element can be adjoined to any semigroup in the same way as the identity. For any semigroup $S$

$$S^0 = \begin{cases} S & \text{if } S \text{ has a zero element,} \\ S \cup \{0\} & \text{otherwise ,} \end{cases}$$

where the multiplication in $S$ is extended to $S^0$ by defining, if necessary, $s0 = 0s = 0$ for all $s \in S^0$. Then $S^0$ is a semigroup with a zero.

Let $A_1, A_2, \ldots, A_n$ be nonempty subsets of a semigroup $S$. Then $A_1A_2 \cdots A_n = \{a_1a_2 \cdots a_n | a_i \in A_i, i = 1, 2, \ldots, n\}$. If $A_1 = A_2 = \cdots = A_n = A$, we write $A^n = A_1A_2 \cdots A_n$. If $A_i = \{a_i\}$, we write $a_i$ in stead of $\{a_i\}$ in $A$. So for example, $S^1a = Sa \cup a$. A nonempty subset $T$ of $S$ is called a *subsemigroup* if it is closed under the multiplication of $S$. If $T$ is a group under the multiplication of $S$, then it is a *subgroup* of $S$.

For any nonempty subset $A$ of a semigroup $S$, the intersection of all subsemigroups of $S$ containing $A$ is the *subsemigroup* $\langle A \rangle$ of $S$ *generated by* $A$. If $\langle A \rangle = S$, then $S$ is *generated by* $A$ and $A$ is a *set of generators* of $S$. The subsemigroup of $S$ generated by a singleton $\{s\}$ is the *cyclic semigroup* generated by $s$ and denoted by $\langle s \rangle$. The *cardinality* of a set $X$ is denoted by $|X|$. The order of $S$ (respectively, $s$) is $|S|$ (respectively, $|\langle s \rangle|$). An element $s$ of $S$ has *finite order* if $\langle s \rangle$ is finite, otherwise $s$ has *infinite order*. If every element of $S$ has finite order, then $S$ is *periodic*. For any subgroup $K$ of $G$, $G/K$ denotes the set of all distinct left cosets of $K$ in $G$.

An element $e \in S$ is called *idempotent* if it satisfies $e^2 = e$. We use $E(S)$ to denote the set of all idempotents of $S$. A *band* is a semigroup such that every element is idempotent. Let $J$ be a nonempty set. The *direct product* of a family of semigroup $\{S_\alpha\}_{\alpha \in J}$ is the semigroup defined on the cartesian product $\prod_{\alpha \in J} S_\alpha$ with coordinatewise multiplication. We denote by $p_\alpha$ the projection of the direct product $\prod_{\alpha \in J} S_\alpha$ onto the $\alpha$-component $S_\alpha$. Then for any right group $S = G \times R_n$, $p_1$ and $p_2$ are the natural projections of $S$ onto $G$ and onto $R_n$, respectively.

A binary relation $\leq$ on a set $X$ is called a *partial order* on $X$ if

(1) $(x, x) \in \leq$ for all $x$ in $X$ (that is, $\leq$ is *reflexive*);

(2) $\forall x, y \in X$, if $(x, y) \in \leq$ and $(y, x) \in \leq$ then $x = y$ (that is, $\leq$ is *antisymmetric*);

(3) $\forall x, y, z \in X$, if $(x, y) \in \leq$ and $(y, z) \in \leq$ then $(x, z) \in \leq$ (that is, $\leq$ is *transitive*).



### 2.1.1 Homomorphisms and Ideals

Let $S$ and $T$ be semigroups. A mapping $\phi : S \to T$ is called a (*semigroup*) *homomorphism* if, for all $x, y \in S$,

$$(xy)\phi = x\phi y\phi.$$

A homomorphism that is injective will be called a *monomorphism*. And if it is surjective, it will be called an *epimorphism*. A homomorphism is called an *isomorphism* if it is bijective. If there exists an epimorphism from $S$ onto $T$, we say that $T$ is a *homomorphic image* of $S$. If there is an isomorphism $\phi : S \to T$, we say that $S$ and $T$ are isomorphic and write $S \cong T$. A homomorphism from $S$ to itself is called an *endomorphism* and an isomorphism from $S$ to itself is called an *automorphism*. The set of all endomorphisms of $S$, under composition of mappings, forms a monoid. We call this monoid the *endomorphism monoid* of $S$ and denote its by $End(S)$. Similarly, the set of all automorphisms forms a group that is denoted by $Aut(S)$ and is called the *automorphism group* of $S$.

A nonempty subset $I$ of a semigroup $S$ is called a *right ideal* of $S$ if $ba \in I$ for all $a \in S, b \in I$. Dually, if $ab \in I$, then $I$ is called a *left ideal* and $I$ is called *(two-sided) ideal* if it is both a left and a right ideal. An (right, left) ideal $I$ of $S$ is called *proper* if $I \neq S$.

### 2.1.2 Green's Relations and Regular Semigroups

Green's relations were first introduced by J.A. Green in [6]. He describes the ideal structure of a semigroup. We now define Green's relations $\mathcal{R}, \mathcal{L}, \mathcal{J}, \mathcal{D}$ and $\mathcal{H}$ as follows.

Let $S$ be a semigroup and let $s \in S$. The principal *right, left* and *two-sided ideals* generated by $s$ are the sets $sS^1 = sS \cup \{s\}, S^1s = Ss \cup \{s\}$ and $S^1sS^1 = sS \cup Ss \cup SsS \cup \{s\}$, respectively. For $s, t \in S$, we say that $s$ and $t$ are $\mathcal{R}$-related, writing $s\mathcal{R}t$, if $s$ and $t$ generate the same principal right ideal (i.e. $sS^1 = tS^1$). We say they are $\mathcal{L}$-related if they generate the same principal left ideal (i.e. $S^1s = S^1t$), in this case we write $s\mathcal{L}t$. Also, we say $s$ and $t$ are $\mathcal{J}$-related, writing $s\mathcal{J}t$, if they generate the same principal two-sided ideal (i.e. $S^1sS^1 = S^1tS^1$). We define $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$ and $\mathcal{D} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$ : the composition of the binary relations $\mathcal{R}$ and $\mathcal{L}$. The relations $\mathcal{L}, \mathcal{R}, \mathcal{H}$ , $\mathcal{D}$ and $\mathcal{J}$ are equivalence relations and called *Green's equivalence relations* on a semigroup. We called the corresponding equivalence classes the $\mathcal{R}$-classes, $\mathcal{L}$-classes, $\mathcal{J}$-classes, $\mathcal{H}$-classes and $\mathcal{D}$-classes, respectively of $S$.



**Proposition 2.1.1.** *[10] Let $S$ be a semigroup and $a, b \in S$.*

*(1)* $a\mathcal{L}b \Leftrightarrow$ *there exist* $x, y \in S^1$ *with* $a = xb, b = ya$.

*(2)* $a\mathcal{R}b \Leftrightarrow$ *there exist* $x, y \in S^1$ *with* $a = bx, b = ay$.

*(3)* $a\mathcal{J}b \Leftrightarrow$ *there exist* $x, y, u, v \in S^1$ *with* $a = xby, b = uav$.

An element $a \in S$ is called *regular* if there exists $x \in S$ such that $axa = a$. The semigroup $S$ is said to be *regular* if all of its elements are regular. If $a \in S$, then we say that $a'$ is an *inverse* of $a$ if

$$aa'a = a \qquad \text{and} \qquad a'aa' = a'.$$

An element has an inverse if and only if that element is regular. A semigroup in which every element has a unique inverse is an *inverse semigroup.*

## 2.2 Completely Regular Semigroups

Let $G$ be a group, $R_n$ an $n$-element right zero semigroup and $S = G \times R_n$. As usual the multiplication on $S$ is defined componentwise by $(g, r)(g', r') = (gg', r')$ for $g, g' \in G, r, r' \in R_n$. We call the semigroup $S$ a *right zero union of the groups* ($RZUG$) over $G$.

Correspondingly, let $L_m$ be an $m$-element left zero semigroup, we set $S = G \times L_m$ and define the multiplication on $S$ componentwise by $(g, l)(g', l') = (gg', l)$ for $g, g' \in G, l, l' \in L_m$. We call this semigroup a *left zero union of the groups* ($LZUG$) over $G$.

A semigroup $S$ is said to be *left(right) cancellative* if $xy = xz(yx = zx)$ implies $y = z$, for all $x, y, z \in S$.

### 2.2.1 Completely (0-)Simple Semigroups

A semigroup $S$ is called *(right, left) simple* if it has no proper (right, left) ideals. A semigroup $S$ with zero element 0 is called *(right 0-simple, left 0-simple) 0-simple* if (i) $S^2 \neq 0$, and (ii) 0 is only proper (right, left) ideal of $S$. A semigroup $S$ is called a *right group* if it is right simple and left cancellative. This is equivalent to saying that, for any element $a$ and $b$ of $S$, there exists one and only one element $x$ of $S$ such that $ax = b$. *Left group* is defined dually.

**Theorem 2.2.1.** *[5] The following assertions concerning a semigroup $S$ are equivalent :*

*(1) $S$ is a right group.*

*(2) $S$ is a right simple, and contains an idempotent.*

*(3) $S$ is the direct product $G \times R$ of a group $G$ and a right zero semigroup $R$.*

From now on, the direct product of a group and a right zero semigroup, will represent the right group. The following lemma (Lemma 2.6 in [23]) will be used in the proof of theorem in chapter 4.

**Lemma 2.2.2.** *[23] Let $S = G \times R_n$ be a right group, where $G$ is a group, $R_n = \{r_1, r_2, \ldots, r_n\}$ a right zero semigroup, and let $A$ be a nonempty subset of $S$. Then $\langle A \rangle = \langle p_1(A) \rangle \times p_2(A)$.*

A direct product of a left zero and a right zero semigroup is called a *rectangular band.* A *rectangular group* is a direct product of a group and a rectangular band.

Let $E(S)$ be the set of all idempotents of a semigroup $S$. If $e, f \in E(S)$, we define $e \leq f$ if and only if $ef = fe = e$. Clearly, $\leq$ is a partial order on $E(S)$. If $S$ contains a zero element 0, then $0 \leq e$ for every $e \in E(S)$. An idempotent $f$ of $S$ is called *primitive* if $f \neq 0$ and if $e \leq f$ implies $e = 0$ or $e = f$.

A semigroup $S$ is called a *completely simple semigroup* if $S$ is a simple semigroup and containing a primitive idempotent. And a semigroup $S$ is called a *completely 0-simple semigroup* if $S$ is a 0-simple semigroup and containing a primitive idempotent.

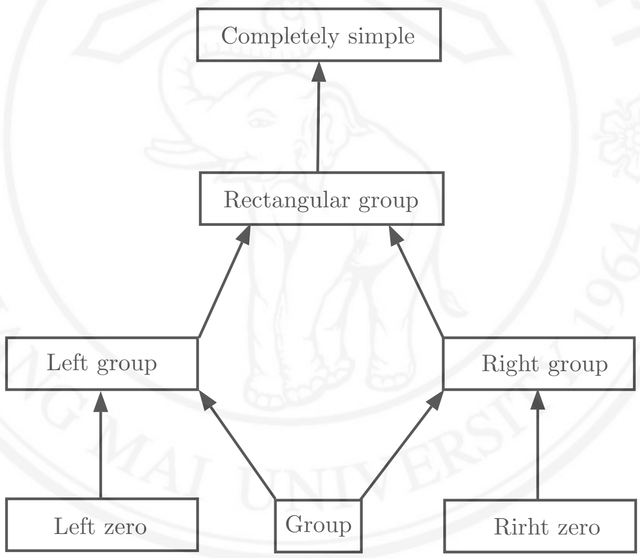



Figure 2.1: Completely simple semigroups.

Note that left(right) zero semigroups, left(right) groups and rectangular groups are just special kinds of a completely simple semigroup. Figure 2.1 shows the relations between completely simple semigroups, rectangular groups, left(right) groups, left(right) zero semigroups, and groups. An arrow A→B means "A implies B".

### 2.2.2 Rees Matrix Semigroups and Brandt Semigroups

Suppose that $G$ is a group, $I$ and $\Lambda$ are nonempty sets, and $P$ is a $\Lambda \times I$ matrix over a group $G$. The *Rees matrix semigroup* $\mathcal{M}(G, I, \Lambda, P)$ *with sandwich matrix* $P$ consists of all triples $(g, i, \lambda)$, where $i \in I, \lambda \in \Lambda$, and $g \in G$ with multiplication defined by the rule $(g_1, i_1, \lambda_1)(g_2, i_2, \lambda_2) = (g_1 p_{\lambda_1 i_2} g_2, i_1, \lambda_2)$.

**Theorem 2.2.3.** *[19] A semigroup $S$ is completely simple if and only if $S$ is isomorphic to a Rees matrix semigroup.*

We denoted the *Rees matrix semigroup over the group with zero $G^0$ with sandwich matrix $P$* by $\mathcal{M}^0(G, I, \Lambda, P)$. $S = \mathcal{M}^0(G, I, \Lambda, P)$ is the $(G \times I \times \Lambda) \cup \{0\}$ with multiplication defined by

$$(g, i, \lambda)(h, j, \mu) = \begin{cases} (g p_{\lambda j} h, i, \mu) & \text{if } p_{\lambda j} \neq 0; \\ 0 & \text{otherwise.} \end{cases}$$

$$(g, i, \lambda)0 = 0(g, i, \lambda) = 00 = 0.$$

**Theorem 2.2.4.** *[19] A semigroup $S$ is completely 0-simple if and only if $S$ is isomorphic to a Rees matrix semigroup over the group with zero.*

In the sequel we will mainly use the term Rees matrix semigroup $\mathcal{M}(G, I, \Lambda, P)$ and the term Rees matrix semigroup over the group with zero $\mathcal{M}^0(G, I, \Lambda, P)$ instead of a completely simple semigroup and a completely 0-simple semigroup, respectively.

The following lemma (Lemma 3.2 in [15]) is one of many of the basic facts which Kelarev and Praeger have collected.

**Lemma 2.2.5.** *[15] Let $S = \mathcal{M}(G, I, \Lambda, P)$ be a completely simple semigroup. Then $S$ is a right group if and only if $|I| = 1$.*

For any semigroup $S$ and a nonempty set $I$, we let

$$B(S, I) = I \times S \times I \cup \{0\}$$

where $0 \notin I \times S \times I$, with the multiplication

$$(i, g, l)(l, h, j) = (i, gh, j)$$

and all other products are equal to 0. Note that when $S$ is a group, $B(S, I)$ is a special case of a Rees matrix semigroup over the group with zero and called a *Brandt semigroup* where $P$ is the identity matrix. Since this semigroup depends only upon the group $G$ and the set $I$, we denote this semigroup by $B(G, I)$. Note that, a completely 0-simple inverse semigroup is a Brandt semigroup.

For a Brandt semigroup $S = B(G, I)$, we put the subsets of $S$

$$
\begin{aligned}
S_{-j} &= \{(i, g, j) \mid i \in I, g \in G\}, \\
S_{i-} &= \{(i, g, j) \mid j \in I, g \in G\}, \\
S_{ij} &= \{(i, g, j) \mid g \in G\}.
\end{aligned}
$$

Note that, the $S_{-j}, S_{i-}$ and $S_{ij}$ are the $\mathcal{L}$-class , $\mathcal{R}$-class and $\mathcal{H}$-class of $S$, respectively.

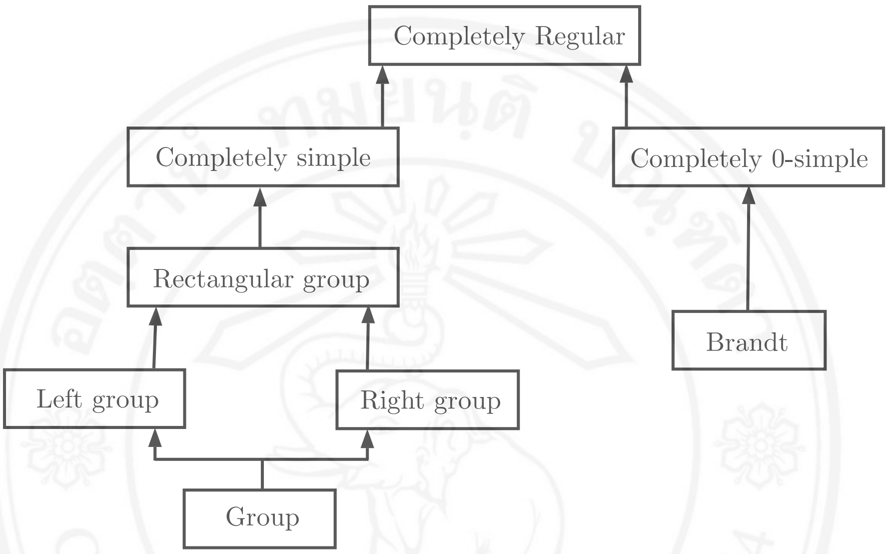



Figure 2.2: Completely Regular Semigroups.

In Figure 2.2. shows the relations between completely simple semigroups, rectangular groups, Brandt semigroups, left groups, right groups and groups. A simple arrow A→B means “A implies B”.

## 2.3 Digraphs

In this section we will present basic definitions of digraphs, which include the isomorphism and basic operations on digraphs.



### 2.3.1 Basic Definitions of Digraphs

A *directed graph* or *digraph* $D$ is a pair $(V(D), E(D))$, where $V(D)$ is a nonempty set whose elements are called the *vertices* and $E(D)$ is the subset of the set of ordered pairs of elements of $V(D)$. The elements of $E(D)$ are called the *arcs* of $D$. We call $V(D)$ the *vertex set* and $E(D)$ the *arc set* of $D$. The *order (size)* of $D$ is the number of vertices (arcs) in $D$.

**Example 2.3.1.** The digraph $D$ in Figure 2.3 is of order 4 and size 6 where $V(D) = \{w, x, y, z\}$ and $E(D) = \{(x, z), (x, y), (y, x), (y, z), (z, w), (w, w)\}$.

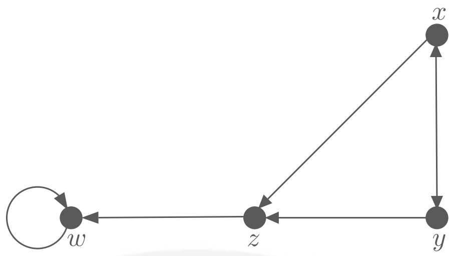



Figure 2.3: A digraph $D$.

For $u, v \in V(D)$, an arc $a = (u, v) \in E(D)$ is denoted by $uv$ and implies that $a$ is directed from $u$ to $v$. Here $u$ is the *initial* vertex (*tail*) and $v$ is the *terminal* vertex (*head*). An arc from a vertex to itself such as $(u, u)$, is called a *loop*. Arcs with the same tail and the same head are called *multiple arcs*. Digraphs considered in this thesis are digraphs without multiple arcs. The head and tail of an arc are its *end-vertices*, we say that the end-vertices are *adjacent*. Also we say that $a$ joins $u$ to $v$, $a$ is *incident* with $u$ and $v$, $a$ is incident from $u$ and $a$ is incident to $v$. For any $v \in V(D)$, the number of arcs incident to $v$ is the *indegree* of $v$ and is denoted by $\overrightarrow{d}(v)$. The number of arcs incident from $v$ is called the *outdegree* of $v$ and is denoted by $\overleftarrow{d}(v)$. The *total degree* (or simply degree) of $v$ is $d(v) = \overrightarrow{d}(v) + \overleftarrow{d}(v)$. If $\overrightarrow{d}(v) = \overleftarrow{d}(v) = k$ for every $v \in V(D)$, then $D$ is said to be a *k-regular* digraph.

Since the number of arcs in a digraph equals the number of their tails and the number of their heads, we obtain the following very basic result.

**Proposition 2.3.2.** *[2] For any digraph $D$,*

$$\sum_{x \in V(D)} \overleftarrow{d}(x) = |E(D)| = \sum_{x \in V(D)} \overrightarrow{d}(x)$$

.

A digraph $D'$ is a *subdigraph* of a digraph $D$ if $V(D') \subseteq V(D)$, $E(D') \subseteq E(D)$. If $V(D') = V(D)$, we say that $D'$ is a *spanning subdigraph* of $D$. In Figure 2.3, the digraph $H$ with vertex set $\{w, x, y, z\}$ and arc set $\{(w, x), (w, y), (z, y), (x, z)\}$ is a spanning subdigraph of the digraph $D$. If every arc of $E(D)$ with both end-vertices in $V(D')$ is in $E(D')$, we say that $D'$ is *induced* by $V(D')$ and call $D'$ an *induced* (or a *strong*) subdigraph of $D$.

**Example 2.3.3.** The digraph $L$ with vertex set $\{x, y, z\}$ and arc set $\{(x, z), (x, y), (y, z)\}$ in Figure 2.4 is a subdigraph of the digraph $D$ in Figure 2.3; $L$ is neither a spanning subdigraph nor an induced subdigraph of the digraph $D$ in Figure 2.3. The digraph $M$ along with the arc $yx$ in Figure 2.4 is an induced subdigraph of the digraph $D$ in Figure 2.3.


Figure 2.4: Induced subdigraphs.

A *walk* in $D$ is an alternating sequence $W = x_1a_1x_2a_2x_3 \ldots x_{k-1}a_{k-1}x_k$ of vertices $x_i$ and arcs $a_j$ from $D$ such that the tail of $a_i$ is $x_i$ and the head of $a_i$ is $x_{i+1}$ for every $i = 1, 2, \ldots, k-1$. A walk $W$ is *closed* if $x_1 = x_k$, and open otherwise. The set of vertices $\{x_1, x_2, \ldots, x_k\}$ is denoted by $V(W)$, the set of arcs $\{a_1, a_2 \ldots, a_{k-1}\}$ is denoted by $E(W)$. We say that $W$ is a walk from $x_1$ to $x_k$ or an $(x_1, x_k)$-*walk*. If $W$ is open, then we say that the vertex $x_1$ is the *initial vertex* of $W$, the vertex $x_k$ is the *terminal vertex* of $W$, and $x_1$ and $x_k$ are *end-vertices* of $W$. The *length* of a walk is the number of its arcs. Hence the walk $W$ above has length $k-1$. The arcs of $W$ are defined from the context or simply unimportant, we will denote $W$ by $x_1x_2 \ldots x_k$.

A *trail* is a walk in which all arcs are distinct. If the vertices of $W$ are distinct, $W$ is a *path*. If the vertices $x_1, x_2, \ldots, x_{k-1}$ are distinct, $k \geq 3$ and $x_1 = x_k$, $W$ is a *cycle*. Since paths and cycles are special cases of walks, the length of a path and a cycle are already defined. The same remark is valid for other parameters and notions, e.g. an $(x, y)$-*path*.

If $W$ is a cycle and $x$ is a vertex of $W$, we say that $W$ is a *cycle through* $x$. A loop is also considered a cycle (of length one). A $k$-*cycle* is a cycle of length $k$.

A *semi-walk* $W'$ is an alternating sequence $W' = x_1a_1x_2a_2x_3 \ldots x_{k-1}a_{k-1}x_k$ of vertices $x_i$ and arcs $a_j$ such that the tail of $a_i$ is $x_i$ or $x_{i+1}$ for every $i = 1, 2, \ldots, k-1$. A *semi-trail*, a *semi-path* and a *semi-cycle* are defined similarly.

**Proposition 2.3.4.** *[2] Let $D$ be a digraph and let $x, y$ be a pair of distinct vertices in $D$. If $D$ has an $(x, y)$-walk $W$, then $D$ contains an $(x, y)$-path $P$ such that $E(P) \subseteq E(W)$. If $D$*

*has a closed $(x, x)$-walk $W$, then $D$ contains a cycle $C$ through $x$ such that $E(C) \subseteq E(W)$.*

**Remark 2.3.5.** *[2] Let $D$ be a digraph in which every vertex $v \in D$ has $\overleftarrow{d}(v) \geq 1$. Then $D$ contains a cycle.*

A digraph $H = (V, E)$ is called an *undirected digraph* if and only if, for every $(u, v) \in E$, $(v, u) \in E$. We called $H$ is *connected* digraph if there is a semipath between any two vertices. A *complete digraph* on $n$ vertices, denoted by $K_n$, is a digraph with $n$ vertices in which each vertex is an arc to each of the others. An *$n$-partite digraph* is a digraph such that the vertices are partitioned into $n$ disjoint subsets $V_1, V_2, \ldots, V_n$ and there is no arc between two vertices in one subset. An $n$- partite digraph is called *complete $n$-partite*, if there is an arc between all pairs of vertices from different subsets.

### 2.3.2 Isomorphism and Basic Operations on Digraphs

Let $(V_1, E_1)$ and $(V_2, E_2)$ be digraphs. A mapping $\phi : V_1 \to V_2$ is called a *digraph homomorphism* if $(u, v) \in E_1$ implies $(\phi(u), \phi(v)) \in E_2$, i.e. $\phi$ preserves arcs. We write $\phi : (V_1, E_1) \to (V_2, E_2)$. A digraph homomorphism $\phi : (V, E) \to (V, E)$ is called a *digraph endomorphism.* If $\phi : (V_1, E_1) \to (V_2, E_2)$ is a bijective digraph homomorphism and $\phi^{-1}$ is also a digraph homomorphism, then $\phi$ is called a *digraph isomorphism.* In this case $(V_1, E_1)$ and $(V_2, E_2)$ are said to be *isomorphic* and we write $(V_1, E_1) \cong (V_2, E_2)$. For example, the digraphs in Figure 2.5 are isomorphic under the bijection $\phi : V(G) \to V(H)$ is given by $\phi(w) = a, \phi(x) = b, \phi(y) = c$, and $\phi(z) = d$. A digraph isomorphism $\phi : (V, E) \to (V, E)$ is called a *digraph automorphism.*


Figure 2.5: Isomorphic digraphs.

For any family of nonempty set $\{X_i | i \in I\}$, let $\dot{\cup}_{i \in I} X_i$ denote the *disjoint union* of $X_i, i \in I$. Let $(V_1, E_1), (V_2, E_2), \ldots, (V_n, E_n)$ be digraphs. The *union* of $(V_1, E_1)$,

$(V_2, E_2), \ldots, (V_n, E_n)$ is defined as $\bigcup_{i=1}^{n}(V_i, E_i) = \left(\cup_{i=1}^{n} V_i, \cup_{i=1}^{n} E_i\right)$. An example is shown in Figure 2.6. If $V_i \cap V_j = \emptyset$ for all $i \neq j$, then $\bigcup_{i=1}^{n}(V_i, E_i)$ is called the *disjoint union* of $(V_1, E_1), (V_2, E_2), \ldots, (V_n, E_n)$ denoted by $\dot{\bigcup}_{i=1}^{n}(V_i, E_i)$. If $V_i = V_j = V$ for all $i \neq j$, then the *edge sum* of $(V_1, E_1), (V_2, E_2), \ldots, (V_n, E_n)$ is defined as $\oplus_{i=1}^{n}(V_i, E_i) = (V, \cup_{i=1}^{n} E_i)$. An example is shown in Figure 2.7.

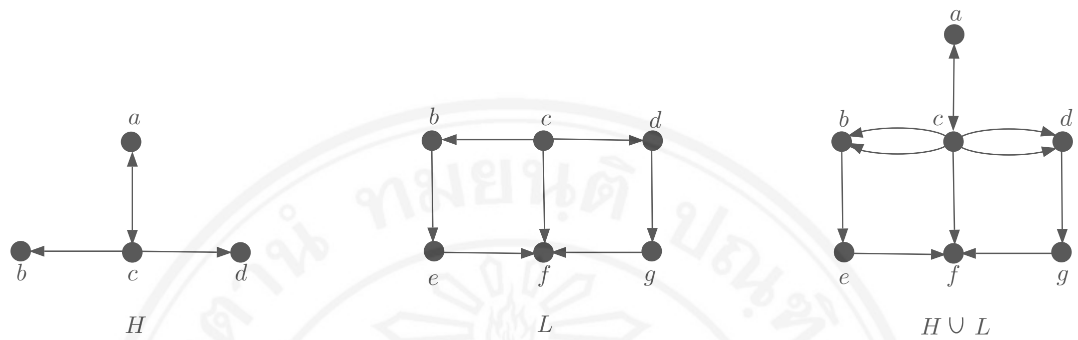



Figure 2.6: The union of digraphs $H$ and $L$.

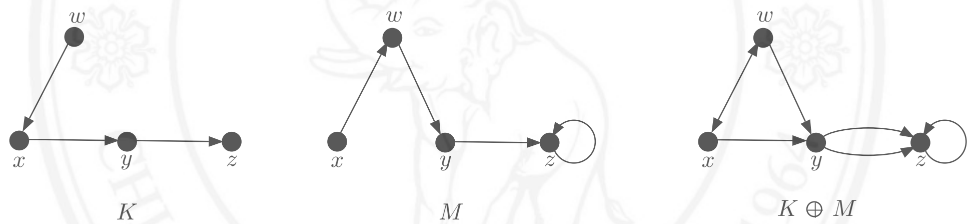


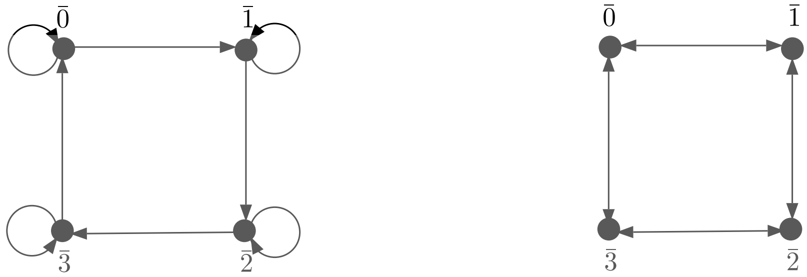

Figure 2.7: The edge sum of digraphs.

## 2.4 Cayley Digraphs

In this section, we give definitions and results about Cayley digraphs that will be used in the later chapters.

Let $S$ be a semigroup (group) and $A \subseteq S$. We define the *Cayley digraph* $D = Cay(S, A)$ as follows: $V(D) = S$ is the *vertex set* and $E(D) = \{(u, ua) | u \in S, a \in A\}$ is the set of *arc* in $Cay(S, A)$. The set $A$ is called the *connection set* of $Cay(S, A)$. We will say that $Cay(S, A)$ is the *Cayley digraph of $S$ relative to $A$*. Certainly, if $A$ is an empty set, then $Cay(S, A)$ is an *empty digraph*, i.e. $E(Cay(S, A)) = \emptyset$.

**Example 2.4.1.** We consider $\mathbb{Z}_4 = (\{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}, +)$ we have $Cay(\mathbb{Z}_4, \{\bar{0}, \bar{1}\})$ and $Cay(\mathbb{Z}_4, \{\bar{1}, \bar{2}\})$ in Figure 2.8.




$Cay(Z_4, \{\bar{0}, \bar{1}\})$ $Cay(Z_4, \{\bar{1}, \bar{2}\})$

Figure 2.8: Cayley digraphs $Cay(\mathbb{Z}_4, \{\bar{0}, \bar{1}\})$ and $Cay(\mathbb{Z}_4, \{\bar{1}, \bar{2}\})$.

A digraph $(V, E)$ is called a *semigroup (group) digraph* or *digraph of a semigroup (group)* if there exists a semigroup (group) $S$ and a connection set $A \subseteq S$ such that $(V, E)$ is isomorphic to the Cayley digraph $Cay(S, A)$.

**Theorem 2.4.2.** *[3] A finite digraph $\Gamma = (V, E)$ is a Cayley digraph of a group $G$ if and only if automorphism group of $\Gamma$ contains a subgroup $\Delta$ isomorphic to $G$ such that for every two vertices $u, v \in V$ there exists a unique $\sigma \in \Delta$ such that $\sigma(u) = v$.*

Sr. Arworn, U. Knauer and N. Na Chiangmai [1], they characterized Cayley digraphs of right zero unions of groups as following theorem.

**Theorem 2.4.3.** *[1] Let $(V, E)$ be a digraph. If it is the vertex disjoint union of $k$ strong $G$-group subdigraphs $(V_1, E_1), (V_2, E_2), \ldots, (V_k, E_k)$ for some group $G$, $V = \cup_{i=1}^{k} Vi, k \geq 2$, with digraph isomorphisms $\varphi_i : Cay(G, A_i) \to (V_i, E_i)$ for each $i \in \{1, 2, \ldots, k\}$, $A_i \subseteq G$ and if $\forall i, j \in \{1, 2, \ldots, k\}[(u_i, v_i) \in E_i \leftrightarrow (u_j, v_i) \in E]$ then $(V, E)$ is an RZUG digraph of $G \times R_k$.*

To provide a criterion for Cayley digraphs of rectangular groups S. Panma [22] presented the characterization of Cayley digraphs of rectangular groups as following theorem.

**Theorem 2.4.4.** *[22] A digraph $(V, E)$ is a Cayley digraph of a rectangular group if and only if the following conditions hold:*

*(1) $(V, E)$ is the disjoint union of $n$ isomorphic subdigraphs $(V_1, E_1), (V_2, E_2), \ldots, (V_n, E_n)$ for some $n \in \mathbb{N}$,*

*(2) there exists a group $G$ and $m \in \mathbb{N}$ such that for each $i \in \{1, 2, \ldots, n\}$, $(V_i, E_i)$ contains $m$ disjoint strong subdigraphs $(V_{i1}, E_{i1}), (V_{i2}, E_{i2}), \ldots, (V_{im}, E_{im})$ which are Cayley digraphs of $G$, and $V_i = \cup_{\alpha=1}^{m} V_{i\alpha}$,*

*(3) for each* $\alpha \in \{1, 2, \ldots, m\}$, *there exists a digraph isomorphism* $\varphi_{i\alpha} : (V_{i\alpha}) \rightarrow Cay(G, A_{i\alpha})$ *for some* $A_{i\alpha} \subseteq G$, *such that* $A_{j\alpha} = A_{k\alpha}$ *for all* $j, k \in \{1, 2, \ldots, n\}$,

*(4) for each* $\alpha, \beta \in \{1, 2, \ldots, m\}$, *and for each* $u \in V_{i\alpha}, v \in V_{i\beta}, (u, v) \in E$ *if and only if* $\varphi_{i\beta}(v) = \varphi_{i\alpha}(u)a$ *for some* $a \in A_{i\beta}$.

Y. Hao, X. Gao and Y. Luo [7], they gave the basic results about Cayley digraphs of Brandt semigroups as the two following lemmas.

**Lemma 2.4.5.** *[7] Let* $S = B(G, I)$ *be a Brandt semigroup and* $A$ *a nonempty subset of* $S$. *Then* $\big((i_1, g_1, j_1), (i_2, g_2, j_2)\big)$ *is an arc in* $Cay(S, A)$ *if and only if* $i_1 = i_2$ *and* $(j_1, g_1^{-1}g_2, j_2) \in A$.

Let $R_A = \{i \in I | (i, g, j) \in A$ for some $g \in G, j \in I\}$. The next lemma shows that when there is an arc between a nonzero vertex and the zero vertex in $Cay(S, A)$.

**Lemma 2.4.6.** *[7] Let* $S = B(G, I)$ *be a Brandt semigroup and* $A$ *a nonempty subset of* $S$. *Then*

*(1) If* $|R_A| \geq 2$, *then there is an arc from each nonzero vertex to the vertex* $0$ *in* $Cay(S, A)$.

*(2) If* $R_A = \{i_0\}$ *for some* $i_0 \in I$, *then for all* $(i, g, j) \in S \setminus \{0\}$, $\big((i, g, j), 0\big)$ *is an arc in* $Cay(S, A)$ *if and only if* $j \neq i_0$.