CHAPTER 3

Characterizations of Cayley Digraphs of Completely Simple Semigroups

In this chapter, we shall characterize digraphs which are Cayley digraphs of completely simple semigroups. We also describe the structure of Cayley digraphs of Rees matrix semigroup with a one-element connection set. Moreover, we introduce the conditions for which they are isomorphic and connected.

3.1 Characterizations of Cayley Digraphs of Completely Simple semigroups

In this section, we shall describe Cayley digraphs of completely simple semigroups. By definition of a completely simple semigroup, we have the following lemma.

Lemma 3.1.1. *Let G be a group, $S = \mathcal{M}(G, I, \Lambda, P)$ a completely simple semigroup, $A \subseteq S$, and let $(g_1, i_1, \lambda_1), (g_2, i_2, \lambda_2) \in S$. Then $((g_1, i_1, \lambda_1), (g_2, i_2, \lambda_2))$ is an arc in $Cay(S, A)$ if and only if there exists $a = (g, l, \lambda_2) \in A$ such that $g_2 = g_1 p_{\lambda_1 l} g$ and $i_1 = i_2$.*

Proof. ($\Rightarrow$) Let $(g_1, i_1, \lambda_1), (g_2, i_2, \lambda_2) \in S$ and let $((g_1, i_1, \lambda_1), (g_2, i_2, \lambda_2))$ be an arc in $Cay(S, A)$. Then there exists $a = (g, l, \mu) \in A$ such that $(g_2, i_2, \lambda_2) = (g_1, i_1, \lambda_1)(g, l, \mu) = (g_1 p_{\lambda_1 l} g, i_1, \mu)$. Hence $g_2 = g_1 p_{\lambda_1 l} g$, $i_1 = i_2$, and thus $\lambda_2 = \mu$.

($\Leftarrow$) Let $(g_1, i_1, \lambda_1), (g_2, i_2, \lambda_2) \in S$, and suppose that there exists $a = (g, l, \lambda_2) \in A$ such that $g_2 = g_1 p_{\lambda_1 l} g$ and $i_1 = i_2$. Then $(g_2, i_2, \lambda_2) = (g_1 p_{\lambda_1 l} g, i_2, \lambda_2) = (g_1 p_{\lambda_1 l} g, i_1, \lambda_2) = (g_1, i_1, \lambda_1)(g, l, \lambda_2)$. Hence $((g_1, i_1, \lambda_1), (g_2, i_2, \lambda_2))$ is an arc in $Cay(S, A)$. □

In view of Lemma 3.2 in [15], we know that a completely simple semigroup $S = \mathcal{M}(G, I, \Lambda, P)$ is right group if and only if $|I| = 1$. In [1] Cayley digraphs which represent right groups are characterized. Then we obtain the following proposition.

Proposition 3.1.2. *Let G be a group, $S = \mathcal{M}(G, \{i\}, \Lambda, P)$. Take $a = (g, i, \beta) \in S$. Then*

(1) the Cayley digraph $Cay(S, \{a\})$ contains a strong group subdigraph $Cay(G, \{p_{\beta i} g\})$;

(2) $((g_1, i, \lambda_1), (g_2, i, \lambda_2))$ is an arc in $Cay(S, \{a\})$ if and only if $g_2 = g_1 p_{\lambda_1 i} g$ and $\lambda_2 = \beta$.

Proof. (1) Let (V, E) be an strong group subdigraph of $Cay(S, \{a\})$ where $V = G \times \{i\} \times \{\beta\}$. We will show that $(V, E) \cong Cay(G, \{p_{\beta i}\}g)$. Define $f : V \to G$ by $f(x, i, \beta) = x$ for all $x \in G$. It is clear that f is a well defined bijection. We will show that f and f^{-1} are digraph homomorphisms.

Let $(x, i, \beta), (y, i, \beta) \in V$ and $((x, i, \beta), (y, i, \beta))$ is an arc in (V, E). It is also an arc in $Cay(S, \{a\})$. Then $(y, i, \beta) = (x, i, \beta)(g, i, \beta)$ and hence $y = xp_{\beta i}g$. Since $x, y \in G$, (x,y) is an arc in $Cay(G, \{p_{\beta i}g\})$. This shows that f is a digraph homomorphism.

Let (x, y) is an arc in $Cay(G, \{p_{\beta i}g\})$. We have that $y = xp_{\beta i}g$. Consider now $(y, i, \beta) = (xp_{\beta i}g, i, \beta) = (x, i, \beta)(g, i, \beta)$ it follows that $((x, i, \beta), (y, i, \beta))$ is an arc in $Cay(S, \{a\})$. Since $(x, i, \beta), (y, i, \beta) \in V$, $((x, i, \beta), (y, i, \beta))$ is an arc in (V, E). This shows that f^{-1} is also a digraph homomorphism.

(2) If $((g_1, i, \lambda_1), (g_2, i, \lambda_2))$ is an arc in $Cay(S, \{a\})$, then $(g_2, i, \lambda_2) = (g_1, i, \lambda_1)(g, i, \beta) = (g_1 p_{\lambda_1 i} g, i, \beta)$. It is easy to see that $g_2 = g_1 p_{\lambda_1 i} g$ and $\lambda_2 = \beta$. Conversely, consider $(g_2, i, \lambda_2) = (g_1 p_{\lambda_1 i} g, i, \beta) = (g_1, i, \lambda_1)(g, i, \beta)$ and also $((g_1, i, \lambda_1), (g_2, i, \lambda_2))$ is an arc in $Cay(S, \{a\})$. □

In the next theorem, we characterize a Cayley digraph of a completely simple semigroup.

Theorem 3.1.3. *A digraph (V, E) is a Cayley digraph of a completely simple semigroup if and only if the following conditions hold:*

(1) (V, E) is the disjoint union of n isomorphic subdigraphs $(V_1, E_1), (V_2, E_2), ..., (V_n, E_n)$ for some $n \in \mathbb{N}$;

(2) (V_i, E_i) has n subdigraphs $(V_{i1}, E_{i1}), (V_{i2}, E_{i2}), ..., (V_{in}, E_{in})$ such that $(V_i, E_i) = \oplus_{j=1}^{n}(V_{ij}, E_{ij})$ with $V_i = V_{ij}$ for every $j \in \{1, 2, .., n\}$;

(3) (V_{ij}, E_{ij}) contains m disjoint strong subdigraphs $(V_{ij}^1, E_{ij}^1), (V_{ij}^2, E_{ij}^2), ..., (V_{ij}^m, E_{ij}^m)$ such that $V_{ij} = \cup_{\alpha=1}^{m} V_{ij}^{\alpha}$;

(4) there exist a group G and a family of digraph isomorphisms $\{f_{ij}^{\alpha}\}_{\alpha=1}^{m}$ such that $f_{ij}^{\alpha} : (V_{ij}^{\alpha}, E_{ij}^{\alpha}) \to Cay(G, a_{ij}^{\alpha} A_{ij}^{\alpha})$ for some $a_{ij}^{\alpha} \in G$, $A_{ij}^{\alpha} \subseteq G$ with $A_{kj}^{\alpha} = A_{tj}^{\alpha}$, $a_{kj}^{\alpha} = a_{tj}^{\alpha}$ for all $k, t \in \{1, 2, ..., n\}$;

(5) for each $u \in V_{ij}^{\alpha}, v \in V_{ij}^{\beta}$, $(u, v) \in E$ if and only if $f_{ij}^{\beta}(v) = f_{ij}^{\alpha}(u) a_{ij}^{\alpha} a$ for some $a \in A_{ij}^{\beta}$.

Proof. (⇒) Let (V, E) be a Cayley digraph of a completely simple semigroup. Then there exists a completely simple semigroup $S = \mathcal{M}(G, I, \Lambda, P)$ where G is a group, $I = \{1, 2, ..., n\}$, $\Lambda = \{1, 2, ..., m\}$, and P is a $\Lambda \times I$ matrix over a group G, such that $(V, E) \cong Cay(S, A)$ for some $A \subseteq S$. Hence we will prove that $(1), (2), (3), (4)$ and (5) are true for $Cay(S, A)$.

(1) For each $i \in I$, set $V_i = G \times \{i\} \times \Lambda$, and $E_i = E(Cay(S, A)) \cap (V_i \times V_i)$. Hence (V_i, E_i) is a strong subdigraph of $Cay(S, A)$ and $Cay(S, A) = \cup_{i=1}^{n}(V_i, E_i)$. We show that $(V_1, E_1), (V_2, E_2), ..., (V_n, E_n)$ are isomorphic. Let $p, q \in I$, $p \neq q$, define a map ϕ from (V_p, E_p) to (V_q, E_q) by $\phi((g, p, r)) = (g, q, r)$. Since $|V_p| = |V_q|$, ϕ is a well defined bijection. To prove that ϕ and ϕ^{-1} are digraph homomorphisms. For $(g, p, r), (g', p, r')$ $\in V_p$, take $((g, p, r), (g', p, r')) \in E_p$. Since $E_p \subseteq E(Cay(S, A)), ((g, p, r), (g', p, r'))$ is an arc in $Cay(S, A)$. By Lemma 3.1.1, there exists $(a, l, r'') \in A$ such that $g' = gp_{rl}a$, $r' = r''$, and thus $(g', q, r') = (gp_{rl}a, q, r'') = (g, q, r)\ (a, l, r'')$. Then $((g, q, r), (g', q, r'))$ is an arc in $Cay(S, A)$. It follows that $((g, q, r), (g', q, r')) \in E_q$. This shows that ϕ is a digraph homomorphism. Similarly, ϕ^{-1} is a digraph homomorphism. Hence ϕ is a digraph isomorphism. Now we prove that $Cay(S, A)$ is the disjoint union of $(V_1, E_1), (V_2, E_2), \ldots, (V_n, E_n)$. By definition of V_i, $S = \dot{\cup} V_i$. Since $E_i \subseteq E\big(Cay(S, A)\big), \dot{\cup} E_i \subseteq E\big(Cay(S, A)\big)$. Let $\big((g, j, r), (\acute{g}, k, \acute{r})\big) \in E(Cay(S, A)$. By Lemma 3.1.1, $j = k$, and thus $\big((g, j, r), (\acute{g}, k, \acute{r})\big) \in E_k$. Then $\big((g, j, r), (\acute{g}, k, \acute{r})\big) \in \dot{\cup} E_i$. Hence $E\big(Cay(S, A)\big) \subseteq \dot{\cup} E_i$, and so $E\big(Cay(S, A)\big) = \dot{\cup} E_i$. Therefore $Cay(S, A) = \dot{\cup}(V_i, E_i)$.

(2) Let $S_{ij} = \mathcal{M}(G, \{i\}, \Lambda, P_j)$ where $i, j \in \{1, 2, .., n\}$,

$$P_j = \begin{bmatrix} p_{1j} \\ p_{2j} \\ . \\ . \\ . \\ p_{mj} \end{bmatrix},$$

and P_j is the j^{th} column of P, and let $A_{ij} = \{(g, i, \beta) | (g, j, \beta) \in A\}$. Take $(V_{ij}, E_{ij}) = Cay(S_{ij}, A_{ij})$. Hence $V_{ij} = \mathcal{M}(G, \{i\}, \Lambda, P_j) = V_i$. To prove that $(V_i, E_i) = \oplus_{j=1}^{n}(V_{ij}, E_{ij})$. Let $((g, i, \alpha), (g', i, \beta)) \in E_i$. Therefore $((g, i, \alpha), (g', i, \beta))$ is an arc in $Cay(S, A)$. By Lemma 3.1.1, there exists $(g'', l, \gamma) \in A$ such that $\gamma = \beta$ and $g' = gp_{\alpha l}g''$. Since $(g'', l, \beta) \in A$, $(g'', i, \beta) \in A_{il}$. Thus $((g, i, \alpha), (g', i, \beta))$ is an arc in $Cay(S_{il}, A_{il})$ as $(g, i, \alpha)(g'', i, \beta) = (gp_{\alpha l}g'', i, \beta) = (g', i, \beta)$. Therefore $((g, i, \alpha)\ , (g', i, \beta)) \in E_{il} \subseteq \cup_{j=1}^{n} E_{ij}$, and so $E_i \subseteq \cup_{j=1}^{n} E_{ij}$. Let $((g, i, \alpha), (g', i, \beta)) \in \cup_{j=1}^{n} E_{ij}$. Therefore

$((g,i,\alpha),(g',i,\beta)) \in E_{il}$ for some $l \in \{1,2,..,n\}$. It follows that $((g,i,\alpha),(g',i,\beta))$ is an arc in $Cay(S_{il},A_{il})$. Then there exists $(g'',i,\gamma) \in A_{il}$ such that $\gamma = \beta$ and $g' = gp_{\alpha l}g''$ by Lemma 3.1.1. Hence $(g'',l,\beta) \in A$ because $(g'',i,\beta) \in A_{il}$. Then $(g',i,\beta) = (gp_{\alpha l}g'',i,\beta) = (g,i,\alpha)(g'',l,\beta)$, $((g,i,\alpha),(g',i,\beta))$ is an arc in $Cay(S,A)$. Therefore $((g,i,\alpha),\ (g',i,\beta)) \in E_i$, and thus $\cup_{j=1}^{n} E_{ij} \subseteq E_i$. Hence $E_i = \cup_{j=1}^{n} E_{ij}$. This show that $(V_i,E_i) = \oplus_{j=1}^{n}(V_{ij},E_{ij})$.

(3) Set $V_{ij}^{\alpha} = \mathcal{M}(G,\{i\},\{\alpha\},P_j^{\alpha})$, $E_{ij}^{\alpha} = E(Cay(S_{ij},A_{ij})) \cap (V_{ij}^{\alpha} \times V_{ij}^{\alpha})$ where $P_j^{\alpha} = [p_{\alpha j}]$. Therefore $V_{ij}^{\alpha} \subseteq V_{ij}$, and thus $(V_{ij}^{\alpha},E_{ij}^{\alpha})$ is a strong subdigraph of (V_{ij},E_{ij}). Let $\alpha,\beta \in \Lambda$ and $\alpha \neq \beta$. To prove that $(V_{ij}^{\alpha},E_{ij}^{\alpha})$ and $(V_{ij}^{\beta},E_{ij}^{\beta})$ are disjoint. Since $V_{ij}^{\alpha} \cap V_{ij}^{\beta} = \emptyset$, by the definition of E_{ij}^{α} and E_{ij}^{β}, $E_{ij}^{\alpha} \cap E_{ij}^{\beta} = \emptyset$. Therefore $(V_{ij}^{\alpha},E_{ij}^{\alpha})$ and $(V_{ij}^{\beta},E_{ij}^{\beta})$ are disjoint subdigraphs of (V_{ij},E_{ij}). Hence $\cup_{\alpha=1}^{m} V_{ij}^{\alpha} = \cup_{\alpha=1}^{m} \mathcal{M}(G,\{i\},\{\alpha\},[p_{\alpha j}]) = \mathcal{M}(G,\{i\},\Lambda,P_j) = V_i$.

(4) Let $A_{ij}^{\alpha} = \{g|(g,i,\alpha) \in A_{ij}\}$. To prove that $(V_{ij}^{\alpha},E_{ij}^{\alpha}) \cong Cay(G,a_{ij}^{\alpha}A_{ij}^{\alpha})$ where $a_{ij}^{\alpha} = p_{\alpha j}$. Let $f_{ij}^{\alpha} : (V_{ij}^{\alpha},E_{ij}^{\alpha}) \to Cay(G,p_{\alpha j}A_{ij}^{\alpha})$ be the projection of V_{ij}^{α} on to its first coordinate, i.e. $f_{ij}^{\alpha} = p_1$. We first show that f_{ij}^{α} and $f_{ij}^{\alpha\,-1}$ are digraph homomorphisms. For (g,i,α), $(g',i,\alpha) \in V_{ij}^{\alpha}$, take $((g,i,\alpha),(g',i,\alpha)) \in E_{ij}^{\alpha}$. By the definition of E_{ij}^{α}, $((g,i,\alpha),(g',i,\alpha))$ is an arc in $Cay(S_{ij},A_{ij})$. Then there exists $(g'',i,\gamma) \in A_{ij}$ such that $g' = gp_{\alpha j}g''$ and $\alpha = \gamma$ by Lemma 3.1.1. Thus there is $g'' \in A_{ij}^{\alpha}$. It follows that $(f_{ij}^{\alpha}(g,i,\alpha),f_{ij}^{\alpha}(g',i,\alpha)) = (g,g')$ is an arc in $Cay(G,p_{\alpha j}A_{ij}^{\alpha})$. Hence f_{ij}^{α} is a digraph homomorphism. For $g,g' \in G$, let (g,g') be an arc in $Cay(G,p_{\alpha j}A_{ij}^{\alpha})$. Therefore $g' = gp_{\alpha j}g''$ for some $g'' \in A_{ij}^{\alpha}$. By the definition of A_{ij}^{α}, there is $(g'',i,\alpha) \in A_{ij}$ and so $(g',i,\alpha) = (gp_{\alpha j}g'',i,\alpha) = (g,i,\alpha)(g'',i,\alpha)$. Therefore $((g,i,\alpha),(g',i,\alpha))$ is an arc in $Cay(S_{ij},A_{ij})$. This shows that $f_{ij}^{\alpha\,-1}$ is a digraph homomorphism. Let $k,t \in \{1,2,...,n\}$. We show that $A_{kj}^{\alpha} = A_{tj}^{\alpha}$. Take $g \in A_{kj}^{\alpha}$. Then $(g,k,\alpha) \in A_{kj}$ and $(g,j,\alpha) \in A$. By the definition of A_{tj}, $(g,t,\alpha) \in A_{tj}$, and thus $g \in A_{tj}^{\alpha}$. This shows that $A_{kj}^{\alpha} \subseteq A_{tj}^{\alpha}$. Similarly $A_{tj}^{\alpha} \subseteq A_{kj}^{\alpha}$. Thus $A_{kj}^{\alpha} = A_{tj}^{\alpha}$ for all $k,t \in \{1,2,...,n\}$. Since $a_{kj}^{\alpha} = p_{\alpha j}$ and $a_{tj}^{\alpha} = p_{\alpha j}$, $a_{kj}^{\alpha} = a_{tj}^{\alpha}$ for all $k,t \in \{1,2,...,n\}$.

(5) For each $u = (g,i,\alpha) \in V_{ij}^{\alpha}$, and $v = (g',i,\beta) \in V_{ij}^{\beta}$. We prove that $((g,i,\alpha),(g',i,\beta)) \in E$ if and only if $f_{ij}^{\beta}(v) = f_{ij}^{\alpha}(u)a_{ij}^{\alpha}a$ for some $a \in A_{ij}^{\beta}$.

Let $((g,i,\alpha),(g',i,\beta)) \in E$. Then $((g,i,\alpha),(g',i,\beta))$ is an arc in $Cay(S,A)$. Hence there exists $(a,j,\xi) \in A$ such that $(g',i,\beta) = (g,i,\alpha)(a,j,\xi) = (gp_{\alpha j}a,i,\xi)$. Therefore $g' = gp_{\alpha j}a$ and $\beta = \xi$. Then we have that $(a,j,\beta) = (a,j,\xi) \in A$. By the definition of A_{ij}, there exists $(a,i,\beta) \in A_{ij}$, and hence $a \in A_{ij}^{\beta}$. Therefore $f_{ij}^{\beta}(v) = g' = gp_{\alpha j}a = f_{ij}^{\alpha}(u)a_{ij}^{\alpha}a$ where $a_{ij}^{\alpha} = p_{\alpha j}$.

Conversely, let $f_{ij}^{\beta}(v) = f_{ij}^{\alpha}(u)a_{ij}^{\alpha}a$ for some $a \in A_{ij}^{\beta}$. Therefore $g' = f_{ij}^{\beta}(v) =$

$f_{ij}^{\alpha}(u)a_{ij}^{\alpha}a = ga_{ij}^{\alpha}a$ and there exists $(a,i,\beta) \in A_{ij}$. By the definition of A_{ij}, there is $(a,j,\beta) \in A$. Since $a_{ij}^{\alpha} = p_{\alpha j}$, $(g',i,\beta) = (ga_{ij}^{\alpha}a,i,\beta) = (gp_{\alpha j}a,i,\beta) = (g,i,\alpha)(a,j,\beta)$, and thus $((g,i,\alpha),(g',i,\beta))$ is an arc in $Cay(S,A)$.

($\Leftarrow$) Choose $k \in \{1,2,...,n\}$, by (1), (2), and (3), we get $V = \cup_{i=1}^{n} \cup_{\alpha=1}^{m} V_{ik}^{\alpha}$ is the disjoint union. Let $S = \mathcal{M}(G,I,\Lambda,P)$ where $I = \{1,2,...,n\}$, $\Lambda = \{1,2,...,m\}$,

$$P = \begin{bmatrix} a_{k1}^{1} & a_{k2}^{1} & \cdots & a_{kn}^{1} \\ a_{k1}^{2} & a_{k2}^{2} & \cdots & a_{kn}^{2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{k1}^{m} & a_{k2}^{m} & \cdots & a_{kn}^{m} \end{bmatrix},$$

and let $A = \cup_{\alpha=1}^{m}\cup_{j=1}^{n}(A_{jk}^{\alpha}\times\{k\}\times\{\alpha\})$. We show that $(V,E) \cong Cay(S,A)$. Define a map f from (V,E) to $Cay(S,A)$ by $f(v) = (f_{ik}^{\alpha}(v),i,\alpha)$ for any $v \in V_{ik}^{\alpha}, i \in \{1,2,...,n\}$, and $\alpha \in \{1,2,...,m\}$. Let $u,v \in V$ and $u = v$. Then $u = v \in V_{ik}^{\beta}$ for some $i \in \{1,2,...,n\}$ and $\beta \in \{1,2,...,m\}$. Hence $f_{ik}^{\beta}(u) = f_{ik}^{\beta}(v)$ and $(f_{ik}^{\beta}(u),i,\beta) = (f_{ik}^{\beta}(v),i,\beta)$. Therefore f is well defined. Let $u,v \in V$ and $f(u) = f(v)$. Then $u \in V_{lk}^{\beta}$ and $v \in V_{tk}^{\delta}$ for some $l,t \in \{1,2,...,n\}$ and $\beta,\delta \in \{1,2,...,m\}$, and so $(f_{lk}^{\beta}(u),l,\beta) = f(u) = f(v) = (f_{tk}^{\delta}(v),t,\delta)$. Hence $f_{lk}^{\beta}(u) = f_{tk}^{\delta}(v)$, $l = t$, and $\beta = \delta$. Then $u,v \in V_{lk}^{\beta}$ and $f_{lk}^{\beta}(u) = f_{lk}^{\beta}(v)$. Since f_{lk}^{β} is an isomorphism, $u = v$. This shows that f is an injection. By (4), $|G \times \{i\} \times \{\alpha\}| = |G| = |V_{ik}^{\alpha}|$ for all $i \in \{1,2,...,n\}$ and $\alpha \in \{1,2,...,m\}$. Thus $|S = \mathcal{M}(G,I,\Lambda,P)| = |\cup_{i=1}^{n} \cup_{\alpha=1}^{m}(G \times \{i\} \times \{\alpha\})| = |\cup_{i=1}^{n} \cup_{\alpha=1}^{m} V_{ik}^{\alpha}| = |V|$. Hence f is a surjection. Now we must prove that f and f^{-1} are digraph homomorphisms. Let $u,v \in V$ and $(u,v) \in E$. By (1), $u,v \in V_t$ for some $t \in \{1,2,...,n\}$. Then $u \in V_{tk}^{\beta}$ and $v \in V_{tk}^{\delta}$ for some $\beta,\delta \in \{1,2,...,m\}$. From (5), $f_{tk}^{\delta}(v) = f_{tk}^{\beta}(u)a_{tk}^{\beta}a$ for some $a \in A_{tk}^{\delta}$. By (4), $A_{tk}^{\delta} = A_{ik}^{\delta}$ and $a_{tk}^{\beta} = a_{ik}^{\beta}$. Hence $a \in A_{ik}^{\delta}$, and thus $(a,k,\delta) \in (A_{ik}^{\delta}\times\{k\}\times\{\delta\}) \subseteq A$. Since a_{tk}^{β} is the entry in the β^{th} row and k^{th} column of P, $a_{ik}^{\beta} = a_{kk}^{\beta} = p_{\beta k}$. Therefore $f(v) = (f_{tk}^{\delta}(v),t,\delta) = (f_{tk}^{\beta}(u)a_{tk}^{\beta}a,t,\delta) = (f_{tk}^{\beta}(u)a_{ik}^{\beta}a,t,\delta) = (f_{tk}^{\beta}(u)p_{\beta k}a,t,\delta) = (f_{tk}^{\beta}(u),t,\beta)(a,k,\delta) = f(u)(a,k,\delta)$. Then we get that $(f(u),f(v))$ is an arc in $Cay(G \times L_n \times R_m, A)$. This shows that f is a digraph homomorphism. Let $g,g' \in G$, $j,t \in \{1,2,...,n\}$, $\beta,\delta \in \{1,2,...,m\}$, and let $((g,j,\beta),(g',t,\delta))$ be an arc in $Cay(S,A)$. Then there exists $(a,k,\xi) \in A$ such that $g' = gp_{\beta k}a$, $t = j$, and $\delta = \xi$. By (4) and $g,g' \in G$, there exists $u \in V_{jk}^{\beta}$ and $v \in V_{jk}^{\delta}$ such that $f_{jk}^{\beta}(u) = g$ and $f_{jk}^{\delta}(v) = g'$. Therefore $f_{jk}^{\delta}(v) = f_{jk}^{\beta}(u)p_{\beta k}a$. By the definition of P and (4), $p_{\beta k} = a_{kk}^{\beta} = a_{jk}^{\beta}$. Hence $(f^{-1}(g,l_j,r_\beta), f^{-1}(g',l_t,r_\delta)) = (f^{-1}(f_{jk}^{\beta}(u),j,\beta), f^{-1}(f_{jk}^{\delta}(v),j,\delta)) = (u,v) \in E$ by (5). Thus f^{-1} is a digraph homomorphism. □

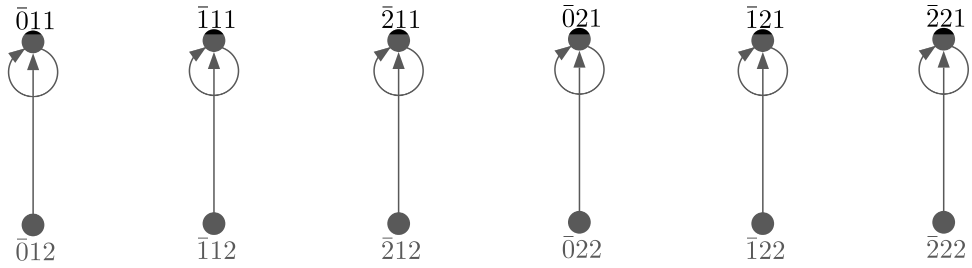

Figure 3.1: Cayley digraph $Cay(\mathcal{M}(\mathbb{Z}_3, \{1,2\}, \{1,2\}, P), \{(\bar{0}, 1, 1)\})$.

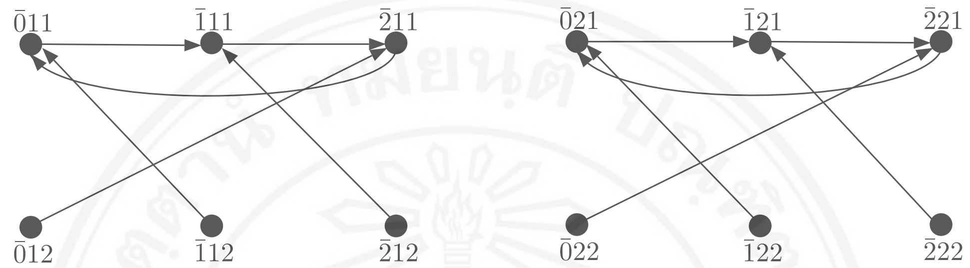

Figure 3.2: Cayley digraph $Cay(\mathcal{M}(\mathbb{Z}_3, \{1,2\}, \{1,2\}, P), \{(\bar{1}, 2, 1)\})$.

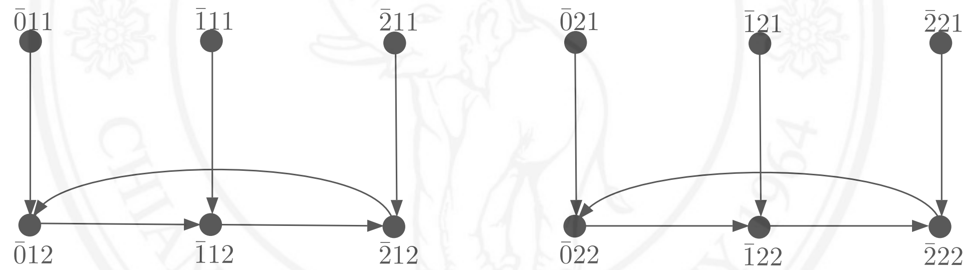

Figure 3.3: Cayley digraph $Cay(\mathcal{M}(\mathbb{Z}_3, \{1,2\}, \{1,2\}, P), \{(\bar{0}, 2, 2)\})$.

Example 3.1.4. Consider the completely simple semigroup $S = \mathcal{M}(\mathbb{Z}_3, I, \Lambda, P), \mathbb{Z}_3 = \{\bar{0}, \bar{1}, \bar{2}\}$ with $I = \{1,2\}$, $\Lambda = \{1,2\}$,

$$P = \begin{bmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{1} \end{bmatrix}, \text{ and thus } P_1 = \begin{bmatrix} \bar{0} \\ \bar{0} \end{bmatrix}, P_2 = \begin{bmatrix} \bar{0} \\ \bar{1} \end{bmatrix},$$

and let $a_1 = (\bar{0}, 1, 1)$, $a_2 = (\bar{1}, 2, 1)$, $a_3 = (\bar{0}, 2, 2)$. Then we give the Cayley digraphs $Cay(S, A)$ for all the three different one-element connection sets A, as indicated in Figures 3.1-3.3.

So we have

$Cay(S, \{a_1\}) = Cay(\mathcal{M}(\mathbb{Z}_3, \{1\}, \Lambda, P_1), \{(\bar{0}, 1, 1)\}) \cup Cay(\mathcal{M}(\mathbb{Z}_3, \{2\}, \Lambda, P_1), \{(\bar{0}, 2, 1)\})$,
$Cay(S, \{a_2\}) = Cay(\mathcal{M}(\mathbb{Z}_3, \{1\}, \Lambda, P_2), \{(\bar{1}, 1, 1)\}) \cup Cay(\mathcal{M}(\mathbb{Z}_3, \{2\}, \Lambda, P_2), \{(\bar{1}, 2, 1)\})$,
$Cay(S, \{a_3\}) = Cay(\mathcal{M}(\mathbb{Z}_3, \{1\}, \Lambda, P_2), \{(\bar{0}, 1, 2)\}) \cup Cay(\mathcal{M}(\mathbb{Z}_3, \{2\}, \Lambda, P_2), \{(\bar{0}, 2, 2)\})$.

Therefore $Cay(S, \{a_1, a_2, a_3\}) =$
$\big[Cay(\mathcal{M}(\mathbb{Z}_3, \{1\}, \Lambda, P_1), \{(\bar{0}, 1, 1)\}) \oplus Cay(\mathcal{M}(\mathbb{Z}_3, \{1\}, \Lambda, P_2), \{(\bar{1}, 1, 1), (\bar{0}, 1, 2)\})\big]$
$\cup \big[Cay(\mathcal{M}(\mathbb{Z}_3, \{2\}, \Lambda, P_1), \{(\bar{0}, 2, 1)\}) \oplus Cay(\mathcal{M}(\mathbb{Z}_3, \{2\}, \Lambda, P_2), \{(\bar{1}, 2, 1), (\bar{0}, 2, 2)\})\big]$.

We see that $Cay(S, \{a_1, a_2, a_3\}) = (V_1, E_1) \cup (V_2, E_2)$ where
$(V_1, E_1) = \big[Cay(\mathcal{M}(\mathbb{Z}_3, \{1\}, \Lambda, P_1), \{(\bar{0}, 1, 1)\})$
$\oplus Cay(\mathcal{M}(\mathbb{Z}_3, \{1\}, \Lambda, P_2)\ , \{(\bar{1}, 1, 1), (\bar{0}, 1, 2)\})\big]$ and
$(V_2, E_2) = \big[Cay(\mathcal{M}(\mathbb{Z}_3, \{2\}, \Lambda, P_1), \{(\bar{0}, 2, 1)\})$
$\oplus Cay(\mathcal{M}(\mathbb{Z}_3, \{2\}, \Lambda, P_2)\ , \{(\bar{1}, 2, 1), (\bar{0}, 2, 2)\})\big]$.

Let $(V_{11}, E_{11}) = Cay(\mathcal{M}(\mathbb{Z}_3, \{1\}, \Lambda, P_1), \{(\bar{0}, 1, 1)\})$,
$(V_{12}, E_{12}) = Cay(\mathcal{M}(\mathbb{Z}_3, \{1\}, \Lambda, P_2), \{(\bar{1}, 1, 1), (\bar{0}, 1, 2)\})$,
$(V_{21}, E_{21}) = Cay(\mathcal{M}(\mathbb{Z}_3, \{2\}, \Lambda, P_1), \{(\bar{0}, 2, 1)\})$, and
$(V_{22}, E_{22}) = Cay(\mathcal{M}(\mathbb{Z}_3, \{2\}, \Lambda, P_2), \{(\bar{1}, 2, 1), (\bar{0}, 2, 2)\})$.

Then we get that $(V_i, E_i) = \oplus_{j=1}^{2}(V_{ij}, E_{ij})$ for every $i \in \{1, 2\}$.

3.2 The Structure of Cayley Digraphs of Completely Simple Semigroups with one-element Connection Sets

In the following results, we describe the structure of Cayley digraphs of a completely simple semigroup with a one-element connection set. By the proof of Theorem 3.1.3(1-2) we have the following lemma.

Lemma 3.2.1. *Let $S = \mathcal{M}(G, I, \Lambda, P)$ be a completely simple semigroup, $I = \{1, 2 \ldots, n\}$, $\Lambda = \{1, 2 \ldots, m\}$, $a = (g, j, \beta) \in S$, P_i the i^{th} column of P. Then $Cay(S, \{a\})$ is the disjoint union of n isomorphic strong subdigraphs $Cay(S_1, \{(g, 1, \beta)\}), Cay(S_2, \{(g, 2, \beta)\})$, $\ldots, Cay(S_n, \{(g, n, \beta)\})$ where $S_i = \mathcal{M}(G, \{i\}, \Lambda, P_j)$.*

Lemma 3.2.2. *Let $S = \mathcal{M}(G, I, \Lambda, P)$ be a completely simple semigroup, $I = \{1, 2 \ldots, n\}$, $\Lambda = \{1, 2 \ldots, m\}$, $a = (g, j, \beta) \in S$, $G/\langle p_{\beta j} g\rangle = \{g_1\langle p_{\beta j} g\rangle, g_2\langle p_{\beta j} g\rangle, \ldots, g_t\langle p_{\beta j} g\rangle\}$ and let $M_{ik} = \big(g_k\langle p_{\beta j} g\rangle \times \{i\} \times \{\beta\}\big) \bigcup \big(\cup_{\alpha \neq \beta}(g_k\langle p_{\beta j} g\rangle\ g^{-1} p_{\alpha j}^{-1} \times \{i\} \times \{\alpha\})\big)$, where $k \in \{1, 2, \ldots, t\}$ and $i \in I$. Then $M_{i1}, M_{i2}, \ldots, M_{it}$ are disjoint.*

Proof. Since $\{g_1\langle p_{\beta j} g\rangle\ g_2\langle p_{\beta j} g\rangle, \ldots, g_t\langle p_{\beta j} g\rangle\}$ is the set of all distinct left cosets of $\langle p_{\beta j} g\rangle$ in G, we get that $M_{i1}, M_{i2}, \ldots, M_{it}$ are disjoint. □

Lemma 3.2.3. *Let $S = \mathcal{M}(G, I, \Lambda, P)$ be a completely simple semigroup, $I = \{1, 2 \ldots, n\}$, $\Lambda = \{1, 2 \ldots, m\}$, $a = (g, j, \beta) \in S$, (M_{ik}, E_{ik}) a strong subdigraph of $Cay(S, \{a\})$. Then $(M_{ik_1}, E_{ik_1}) \cong (M_{ik_2}, E_{ik_2})$ for all $k_1, k_2 \in \{1, 2, \ldots, t\}$.*

Proof. We define $f : (M_{ik_1}, E_{ik_1}) \to (M_{ik_2}, E_{ik_2})$ by

$$\big(g_{k_1}(p_{\beta j}g)^r, i, \beta\big) \mapsto \big(g_{k_2}(p_{\beta j}g)^r, i, \beta\big)$$
$$\big(g_{k_1}(p_{\beta j}g)^r g^{-1}p_{\alpha j}^{-1}, i, \alpha\big) \mapsto \big(g_{k_2}(p_{\beta j}g)^r g^{-1}p_{\alpha j}^{-1}, i, \alpha\big) \quad for\ \alpha \neq \beta.$$

Since, for all $k \in \{1, 2, \ldots, t\}$, $g_k\langle p_{\beta j}g\rangle = \{g_k(p_{\beta j}g),\ g_k(p_{\beta j}g)^2, \ldots,\ g_k(p_{\beta j}g)^{|\langle p_{\beta j}g\rangle|}\}$, f is a well defined bijection.

We must prove that f and f^{-1} are homomorphisms. For $x, y \in M_{ik_1}$, let $(x, y) \in E_{ik_1}$. Then (x, y) is an arc in $Cay(S, \{a\})$, so $y = xa$. By Proposition 3.1.2(2), $p_3(y) = \beta$. Therefore $y \in (g_{k_1}\langle p_{\beta j}g\rangle \times \{i\} \times \{\beta\})$, and so $y = \big(g_{k_1}(p_{\beta j}g)^d, i, \beta\big)$ for some $d \in \{1, 2, \ldots, |\langle p_{\beta j}g\rangle|\}$. We consider the following two cases.

(case1) If $x = \big(g_{k_1}(p_{\beta j}g)^h, i, \beta\big)$ for some $h \in \{1, 2, \ldots, |\langle p_{\beta j}g\rangle|\}$, then

$$\begin{aligned}\big(g_{k_1}(p_{\beta j}g)^d, i, \beta\big) &= \big(g_{k_1}(p_{\beta j}g)^h, i, \beta\big)(g, j, \beta)\\ &= \big(g_{k_1}(p_{\beta j}g)^h p_{\beta j}g, i, \beta\big).\end{aligned}$$

Thus $(p_{\beta j}g)^d = (p_{\beta j}g)^h p_{\beta j}g$, so

$$\begin{aligned}f(y) &= f\big(g_{k_1}(p_{\beta j}g)^d, i, \beta\big)\\ &= \big(g_{k_2}(p_{\beta j}g)^d, i, \beta\big)\\ &= \big(g_{k_2}(p_{\beta j}g)^h p_{\beta j}g, i, \beta\big)\\ &= \big(g_{k_2}(p_{\beta j}g)^h, i, \beta\big)(g, j, \beta)\\ &= f\big(g_{k_1}(p_{\beta j}g)^h, i, \beta\big)a = f(x)a.\end{aligned}$$

Therefore $\big(f(x),\ f(y)\big)$ is an arc in (M_{ik_2}, E_{ik_2}).

(case2) If $x = \big(g_{k_1}(p_{\beta j}g)^{h'} g^{-1}p_{\alpha j}^{-1}, i, \alpha\big)$ for some $\alpha \neq \beta$, and $h' \in \{1, 2, \ldots, |\langle p_{\beta j}g\rangle|\}$, then

$$\begin{aligned}\big(g_{k_1}(p_{\beta j}g)^d, i, \beta\big) &= \big(g_{k_1}(p_{\beta j}g)^{h'} g^{-1}p_{\alpha j}^{-1}, i, \alpha\big)(g, j, \beta)\\ &= \big(g_{k_1}(p_{\beta j}g)^{h'} g^{-1}p_{\alpha j}^{-1}p_{\alpha j}g, i, \beta\big)\\ &= \big(g_{k_1}(p_{\beta j}g)^{h'}, i, \beta\big).\end{aligned}$$

Hence $(p_{\beta j}g)^d = (p_{\beta j}g)^{h'}$, and so

$$\begin{aligned}f(y) &= f\big(g_{k_1}(p_{\beta j}g)^d, i, \beta\big)\\ &= \big(g_{k_2}(p_{\beta j}g)^d g^{-1}p_{\alpha j}^{-1}p_{\alpha j}g, i, \beta\big)\\ &= \big(g_{k_2}(p_{\beta j}g)^{h'} g^{-1}p_{\alpha j}^{-1}, i, \alpha\big)(g, j, \beta)\\ &= f\big(g_{k_1}(p_{\beta j}g)^{h'} g^{-1}p_{\alpha j}^{-1}, i, \alpha\big)a = f(x)a.\end{aligned}$$

Therefore $\big(f(x), f(y)\big)$ is an arc in (M_{ik_2}, E_{ik_2}).

This means that f is a digraph homomorphism. Similarly, f^{-1} is a digraph homomorphism. Hence $(M_{ik_1}, E_{ik_1}) \cong (M_{ik_2}, E_{ik_2})$. $\square$

Lemma 3.2.4. *Let $S = \mathcal{M}(G, I, \Lambda, P)$ be a completely simple semigroup, $I = \{1, 2 \ldots, n\}$, $\Lambda = \{1, 2 \ldots, m\}$, $a = (g, j, \beta) \in S$. Then $Cay(S_i, \{(g, i, \beta)\}) = \dot{\cup}_{k=1}^{t}(M_{ik}, E_{ik})$.*

Proof. We first show that $S_i = \dot{\cup}_{k=1}^{t} M_{ik}$. Since $M_{ik} \subseteq S_i$ for all $k \in \{1, 2, ..., t\}$, $\dot{\cup}_{k=1}^{t} M_{ik} \subseteq S_i$. We will show that $S_i \subseteq \dot{\cup}_{k=1}^{t} M_{ik}$. Let $x = (g', i, \lambda) \in S_i$, we get $g' \in G = \cup_{k=1}^{t} g_k \langle p_{\beta j} g\rangle$, and thus $g' = g_w (p_{\beta j} g)^v$ for some $v \in \mathbb{N}$ and $w \in \{1, 2, \ldots, t\}$. We need to consider the following two cases.

(case1) If $\lambda = \beta$, then $x = \big(g_w (p_{\beta j} g)^v, i, \beta\big) \in \big(g_w \langle p_{\beta j} g\rangle \times \{i\} \times \{\beta\}\big) \subseteq M_{iw} \subseteq \dot{\cup}_{k=1}^{t} M_{ik}$.

(case2) If $\lambda \neq \beta$, then $xa = \big(g_w (p_{\beta j} g)^v, i, \lambda\big)(g, j, \beta) = \big(g_w (p_{\beta j} g)^v\ p_{\lambda j} g, i, \beta\big)$. Since $g_w (p_{\beta j} g)^v p_{\lambda j} g \in G = \cup_{k=1}^{t} g_k \langle p_{\beta j} g\rangle$, $g_w (p_{\beta j} g)^v p_{\lambda j} g \in g_u \langle p_{\beta j} g\rangle$ for some $u \in \{1, 2, \ldots, t\}$, we get that $g_w (p_{\beta j} g)^v p_{\lambda j} g = g_u (p_{\beta j} g)^{v'}$ for some $\acute{v} \in \mathbb{N}$, and thus $g_w (p_{\beta j} g)^v = g_u (p_{\beta j} g)^{v'} g^{-1} p_{\lambda j}^{-1}$. Therefore $x = \big(g_w (p_{\beta j} g)^v, i, \lambda\big) = \big(g_u (p_{\beta j} g)^{v'} g^{-1} p_{\lambda j}^{-1}, i, \lambda\big) \in \big(g_u \langle p_{\beta j} g\rangle g^{-1} p_{\lambda j}^{-1} \times \{i\} \times \{\lambda\}\big) \subseteq M_{iu} \subseteq \dot{\cup}_{k=1}^{t} M_{ik}$.

Hence $S_i \subseteq \dot{\cup}_{k=1}^{t} M_{ik}$. Then we conclude that $S_i = \dot{\cup}_{k=1}^{t} M_{ik}$.

Since $(M_{i1}, E_{i1}), (M_{i2}, E_{i2}), \ldots, (M_{it}, E_{it})$ are strong subdigraphs of $Cay(S_i, \{(g, i, \beta)\})$, $E\big(\dot{\cup}_{k=1}^{t}(M_{ik}, E_{ik})\big) \subseteq E\big(Cay(S_i, \{(g, i, \beta)\})\big)$. Let $x = (u_1, i, \lambda_1), y = (u_2, i, \lambda_2) \in S_i$ and (x, y) be an arc in $Cay(S_i, \{(g, i, \beta)\})$. Therefore $u_2 = u_1 p_{\lambda_1 j} g$ and $\lambda_2 = \beta$ by Proposition 3.1.2(2). Since $S_i = \dot{\cup}_{k=1}^{t} M_{ik}$, $x \in M_{ib_1}$ and $y \in M_{ib_2}$ for some $b_1, b_2 \in \{1, 2, \ldots, t\}$. Hence $y \in \big(g_{b_2} \langle p_{\beta j} g\rangle \times \{i\} \times \{\beta\}\big)$, and thus $y = (g_{b_2} (p_{\beta j} g)^{d'}, i, \beta)$ for some $d' \in \{1, 2, \ldots, |\langle p_{\beta j} g\rangle|\}$. Then $u_2 = g_{b_2} (p_{\beta j} g)^{d'}$. We consider the following two cases.

(case1) If $\lambda_1 \neq \beta$, then $x \in \big(g_{b_1} \langle p_{\beta j} g\rangle g^{-1} p_{\lambda_1 j}^{-1} \times \{i\} \times \{\lambda_1\}\big)$. Hence $x = \big(g_{b_1} (p_{\beta j} g)^c g^{-1} p_{\lambda_1 j}^{-1}, i, \lambda_1\big)$ for some $c \in \{1, 2, ..., |\langle p_{\beta j} g\rangle|\}$, and thus $u_1 = g_{b_1} (p_{\beta j} g)^c g^{-1} p_{\lambda_1 j}^{-1}$. Since $u_2 = u_1 p_{\lambda_1 j} g$,

$$\begin{aligned} g_{b_2} (p_{\beta j} g)^{d'} &= g_{b_1} (p_{\beta j} g)^c g^{-1} p_{\lambda_1 j}^{-1} p_{\lambda_1 j} g \\ &= g_{b_1} (p_{\beta j} g)^c. \end{aligned}$$

Then $b_1 = b_2$, and thus $x, y \in M_{ib_1}$. Hence $(x, y) \in E_{ib_1}$. We get that (x, y) is an arc in $\dot{\cup}_{k=1}^{t}(M_{ik}, E_{ik})$.

(case2) If $\lambda_1 = \beta$, then $x \in \big(g_{b_1}\langle p_{\beta j}g\rangle \times \{i\} \times \{\beta\}\big)$. Hence $x = \big(g_{b_1}(p_{\beta j}g)^{c'}, i, \beta\big)$ for some $c' \in \{1, 2, ..., |\langle p_{\beta j}g\rangle|\}$, and so $u_1 = g_{b_1}(p_{\beta j}g)^{c'}$. Since $u_2 = u_1 p_{\lambda_1 j} g$,

$$\begin{aligned} g_{b_2}(p_{\beta j}g)^{d'} &= g_{b_1}(p_{\beta j}g)^{c'} p_{\lambda_1 j} g \\ &= g_{b_1}(p_{\beta j}g)^{c'} p_{\beta j} g \\ &= g_{b_1}(p_{\beta j}g)^{c'+1}. \end{aligned}$$

Therefore $b_1 = b_2$ and thus $x, y \in M_{ib_1}$. Hence $(x, y) \in E_{ib_1}$. We get that (x, y) is an arc in $\dot\cup_{k=1}^{t}(M_{ik}, E_{ik})$.

Hence $E\big(\dot\cup_{k=1}^{t}(M_{ik}, E_{ik})\big) = E(Cay(S_i, \{(g, i, \beta)\}))$. We conclude that $Cay(S_i, \{(g, i, \beta)\}) = \dot\cup_{k=1}^{t}(M_{ik}, E_{ik})$. □

Lemma 3.2.5. *Let $S = \mathcal{M}(G, I, \Lambda, P)$ be a completely simple semigroup, $I = \{1, 2, \ldots, n\}$, $\Lambda = \{1, 2 \ldots, m\}$, $a = (g, j, \beta) \in S$. Then $(M_{ik}, E_{ik}) \cong Cay\big(\langle p_{\beta j}g\rangle \times R_m, \{(p_{\beta j}g, r_\beta)\}\big)$ where $R_m = \{r_1, r_2, \ldots, r_m\}$ is a right zero semigroup.*

Proof. We define $f : (M_{ik}, E_{ik}) \to Cay\big(\langle p_{\beta j}g\rangle \times R_m, \{(p_{\beta j}g, r_\beta)\}\big)$ by

$$\big(g_k(p_{\beta j}g)^q, i, \beta\big) \mapsto \big(g_k(p_{\beta j}g)^q, r_\beta\big)$$
$$\big(g_k(p_{\beta j}g)^q g^{-1} p_{\alpha j}^{-1}, i, \alpha\big) \mapsto \big(g_k(p_{\beta j}g)^{q-1}, r_\alpha\big) \ for \alpha \neq \beta.$$

Clearly, f is well defined. We will show that f is a bijection. Let $x, y \in M_{ik}$, so $x = \big(g_k(p_{\beta j}g)^{t_1}, i, \beta\big)$ or $x = \big(g_k(p_{\beta j}g)^{t_1'} g^{-1} p_{\alpha j}^{-1}, i,\ \alpha\big)$ for some $\alpha \neq \beta$ and $t_1, t_1' \in \{1, 2, ..., |\langle p_{\beta j}g\rangle|\}$, and $y = \big(g_k(p_{\beta j}g)^{t_2}, i, \beta\big)$ or $y = \big(g_k(p_{\beta j}g)^{t_2'} g^{-1} p_{\gamma j}^{-1}, i,\ \gamma\big)$ for some $\gamma \neq \beta$ and $t_2, t_2' \in \{1, 2, ..., |\langle p_{\beta j}g\rangle|\}$. Let $f(x) = f(y)$. Then $p_2(f(x)) = p_2(f(y))$. By the definition of f, we get $p_3(x) = p_3(y)$. We consider the following two cases.

(case1) For $x = \big(g_k(p_{\beta j}g)^{t_1}, i, \beta\big)$ and $y = \big(g_k(p_{\beta j}g)^{t_2}, i, \beta\big)$. Since $f(x) = f(y)$, $\big(g_k(p_{\beta j}g)^{t_1}, r_\beta\big) = \big(g_k(p_{\beta j}g)^{t_2}, r_\beta\big)$. We get that $g_k(p_{\beta j}g)^{t_1} = g_k(p_{\beta j}g)^{t_2}$. Hence $x = y$.

(case2) For $x = \big(g_k(p_{\beta j}g)^{t_1'} g^{-1} p_{\alpha j}^{-1}, i, \alpha\big)$ and $y = \big(g_k(p_{\beta j}g)^{t_2'} g^{-1} p_{\gamma j}^{-1}, i,\ \gamma\big)$ for some $\alpha, \gamma \neq \beta$. Since $f(x) = f(y)$, $\big(g_k(p_{\beta j}g)^{t_1'-1}, r_\alpha\big) = \big(g_k(p_{\beta j}g)^{t_2'-1}, r_\gamma\big)$. Hence $g_k(p_{\beta j}g)^{t_1'-1} = g_k(p_{\beta j}g)^{t_2'-1}$ and $r_\alpha = r_\gamma$. Therefore $\alpha = \gamma$ and $g_k(p_{\beta j}g)^{t_1'} = g_k(p_{\beta j}g)^{t_2'}$. This means that $x = y$.

Then f is an injection . Let $z \in \langle p_{\beta j}g\rangle \times R_m$, so $z = \big(g_k(p_{\beta j}g)^q, r_\beta\big)$ for some $q \in \{1, 2, ..., |\langle p_{\beta j}g\rangle|\}$ or $z = \big(g_k(p_{\beta j}g)^{q'}, r_\alpha\big)$ for some $\alpha \neq \beta$ and $q' \in \{1, 2, ..., |\langle p_{\beta j}g\rangle|\}$. We need to consider the following two cases.

(case1) If $z = \big(g_k(p_{\beta j}g)^q, r_\beta\big)$, there exists $x = \big(g_k(p_{\beta j}g)^q, i, \beta\big) \in M_{ik}$ such that $f(x) = f\big(g_k(p_{\beta j}g)^q,\, i, \beta\big) = \big(g_k(p_{\beta j}g)^q, r_\beta\big) = z$.

(case2) If $z = \big(g_k(p_{\beta j}g)^{q'}, r_\alpha\big)$, there exists $x = \big(g_k(p_{\beta j}g)^{q'+1}g^{-1}p_{\alpha j}^{-1},\, i, \alpha\big) \in M_{ik}$ such that $f(x) = f\big(g_k(p_{\beta j}g)^{q'+1}g^{-1}p_{\alpha j}^{-1}, i, \alpha\big) = \big(g_k(p_{\beta j}g)^{q'}, r_\alpha\big) = z$.

This means that f is a surjection.

We will show that f and f^{-1} are digraph homomorphisms. For $x, y \in M_{ik}$, let $(x, y) \in E_{ik}$. Then (x, y) is an arc in $Cay(S, \{a\})$, and thus $y = xa$. By Proposition 3.1.2(2), $p_3(y) = \beta$. Hence $y \in (g_k\langle p_{\beta j}g\rangle \times \{i\} \times \{\beta\})$, and so $y = (g_k(p_{\beta j}g)^c, i, \beta)$ for some $c \in \{1, 2, \ldots, |\langle p_{\beta j}g\rangle|\}$. We need only consider two cases.

(case1) If $x \in (g_k\langle p_{\beta j}g\rangle \times \{i\} \times \{\beta\})$, then $x = \big(g_k(p_{\beta j}g)^d, i, \beta\big)$ for some $d \in \{1, 2, \ldots, |\langle p_{\beta j}g\rangle|\}$. Since $y = xa$, $\big(g_k(p_{\beta j}g)^c, i, \beta\big) = \big(g_k(p_{\beta j}g)^d, i, \beta\big)(g, j, \beta) = \big(g_k(p_{\beta j}g)^d p_{\beta j}g, i, \beta\big)$. Thus $g_k(p_{\beta j}g)^c = g_k(p_{\beta j}g)^d p_{\beta j}g$, and so

$$\begin{aligned} f(y) &= f\big(g_k(p_{\beta j}g)^c, i, \beta\big) \\ &= \big(g_k(p_{\beta j}g)^c, r_\beta\big) \\ &= \big(g_k(p_{\beta j}g)^d p_{\beta j}g, r_\beta\big) \\ &= \big(g_k(p_{\beta j}g)^d, r_\beta\big)\big(p_{\beta j}g, r_\beta\big) \\ &= f(x)(p_{\beta j}g, r_\beta). \end{aligned}$$

Therefore $\big(f(x), f(y)\big)$ is an arc in $Cay\big(\langle p_{\beta j}g\rangle \times R_m, \{(p_{\beta j}g, r_\beta)\}\big)$.

(case2) If $x \in \big(\cup_{\alpha\neq\beta}\, (g_k\langle p_{\beta j}g\rangle g^{-1}p_{\alpha j}^{-1} \times \{i\} \times \{\alpha\})\big)$, then $x = (g_k(p_{\beta j}g)^{d'} g^{-1}p_{\alpha j}^{-1}, i, \alpha)$ for some $\alpha \neq \beta$ and $d' \in \{1, 2, \ldots, |\langle p_{\beta j}g\rangle|\}$. Since $y = xa$,

$$\begin{aligned} \big(g_k(p_{\beta j}g)^c, i, \beta\big) &= \big(g_k(p_{\beta j}g)^{d'} g^{-1}p_{\alpha j}^{-1}, i, \alpha\big)(g, j, \beta) \\ &= \big(g_k(p_{\beta j}g)^{d'} g^{-1}p_{\alpha j}^{-1}p_{\alpha j}g, i, \beta\big) \\ &= \big(g_k(p_{\beta j}g)^{d'},\ i, \beta\big). \end{aligned}$$

Thus $g_k(p_{\beta j}g)^c = g_k(p_{\beta j}g)^{d'}$, and so

$$\begin{aligned} f(y) &= \big(g_k(p_{\beta j}g)^c, r_\beta\big) \\ &= \big(g_k(p_{\beta j}g)^{d'}, r_\beta\big) \\ &= \big(g_k(p_{\beta j}g)^{d'-1}p_{\beta j}g, r_\beta\big) \\ &= \big(g_k(p_{\beta j}g)^{d'-1}, r_\alpha\big)(p_{\beta j}g, r_\beta) \\ &= f(x)(p_{\beta j}g, r_\beta). \end{aligned}$$

Therefore $\big(f(x), fy)\big)$ is an arc in $Cay\big(\langle p_{\beta j}g\rangle \times R_m, \{(p_{\beta j}g, r_\beta)\}\big)$.

This means that f is a digraph homomorphism.

For $x, y \in M_{ik}$, let $\big(f(x), f(y)\big)$ be an arc in $Cay\big(\langle p_{\beta j}g\rangle \times R_m, \{(p_{\beta j}g, r_\beta)\}\big)$, then $f(y) = f(x)(p_{\beta j}g, r_\beta)$. By the definition of right group, $p_2(f(y)) = r_\beta$, and so $p_3(y) = \beta$. Hence $y \in (g_k\langle p_{\beta j}g\rangle \times \{i\} \times \{\beta\})$, and thus $y = (g_k(p_{\beta j}g)^c, i, \beta)$ for some $c \in \{1, 2, \ldots, |\langle p_{\beta j}g\rangle|\}$. We consider the following two cases.

(case1) If $x \in (g_k\langle p_{\beta j}g\rangle \times \{i\} \times \{\beta\})$, then $x = \big(g_k(p_{\beta j}g)^d, i, \beta\big)$ for some $d \in \{1, 2, \ldots, |\langle p_{\beta j}g\rangle|\}$. Since $f(y) = f(x)(p_{\beta j}g, r_\beta)$,

$$\begin{aligned}\big(g_k(p_{\beta j}g)^c, r_\beta\big) &= \big(g_k(p_{\beta j}g)^d, r_\beta\big)(p_{\beta j}g, r_\beta)\\ &= \big(g_k(p_{\beta j}g)^d p_{\beta j}g, r_\beta\big).\end{aligned}$$

Thus $g_k(p_{\beta j}g)^c = g_k(p_{\beta j}g)^d p_{\beta j}g$. Hence

$$\begin{aligned}y &= \big(g_k(p_{\beta j}g)^c, i, \beta\big)\\ &= \big(g_k(p_{\beta j}g)^d p_{\beta j}g, i, \beta\big)\\ &= \big(g_k(p_{\beta j}g)^d, i, \beta\big)(g, j, \beta)\\ &= xa.\end{aligned}$$

Therefore (x, y) is an arc in (M_{ik}, E_{ik}).

(case2) If $x \in \big(\cup_{\alpha\neq\beta}\,(g_k\langle p_{\beta j}g\rangle g^{-1}p_{\alpha j}^{-1} \times \{i\} \times \{\alpha\})\big)$, then $x = (g_k(p_{\beta j}g)^{d'} g^{-1}p_{\alpha j}^{-1}, i, \alpha)$ for some $\alpha \neq \beta$ and $d' \in \{1, 2, \ldots, |\langle p_{\beta j}g\rangle|\}$. Since $f(y) = f(x)(p_{\beta j}g, r_\beta)$,

$$\begin{aligned}\big(g_k(p_{\beta j}g)^c, r_\beta\big) &= \big(g_k(p_{\beta j}g)^{d'-1}, r_\alpha\big)(p_{\beta j}g, r_\beta)\\ &= \big(g_k(p_{\beta j}g)^{d'}, r_\beta\big).\end{aligned}$$

Thus $g_k(p_{\beta j}g)^c = g_k(p_{\beta j}g)^{d'}$. Hence

$$\begin{aligned}y &= \big(g_k(p_{\beta j}g)^c, i, \beta\big)\\ &= \big(g_k(p_{\beta j}g)^{d'}, i, \beta\big)\\ &= \big(g_k(p_{\beta j}g)^{d'} g^{-1}p_{\alpha j}^{-1}p_{\alpha j}g, i, \beta\big)\\ &= \big(g_k(p_{\beta j}g)^{d'} g^{-1}p_{\alpha j}^{-1}, i, \alpha\big)(g, j, \beta)\\ &= xa,\end{aligned}$$

and so (x, y) is an arc in (M_{ik}, E_{ik}).

This means that f^{-1} is a digraph homomorphism. Hence $(M_{ik}, E_{ik}) \cong Cay\big(\langle p_{\beta j}g\rangle \times R_m, \{(p_{\beta j}g, r_\beta)\}\big)$. □

Example 3.2.6. Consider the completely simple semigroup $S = \mathcal{M}(\mathbb{Z}_2 \times \mathbb{Z}_2, I, \Lambda, P), \mathbb{Z}_2 \times \mathbb{Z}_2 = \{\bar{0}\bar{0}, \bar{0}\bar{1}, \bar{1}\bar{0}, \bar{1}\bar{1}\}$ with $I = \{1, 2\}$, $\Lambda = \{1, 2\}$,

$$P = \begin{bmatrix} \bar{0}\bar{0} & \bar{1}\bar{0} \\ \bar{0}\bar{1} & \bar{1}\bar{1} \end{bmatrix}, \text{ and thus } P_1 = \begin{bmatrix} \bar{0}\bar{0} \\ \bar{0}\bar{1} \end{bmatrix}, P_2 = \begin{bmatrix} \bar{1}\bar{0} \\ \bar{1}\bar{1} \end{bmatrix}.$$

Then we give the Cayley digraph $Cay\big(\mathcal{M}(\mathbb{Z}_2 \times \mathbb{Z}_2, I, \Lambda, P), \{(\bar{1}\bar{0}, 1, 2)\}\big)$ in Figure 3.4.

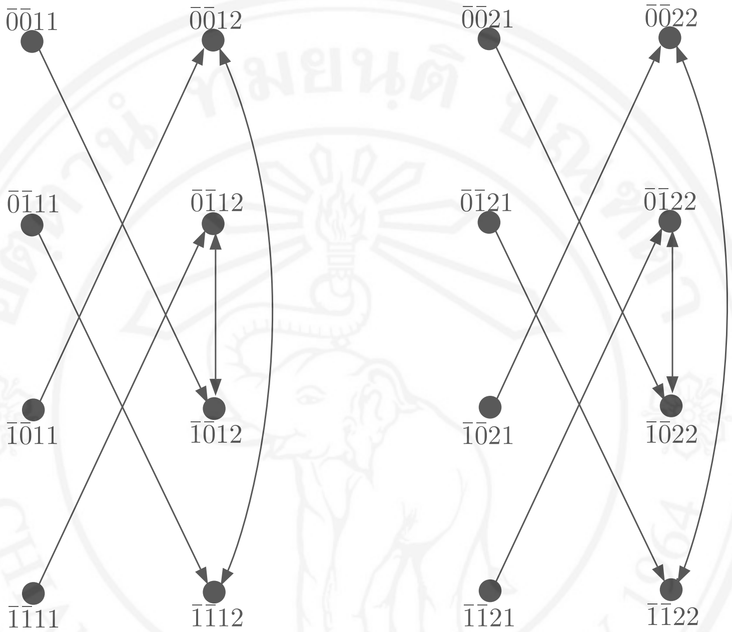

Figure 3.4: Cayley digraph $Cay(\mathcal{M}(\mathbb{Z}_2 \times \mathbb{Z}_2, \{1, 2\}, \{1, 2\}, P), \{(\bar{1}\bar{0}, 1, 2)\})$.

In Figure 3.4, we see that $Cay\big(S, \{(\bar{1}\bar{0}, 1, 2)\}\big) = \dot{\cup}_{i=1}^{2} Cay(S_i, \{(g, i, \beta)\})$ where $S_1 = \mathcal{M}(\mathbb{Z}_2 \times \mathbb{Z}_2, \{1\}, \Lambda, P_1)$, $S_2 = \mathcal{M}(\mathbb{Z}_2 \times \mathbb{Z}_2, \{2\}, \Lambda, P_2)$, $(g, 1, \beta) = \{(\bar{1}\bar{0}, 1, 2)\}$ and $(g, 2, \beta) = \{(\bar{1}\bar{0}, 2, 2)\}$. Then it is the union of right group digraphs.

We have $\langle p_{21}\bar{1}\bar{0}\rangle = \{\bar{1}\bar{1}, \bar{0}\bar{0}\}, \mathbb{Z}_2 \times \mathbb{Z}_2 / \langle p_{21}\bar{1}\bar{0}\rangle = \{g_1\langle p_{21}\bar{1}\bar{0}\rangle, g_2\langle p_{21}\bar{1}\bar{0}\rangle\}$ where $g_1 = \bar{0}\bar{0}, g_2 = \bar{0}\bar{1}$. Hence

$M_{11} = \{(\bar{1}\bar{1}, 1, 2), (\bar{0}\bar{0}, 1, 2), (\bar{0}\bar{1}, 1, 1), (\bar{1}\bar{0}, 1, 1)\}$,

$M_{12} = \{(\bar{1}\bar{0}, 1, 2), (\bar{0}\bar{1}, 1, 2), (\bar{0}\bar{0}, 1, 1), (\bar{1}\bar{1}, 1, 1)\}$,

$M_{21} = \{(\bar{1}\bar{1}, 2, 2), (\bar{0}\bar{0}, 2, 2), (\bar{0}\bar{1}, 2, 1), (\bar{1}\bar{0}, 2, 1)\}$,

$M_{22} = \{(\bar{0}\bar{1}, 2, 2), (\bar{1}\bar{0}, 2, 2), (\bar{0}\bar{0}, 2, 1), (\bar{1}\bar{1}, 2, 1)\}$.

We see that $(M_{11}, E_{11}) \cong (M_{12}, E_{12}) \cong (M_{21}, E_{21}) \cong (M_{22}, E_{22}) \cong Cay\big(\langle p_{21}\bar{1}\bar{0}\rangle \times R_2, \{(p_{21}\bar{1}\bar{0}, r_2)\}\big)$ and $Cay\big(S, \{a\}\big) = \dot{\cup}_{i=1}^{2}\dot{\cup}_{k=1}^{2}(M_{ik}, E_{ik})$ where $R_2 = \{r_1, r_2\}$ is a right zero semigroup.

By Lemma 3.2.1-3.2.5, we get that a Cayley digraph of a completely simple semigroup $\mathcal{M}(G, I, \Lambda, P)$ with a one-element connection set $\{(g, j, \beta)\}$, is the disjoint union of $|I|t$ copies of $Cay\big(\langle p_{\beta j} g\rangle \times R_{|\Lambda|},\ \{(p_{\beta j} g, r_\beta)\}\big)$ where $t = |G/\langle p_{\beta j} g\rangle|$. Then $|I|t$ is the number of connected components of $Cay(S, \{a\})$. In Example 3.2.6, we have $|I| = 2$, $t = 2$ and $Cay(S, \{\bar{1}\bar{0}, 1, 2\})$ has 4 connected components.

Next, we introduce the conditions for Cayley digraphs of a completely simple semigroup with a one-element connection set to be isomorphic and connected. The following theorem gives the conditions for two Cayley digraphs of completely simple semigroups $Cay(S, \{a\})$ and $Cay(S, \{b\})$ being isomorphic.

Theorem 3.2.7. *Let $S = \mathcal{M}(G, I, \Lambda, P)$ be a completely simple semigroup, $I = \{1, 2 \ldots, n\}$, $\Lambda = \{1, 2 \ldots, m\}$, $a = (g, j, \beta), b = (g', i, \lambda) \in S$. Then $Cay(S, \{a\}) \cong Cay(S, \{b\})$ if and only if $|\langle p_{\beta j} g\rangle| = |\langle p_{\lambda i} g'\rangle|$.*

Proof. ($\Rightarrow$) Suppose that $Cay(S, \{a\}) \cong Cay(S, \{b\})$. By Lemma 3.2.5, we get that $Cay\big(\langle p_{\beta j} g\rangle \times R_{|\Lambda|}, \{(p_{\beta j} g, r_\beta)\}\big) \cong Cay\big(\langle p_{\lambda i} g'\rangle \times R_{|\Lambda|}, \{(p_{\lambda i} g', r_\lambda)\}\big)$. Therefore $|\langle p_{\beta j} g\rangle \times R_{|\Lambda|}| = |\langle p_{\lambda i} g'\rangle \times R_{|\Lambda|}|$. Hence $|\langle p_{\beta j} g\rangle| = |\langle p_{\lambda i} g'\rangle|$.

($\Leftarrow$) Assume that $|\langle p_{\beta j} g\rangle| = |\langle p_{\lambda i} g'\rangle| = l$. By Lemma 3.2.1-3.2.5, we only need to show that $Cay\big(\langle p_{\beta j} g\rangle \times R_m, \{(p_{\beta j} g, r_\beta)\}\big) \cong Cay\big(\langle p_{\lambda i} g'\rangle \times R_m, \{(p_{\lambda i} g', r_\lambda)\}\big)$ where $R_m = \{r_1, r_2, \ldots, r_m\}$ is a right zero semigroup. We define

$$f : Cay\big(\langle p_{\beta j} g\rangle \times R_m, \{(p_{\beta j} g, r_\beta)\}\big) \to Cay\big(\langle p_{\lambda i} g'\rangle \times R_m, \{(p_{\lambda i} g', r_\lambda)\}\big)$$

$$\text{by } f\big((p_{\beta j} g)^r, r_\mu\big) = \begin{cases} \big((p_{\lambda i} g')^r, r_\lambda\big) & if \ \ \mu = \beta; \\ \big((p_{\lambda i} g')^r, r_\beta\big) & if \ \ \mu = \lambda; \\ \big((p_{\lambda i} g')^r, r_\alpha\big) & \text{otherwise.} \end{cases}$$

Clearly, f is well defined. Since $|\langle p_{\beta j} g\rangle| = |\langle p_{\lambda i} g'\rangle|$, f is a bijection. To show that f and f^{-1} are homomorphisms. Let $x, y \in \langle p_{\beta j} g\rangle \times R_m$ and let (x, y) be an arc in $Cay\big(\langle p_{\beta j} g\rangle \times R_m, \{(p_{\beta j} g, r_\beta)\}\big)$. Hence $x = \big((p_{\beta j} g)^{t_1}, r_{k_1}\big), y = \big((p_{\beta j} g)^{t_2}, r_{k_2}\big)$ for some $t_1, t_2 \in \{1, 2, ..., |\langle p_{\beta j} g\rangle|\}$, and so $\big((p_{\beta j} g)^{t_2}, r_{k_2}\big) = \big((p_{\beta j} g)^{t_1}, r_{k_1}\big)(p_{\beta j} g, r_\beta) = \big((p_{\beta j} g)^{t_1+1}, r_\beta\big)$. Hence $t_2 \equiv t_1 + 1 \ (mod\ l)$ and $k_2 = \beta$. We will show that $\big(f(x), f(y)\big)$ is an arc in $Cay\big(\langle p_{\lambda i} g'\rangle \times R_m, \{(p_{\lambda i} g', r_\lambda)\}\big)$. Consider three cases:

(case1) If $k_1 = \beta$, then

$$\begin{aligned} f(y) &= f\big((p_{\beta j}g)^{t_2}, r_\beta\big) \\ &= \big((p_{\lambda i}g^{'})^{t_2}, r_\lambda\big) \\ &= \big((p_{\lambda i}g^{'})^{t_1}p_{\lambda i}g^{'}, r_\lambda\big) \\ &= \big((p_{\lambda i}g^{'})^{t_1}, r_\lambda\big)(p_{\lambda i}g^{'}, r_\lambda) \\ &= f(x)(p_{\lambda i}g^{'}, r_\lambda). \end{aligned}$$

(case2) If $k_1 = \lambda$, then

$$\begin{aligned} f(y) &= \big((p_{\lambda i}g^{'})^{t_2}, r_\lambda\big) \\ &= \big((p_{\lambda i}g^{'})^{t_1}p_{\lambda i}g^{'}, r_\lambda\big) \\ &= \big((p_{\lambda i}g^{'})^{t_1}, r_\beta\big)(p_{\lambda i}g^{'}, r_\lambda) \\ &= f\big((p_{\beta j}g)^{t_1}, r_\lambda\big)(p_{\lambda i}g^{'}, r_\lambda) \\ &= f(x)(p_{\lambda i}g^{'}, r_\lambda). \end{aligned}$$

(case3) If $k_1 = \alpha$ where $\alpha \neq \beta, \lambda$, then

$$\begin{aligned} f(y) &= \big((p_{\lambda i}g^{'})^{t_2}, r_\lambda\big) \\ &= \big((p_{\lambda i}g^{'})^{t_1}p_{\lambda i}g^{'}, r_\lambda\big) \\ &= \big((p_{\lambda i}g^{'})^{t_1}, r_\alpha\big)(p_{\lambda i}g^{'}, r_\lambda) \\ &= f\big((p_{\beta j}g)^{t_1}, r_\alpha\big)(p_{\lambda i}g^{'}, r_\lambda) \\ &= f(x)(p_{\lambda i}g^{'}, r_\lambda). \end{aligned}$$

This means that f is a digraph homomorphism. Similarly, f^{-1} is a digraph homomorphism. Hence

$$Cay\big(\langle p_{\beta j}g\rangle \times R_m, \{(p_{\beta j}g, r_\beta)\}\big) \cong Cay\big(\langle p_{\lambda i}g^{'}\rangle \times R_m, \{(p_{\lambda i}g^{'}, r_\lambda)\}\big).$$

By Lemma 3.2.5, we get that $Cay(S, \{a\}) \cong Cay(S, \{b\})$. □

Example 3.2.8. Consider $Cay\big(\mathcal{M}(\mathbb{Z}_2 \times \mathbb{Z}_2, I, \Lambda, P), \{(\bar{1}\bar{0}, 1, 2)\}\big)$ in Example 3.2.6, and $Cay\big(\mathcal{M}(\mathbb{Z}_2 \times \mathbb{Z}_2, I, \Lambda, P), \{(\bar{0}\bar{1}, 2, 2)\}\big)$ (see Figure 3.5).

We see that $|\langle p_{21}\bar{1}\bar{0}\rangle| = |\langle p_{22}\bar{0}\bar{1}\rangle| = 2$ and $Cay\big(\mathcal{M}(\mathbb{Z}_2 \times \mathbb{Z}_2, I, \Lambda, P), \{(\bar{1}\bar{0}, 1, 2)\}\big) \cong Cay\big(\mathcal{M}(\mathbb{Z}_2 \times \mathbb{Z}_2, I, \Lambda, P), \{(\bar{0}\bar{1}, 2, 2)\}\big)$.

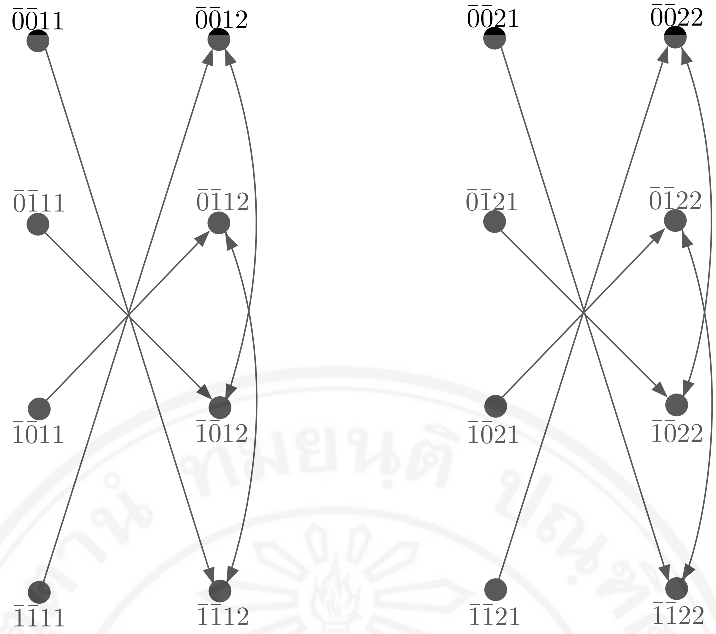

Figure 3.5: Cayley digraph $Cay(\mathcal{M}(\mathbb{Z}_2 \times \mathbb{Z}_2, \{1,2\}, \{1,2\}, P), \{(\bar{0}\bar{1}, 2, 2)\})$.

Note that a rectangular group $S = G \times L_m \times R_n$ is a special case of a completely simple semigroup $\mathcal{M}(G, I, \Lambda, P)$, where $|I| = m$, $|\Lambda| = n$ and P is an identity matrix. The following corollary gives the conditions for two Cayley digraphs of rectangular groups with a one-element connection set $Cay(S, \{a\})$ and $Cay(S, \{b\})$ to be isomorphic.

Corollary 3.2.9. *Let $S = G \times L_m \times R_n$ be a rectangular group, $a = (g_1, l_1, r_1)$, $b = (g_2, l_2, r_2) \in S$. Then $Cay(S, \{a\}) \cong Cay(S, \{b\})$ if and only if $|\langle g_1 \rangle| = |\langle g_2 \rangle|$.*

Now we give the conditions for a Cayley digraph of a completely simple semigroup with a one-element connection set to be connected.

Theorem 3.2.10. *Let $S = \mathcal{M}(G, I, \Lambda, P)$ be a completely simple semigroup, $a = (g, j, \beta) \in S$. Then $Cay(S, \{a\})$ is connected if and only if $G = \langle p_{\beta j} g \rangle$ and $|I| = 1$, in particular this means that S is a right group.*

Proof. ($\Rightarrow$) Let $Cay(S, \{a\})$ be connected. By Lemma 3.2.1, we get $|I| = 1$ and $G = \langle p_{\beta j} g \rangle$.

($\Leftarrow$) Assume that $G = \langle p_{\beta j} g \rangle$ and $|I| = 1$. We will prove that $Cay(S, \{a\})$ is connected. Let $(g_1, i, \lambda_1), (g_2, i, \lambda_2) \in S$. Hence $g_1, g_2 \in \langle p_{\beta j} g \rangle$. Therefore $g_1 = (p_{\beta j} g)^r$ and $g_2 = (p_{\beta j} g)^q$ for some $r, q \in \{1, 2, ..., |\langle p_{\beta j} g \rangle|\}$. Then $r \leq q$ or $r > q$. We consider four cases.

(case1) For $\lambda_1 = \lambda_2 = \beta$. If $r \leq q$, then $q = r + t$ for some $t \in \mathbb{N} \cup \{0\}$. Then we get $(g_1, i, \lambda_1) = \big((p_{\beta j}g)^r, i, \beta\big), \big((p_{\beta j}g)^{r+1}, i, \beta\big), \ldots, \big((p_{\beta j}g)^{r+t}, i, \beta\big) = (g_2, i, \lambda_2)$ is a path from (g_1, i, λ_1) to (g_2, i, λ_2) in $Cay(S, \{a\})$. Similarly, if $r > q$, there is a path from (g_2, i, λ_2) to (g_1, i, λ_1) in $Cay(S, \{a\})$.

(case2) For $\lambda_1 = \beta$, $\lambda_2 \neq \beta$. Then $(g_2, i, \lambda_2)(g, j, \beta) = (g_2 p_{\lambda_2 j} g, i, \beta)$, and thus $\big((g_2, i, \lambda_2), (g_2 p_{\lambda_2 j} g, i, \beta)\big)$ is an arc in $Cay(S, \{a\})$. Since $G = \langle p_{\beta j} g\rangle$ and $g_2 p_{\lambda_2 j} g \in G$, $g_2 p_{\lambda_2 j} g = (p_{\beta j} g)^u$ for some $u \in \{1, 2, ..., |\langle p_{\beta j} g\rangle|\}$. By case 1, there is a path from (g_1, i, λ_1) to $(g_2 p_{\lambda_2 j} g, i, \beta)$ or from $(g_2 p_{\lambda_2 j} g, i, \beta)$ to (g_1, i, λ_1). Therefore we have a semipath between (g_1, i, λ_1) and (g_2, i, λ_2).

(case3) For $\lambda_1 \neq \beta, \lambda_2 = \beta$. Then $(g_1, i, \lambda_1)(g, j, \beta) = (g_1 p_{\lambda_1 j} g, i, \beta)$, and so $\big((g_1, i, \lambda_1), (g_1 p_{\lambda_1 j} g, i, \beta)\big)$ is an arc in $Cay(S, \{a\})$. Since $G = \langle p_{\beta j} g\rangle$ and $g_1 p_{\lambda_1 j} g \in G$, $g_1 p_{\lambda_1 j} g = (p_{\beta j} g)^v$ for some $v \in \{1, 2, ..., |\langle p_{\beta j} g\rangle|\}$. By case 1, there is a path from (g_2, i, λ_2) to $(g_1 p_{\lambda_1 j} g, i, \beta)$ or from $(g_1 p_{\lambda_1 j} g, i, \beta)$ to (g_2, i, λ_2). Therefore we have a semipath between (g_1, i, λ_1) and (g_2, i, λ_2).

(case4) For $\lambda_1 \neq \beta, \lambda_2 \neq \beta$. Then $(g_1, i, \lambda_1)(g, j, \beta) = (g_1 p_{\lambda_1 j} g, i, \beta)$ and $(g_2, i, \lambda_2)(g, j, \beta) = (g_2 p_{\lambda_2 j} g, i, \beta)$. Thus $\big((g_1, i, \lambda_1), (g_1 p_{\lambda_1 j} g, i, \beta)\big)$ and $\big((g_2, i, \lambda_2), (g_2 p_{\lambda_2 j} g, i, \beta)\big)$ are arcs in $Cay(S, \{a\})$. Since $g_1 p_{\lambda_1 j} g, g_2 p_{\lambda_2 j} g \in G = \langle p_{\beta j} g\rangle$, $g_1 p_{\lambda_1 j} g = (p_{\beta j} g)^w$ and $g_2 p_{\lambda_2 j} g = (p_{\beta j} g)^z$ for some $w, z \in \{1, 2, ..., |p_{\beta j} g|\}$. By case 1, there is a path from $(g_1 p_{\lambda_1 j} g, i, \beta)$ to $(g_2 p_{\lambda_2 j} g, i, \beta)$ or from $(g_2 p_{\lambda_2 j} g, i, \beta)$ to $(g_1 p_{\lambda_1 j} g, i, \beta)$. Therefore we have a semipath between (g_1, i, λ_1) and (g_2, i, λ_2).

By the above four cases we conclude that $Cay(S, \{a\})$ is connected. $\square$

Example 3.2.11. Consider the completely simple semigroup $S = \mathcal{M}(G, I, \Lambda, P)$, where $G = \{\bar{0}\bar{0}, \bar{1}\bar{2}, \bar{2}\bar{1}\}$ is a subgroup of $\mathbb{Z}_3 \times \mathbb{Z}_3$, $I = \{1\}$, $\Lambda = \{1, 2\}$ and

$$P = \begin{bmatrix} \bar{0}\bar{0} \\ \bar{2}\bar{1} \end{bmatrix}.$$

Let $a = (\bar{2}\bar{1}, 1, 2)$. We see that $\langle p_{\beta j} g\rangle = \langle \bar{2}\bar{1} \cdot \bar{2}\bar{1}\rangle = \langle \bar{1}\bar{2}\rangle = \{\bar{1}\bar{2}, \bar{2}\bar{1}, \bar{0}\bar{0}\} = G$, $|I| = 1$, and Cayley digraph $Cay(S, \{a\})$ is connected (see Figure 3.6).

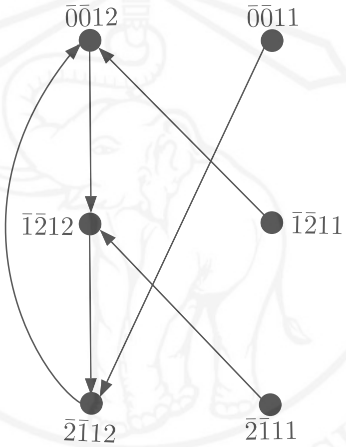

Figure 3.6: Cayley digraph $Cay(\mathcal{M}(\{\bar{0}\bar{0}, \bar{1}\bar{2}, \bar{2}\bar{1}\}, \{1\}, \{1, 2\}, P), \{(\bar{2}\bar{1}, 1, 2)\})$.