CHAPTER 4

Isomorphism Conditions for Cayley Digraphs of Rectangular Groups

In this chapter, we give some equivalent conditions for Cayley digraphs of a rectangular group to be isomorphic to each other. In order to contribute to these objectives, the isomorphism conditions for Cayley digraphs of a right group and for Cayley digraphs of a rectangular band have been presented in this chapter.

4.1 Isomorphism Conditions for Cayley Digraphs of Rectangular Bands

We consider an isomorphism of Cayley digraphs of a given rectangular band in this section. First, we will introduce the conditions for Cayley digraphs of a right zero semigroup to be isomorphic to each other. By the definition of a right zero semigroup, we get the following lemma.

Lemma 4.1.1. *Let $A \subseteq R_n$, and let v be a vertex in $Cay(R_n, A)$. Then*

(1) $\overrightarrow{d}(v) = |R_n|$ *if and only if* $v \in A$*;*

(2) $\overrightarrow{d}(v) = 0$ *if and only if* $v \notin A$.

proof. (1) Let $\overrightarrow{d}(v) = |R_n|$. Then there is an arc from each vertex in R_n to v. It follows that there exists $w \in A$ such that $v = uw$ for $u \in R_n$. Since $uw = w$, $v = w \in A$. Conversely, we assume that $v \in A$. Let $u \in R_n$. Since $uv = v$, there is an arc from u to v. It follows that there is an arc from every vertices in R_n to v. Hence $\overrightarrow{d}(v) = |R_n|$.

(2) Let $\overrightarrow{d}(v) = 0$ and $u \in R_n$. We assume that $v \in A$. Since $uv = v$, (u, v) is an arc in $Cay(R_n, A)$, and thus $\overrightarrow{d}(v) \neq 0$. There is a contradiction, so $v \notin A$. Conversely, we assume that $\overrightarrow{d}(v) \neq 0$. There is $w \in A$ such that $v = uw$ for some $u \in R_n$. Since $uw = w$, $v = w \in A$. There is a contradiction, therefore $\overrightarrow{d}(v) = 0$. $\square$

By Lemma 4.1.1, we have the following theorem which introduces the isomorphism condition for the Cayley digraphs of a given right zero semigroup.

Theorem 4.1.2. *Let $A, B \subseteq R_n$. Then $Cay(R_n, A) \cong Cay(R_n, B)$ if and only if $|A| = |B|$.*

proof. ($\Rightarrow$) Let $Cay(R_n, A) \cong Cay(R_n, B)$. Then the number of vertices with indegree $|R_n|$ of $Cay(R_n, A)$ and $Cay(R_n, B)$ are equal. By Lemma 4.1.1(1), we get $|A| = |B|$.

($\Leftarrow$) Let $|A| = |B| = l$. We assume that $A = \{a_1, a_2, \ldots, a_l\}$ and $B = \{b_1, b_2, \ldots, b_l\}$. Suppose that $R_n \backslash A = \{a'_1, a'_2, \ldots, a'_t\}$ and $R_n \backslash B = \{b'_1, b'_2, \ldots, b'_t\}$. We define a mapping $f : R_n \to R_n$ by

$$f(a) = \begin{cases} b_i & \text{if } a = a_i \text{ for } i \in \{1, 2, \ldots, l\}; \\ b'_i & \text{if } a = a'_i \text{ for } i \in \{1, 2, \ldots, t\}. \end{cases}$$

Clearly, f is well defined and bijective. We will show that f and f^{-1} are digraph homomorphisms. Assume that (x, y) is an arc in $Cay(R_n, A)$. By Lemma 4.1.1(1), we have $y \in A$, and thus $f(y) \in B$. By Lemma 4.1.1(1) again, there exists an arc from every vertices to $f(y)$ in $Cay(R_n, B)$. Then $(f(x), f(y))$ is an arc in $Cay(R_n, B)$. Therefore f is a digraph homomorphism. Similarly, we can show that f^{-1} is a digraph homomorphism. Hence $Cay(R_n, A) \cong Cay(R_n, B)$. $\square$

The following lemma shows that the Cayley digraph of a rectangular band is the union of Cayley digraphs of right zero semigroups.

Lemma 4.1.3. *Let $S = L_m \times R_n$ be a rectangular band and $A \subseteq S$. Then $Cay(S, A)$ is the disjoint union of m isomorphic strong subdigraphs $Cay(\{l_i\} \times R_n, \{l_i\} \times p_2(A))$ for $i \in \{1, 2, \ldots, m\}$.*

proof. Let $p, q \in \{1, 2, \ldots, m\}$. Since $(\{l_p\} \times R_n) \cap (\{l_q\} \times R_n) = \emptyset$ for all $p \neq q$, we get that $Cay(\{l_p\} \times R_n, \{l_p\} \times p_2(A))$ and $Cay(\{l_q\} \times R_n, \{l_q\} \times p_2(A))$ are disjoint. To show that $Cay(\{l_p\} \times R_n, \{l_p\} \times p_2(A)) \cong Cay(\{l_q\} \times R_n, \{l_q\} \times p_2(A))$ and $Cay(S, A) = \dot{\bigcup}_{i=1}^{m} Cay(\{l_i\} \times R_n, \{l_i\} \times p_2(A))$. We define a mapping $f : \{l_p\} \times R_n \to \{l_q\} \times R_n$ by $f((l_p, r)) = (l_q, r)$ for every $(l_p, r) \in \{l_p\} \times R_n$. Clearly, f is well defined and is a bijection. We will show that f and f^{-1} are digraph homomorphisms.

For $(l_p, r_1), (l_p, r_2) \in \{l_p\} \times R_n$, let $((l_p, r_1), (l_p, r_2))$ be an arc in $Cay(\{l_p\} \times R_n, \{l_p\} \times p_2(A))$. There exists $(l_p, r) \in \{l_p\} \times p_2(A)$ such that $(l_p, r_2) = (l_p, r_1)(l_p, r) = (l_p, r)$. Therefore $r = r_2$, we get that $r_2 \in p_2(A)$. Hence $(l_q, r_2) \in \{l_q\} \times p_2(A)$ and thus $(l_q, r_1)(l_q, r_2) = (l_q, r_2)$. This means that $((l_q, r_1), (l_q, r_2))$ is an arc in $Cay(\{l_q\} \times R_n, \{l_q\} \times p_2(A))$. We have shown that f is a digraph homomorphism. Similarly, we can show that f^{-1} is a digraph homomorphism. Hence $Cay(\{l_p\} \times R_n, \{l_p\} \times p_2(A)) \cong Cay(\{l_q\} \times R_n, \{l_q\} \times p_2(A))$.

We will show that $Cay(S,A) = \dot\cup_{i=1}^{m} Cay(\{l_i\} \times R_n, \{l_i\} \times p_2(A))$. Let $D = \dot\cup_{i=1}^{m} Cay(\{l_i\} \times R_n, \{l_i\} \times p_2(A))$. Since $L_m \times R_n = \dot\cup_{i=1}^{m}(\{l_i\} \times R_n)$, then $S = \dot\cup_{i=1}^{m}(\{l_i\} \times R_n) = V(D)$. We will show that $E(Cay(S,A)) = E(D)$. For $(l_i, r), (l_j, r') \in S$, let $((l_i, r), (l_j, r'))$ be an arc in $Cay(S,A)$. There exists $(l_k, r'') \in A$ such that $(l_j, r') = (l_i, r)(l_k, r'') = (l_i, r'')$, we get $l_i = l_j$ and $r' = r''$. Since $(l_k, r'') \in A$, $(l_j, r'') \in \{l_j\} \times p_2(A)$. Consider $(l_j, r') = (l_j, r'') = (l_j, r)(l_j, r'')$, we have that $((l_j, r), (l_j, r')) = ((l_i, r), (l_j, r'))$ is an arc in $Cay(\{l_j\} \times R_n, \{l_j\} \times p_2(A))$. Hence $((l_i, r), (l_j, r'))$ is an arc in D. Therefore $E(Cay(S,A)) \subseteq E(D)$. Conversely, let $((l_j, r), (l_j, r')) \in E(D)$. This means that it is an arc in $Cay(\{l_j\} \times R_n, \{l_j\} \times p_2(A))$ for some $j \in \{1, 2, \ldots, m\}$. There exists $(l_j, r'') \in \{l_j\} \times p_2(A)$ such that $(l_j, r') = (l_j, r)(l_j, r'') = (l_j, r'')$ and we get that $r' = r''$. Also since $(l_j, r'') \in \{l_j\} \times p_2(A)$, we have $(l, r'') \in A$ for some $l \in L_m$. Consider $(l_j, r') = (l_j, r'') = (l_j, r)(l, r'')$, it follows that $((l_j, r), (l_j, r'))$ is an arc in $Cay(S,A)$. Hence $E(D) \subseteq E(Cay(S,A))$. Therefore $Cay(S,A) = D$. $\square$

Theorem 4.1.4. *Let $S = L_m \times R_n$ be a rectangular band and $A, B \subseteq S$. Then $Cay(S,A) \cong Cay(S,B)$ if and only if $|p_2(A)| = |p_2(B)|$.*

proof. $(\Rightarrow)$ Let $Cay(S,A) \cong Cay(S,B)$. By Lemma 4.1.3, we get $\dot\cup_{i=1}^{m} Cay(\{l_i\} \times R_n, \{l_i\} \times p_2(A)) \cong \dot\cup_{i=1}^{m} Cay(\{l_i\} \times R_n, \{l_i\} \times p_2(B))$. Then $Cay(\{l_i\} \times R_n, \{l_i\} \times p_2(A)) \cong Cay(\{l_i\} \times R_n, \{l_i\} \times p_2(B))$ and thus $Cay(R_n, p_2(A)) \cong Cay(R_n, p_2(B))$. By Theorem 4.1.2, we get $|p_2(A)| = |p_2(B)|$.

$(\Leftarrow)$ Let $|p_2(A)| = |p_2(B)|$. By Theorem 4.1.2, we get $Cay(R_n, p_2(A)) \cong Cay(R_n, p_2(B))$. Then $\dot\cup_{i=1}^{m} Cay(\{l_i\} \times R_n, \{l_i\} \times p_2(A)) \cong \dot\cup_{i=1}^{m} Cay(\{l_i\} \times R_n, \{l_i\} \times p_2(B))$. By Lemma 4.1.3, we get $Cay(S,A) \cong Cay(S,B)$. $\square$

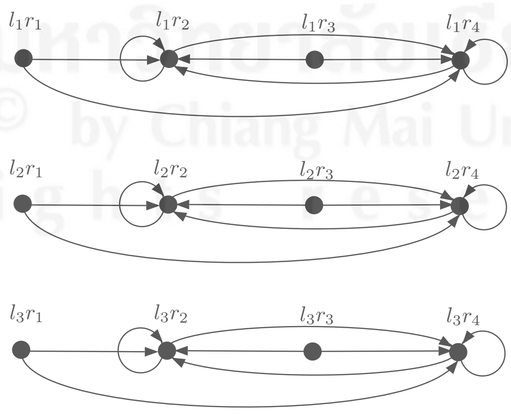

Figure 4.1: Cayley digraph $Cay(L_3 \times R_4, \{(l_1, r_2), (l_3, r_4)\})$.

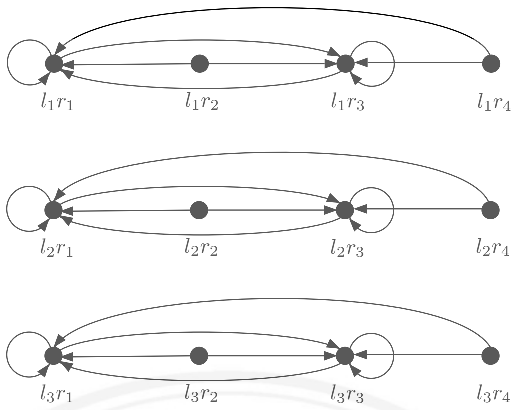

Figure 4.2: Cayley digraph $Cay(L_3 \times R_4, \{(l_2, r_1), (l_3, r_1), (l_3, r_3), (l_1, r_3)\})$.

Example 4.1.5. Let $S = L_3 \times R_4$ be a rectangular band, and let $A = \{(l_1, r_2), (l_3, r_4)\}$, $B = \{(l_2, r_1), (l_3, r_1), (l_3, r_3), (l_1, r_3)\}$ be subsets of S. It is easily seen that $Cay(S, A) \cong Cay(S, B)$ (see Figures. 4.1 and 4.2) , and $|p_2(A)| = |p_2(B)| = 2$.

4.2 Isomorphism Conditions for Cayley Digraphs of Right Groups

In this section, we present the conditions for Cayley digraphs of a given right group to be isomorphic to each other. By the definition of a right group, we get the following lemma.

Lemma 4.2.1. *Let $S = G \times R_n$ be a right group, A a nonempty subset of S, $g, g' \in G$, and $r, r' \in R_n$. Then the following statements are equivalent:*

(1) $((g, r), (g', r'))$ is an arc in $Cay(S, A)$;

(2) there exists $(a, r') \in A$ such that $g' = ga$;

(3) $((g, r'), (g', r'))$ is an arc in $Cay(S, A)$.

Proof. (1)→(2) Let $((g, r), (g', r'))$ be an arc in $Cay(S, A)$. There exists $(a, r'') \in A$ such that $(g', r') = (g, r)(a, r'') = (ga, r'')$. Hence $g' = ga$ and $r' = r''$, therefore $(a, r') \in A$.

(2)→(3) Assume that there exists $(a, r') \in A$ such that $g' = ga$. Since $(g', r') = (ga, r') = (g, r)(a, r')$, we get $((g, r)(g', r'))$ is an arc in $Cay(S, A)$.

(3)→(1) Let $((g, r')(g', r'))$ be an arc in $Cay(S, A)$. There exists $(a, r'') \in A$ such that $(g', r') = (g, r')(a, r'') = (ga, r'')$. Hence $g' = ga$ and $r' = r''$. Since $(g', r') = (ga, r'') = (g, r)(a, r'')$, we get that $((g, r), (g', r'))$ is an arc in $Cay(S, A)$. $\square$

The next result gives some description for Cayley digraphs of right groups.

Lemma 4.2.2. *Let $S = G\times R_n$ be a right group, A a nonempty subset of S, $G/\langle p_1(A)\rangle = \{g_1\langle p_1(A)\rangle, g_2\langle p_1(A)\rangle, \ldots, g_w\langle p_1(A)\rangle\}$, and $(g_i\langle p_1(A)\rangle \times p_2(A), E_i)$ a strong subdigraph of $Cay(S, A)$ for $i = 1, 2, ..., w$. Then $(g_j\langle p_1(A)\rangle \times p_2(A), E_j)$ and $(g_k\langle p_1(A)\rangle \times p_2(A), E_k)$ are disjoint strong subdigraphs of $Cay(S, A)$ for all $j \neq k$.*

proof. Let $u = (g_j g, r) \in g_j\langle p_1(A)\rangle \times p_2(A)$ and $v \in g_k\langle p_1(A)\rangle \times p_2(A)$. Since $g_j\langle p_1(A)\rangle$ and $g_k\langle p_1(A)\rangle$ are distinct left cosets of $\langle p_1(A)\rangle$ in G, $v \notin g_j\langle p_1(A)\rangle \times p_2(A)$. Assume that (u, v) is an arc from $g_j\langle p_1(A)\rangle \times p_2(A)$ to $g_k\langle p_1(A)\rangle \times p_2(A)$. There exists $a = (h, \lambda) \in A$ such that $v = ua = (g_j g, r)(h, \lambda) = (g_j gh, \lambda) \in g_j\langle p_1(A)\rangle \times p_2(A)$. There is a contradiction because $v \notin g_j\langle p_1(A)\rangle \times p_2(A)$. This means that there is no arc between $(g_j\langle p_1(A)\rangle \times p_2(A)\ , E_j)$ and $(g_k\langle p_1(A)\rangle \times p_2(A)\ , E_k)$. $\square$

Theorem 4.2.3. *Let $S = G\times R_n$ be a right group, A a nonempty subset of S, $G/\langle p_1(A)\rangle = \{g_1\langle p_1(A)\rangle, g_2\langle p_1(A)\rangle, \ldots, g_w\langle p_1(A)\rangle\}$, and $(g_i\langle p_1(A)\rangle \times p_2(A), E_i)$ a strong subdigraph of $Cay(S, A)$. Then $Cay(S, A) = \dot\bigcup_{i=1}^{w} (g_i\langle p_1(A)\rangle \times p_2(A), E_i) \bigcup \dot\bigcup_{i=1}^{w} (g_i\langle p_1(A)\rangle \times R_n, E_i')$, where $E_i' = \{((s, t), (u, v)) \mid t \notin p_2(A), ((s, v), (u, v)) \in E_i\}$.*

proof. Let $D = \dot\bigcup_{i=1}^{w} (g_i\langle p_1(A)\rangle \times p_2(A), E_i) \bigcup \dot\bigcup_{i=1}^{w} (g_i\langle p_1(A)\rangle \times R_n, E_i')$. It is clear that $S = \dot\bigcup_{i=1}^{w} (g_i\langle p_1(A)\rangle \times p_2(A)) \bigcup \dot\bigcup_{i=1}^{w} (g_i\langle p_1(A)\rangle \times R_n) = V(D)$. We will prove that $E(Cay(S, A)) = E(D)$. Let $((g, r), (g', r'))$ be an arc in $Cay(S, A)$. By Lemma 4.2.1, there exists $(a, r') \in A$ and $g' = ga$. Hence $g' \in g_{k_1}\langle p_1(A)\rangle$ and $g \in g_{k_2}\langle p_1(A)\rangle$ for some $k_1, k_2 \in \{1, 2, \ldots, w\}$. We have the following cases.

(case1) If $r \in p_2(A)$, then $(g, r), (g', r') \in \dot\bigcup_{i=1}^{w} (g_i\langle p_1(A)\rangle \times p_2(A))$. Since $\dot\bigcup_{i=1}^{w} (g_i\langle p_1(A)\rangle \times p_2(A), E_i)$ is a strong subdigraph of $Cay(S, A)$, $((g, r), (g', r'))$ is an arc in $\dot\bigcup_{i=1}^{w} (g_i\langle p_1(A)\rangle \times p_2(A), E_i)$. Therefore $((g, r), (g', r'))$ is an arc in D.

(case2) If $r \notin p_2(A)$, then $((g, r'), (g', r'))$ is also an arc in $Cay(S, A)$ by Lemma 4.2.1 and $((g, r), (g', r'))$ is an arc in $Cay(S, A)$. This implies that $((g, r'), (g', r')) \in E_i$. Then $((g, r), (g', r')) \in E_i'$. Hence $((g, r), (g', r'))$ is an arc in D.

Therefore $E(Cay(S, A)) \subseteq E(D)$. To show that $E(D) \subseteq E(Cay(S, A))$. Let $((g, r), (g', r'))$ be an arc in D. We consider two cases.

(case1) If $((g, r), (g', r'))$ is an arc in $\dot\bigcup_{i=1}^{w} (g_i\langle p_1(A)\rangle \times p_2(A), E_i)$, then it is an arc in $Cay(S, A)$ because $\dot\bigcup_{i=1}^{w} (g_i\langle p_1(A)\rangle \times p_2(A), E_i)$ is a strong subdigraph of $Cay(S, A)$.

(case2) If $((g,r),(g',r'))$ is an arc in $\dot\cup_{i=1}^{w}(g_i\langle p_1(A)\rangle \times R_n, E'_i)$, then it is an arc in E'_k for some k. We get that $((g,r'),(g',r')) \in E_k$ and this implies that $((g,r'),$ $(g',r'))$ is an arc in $Cay(S,A)$. By Lemma 4.2.1, we have $((g,r),\ (g',r'))$ is also an arc in $Cay(S,A)$.

Then $E(D) \subseteq E(Cay(S,A))$. Hence we prove that $Cay(S,A) = D$. □

By the definition of E'_j in Theorem 4.2.3 we have the next corollary.

Corollary 4.2.4. *Let $S = G \times R_n$ be a right group, A a nonempty subset of S, and $(g_i\langle p_1(A)\rangle \times p_2(A), E_i)$ a strong subdigraph of $Cay(S,A)$. For all $r' \in R_n \setminus p_2(A)$, if $((u,r),(v,r))$ is an arc in $\dot\cup_{j\in I}(g_j\langle p_1(A)\rangle \times p_2(A), E_j)$, then $((u,r'),(v,r)) \in E'_j$ where $E'_j = \{((s,t),\ (u,v)) \mid t \notin p_2(A),\ ((s,v),(u,v)) \in E_j\}$.*

Theorem 4.2.5. *Let $S = G\times R_n$ be a right group, A a nonempty subset of S, $G/\langle p_1(A)\rangle = \{g_1\langle p_1(A)\rangle, g_2\langle p_1(A)\rangle, \dots, g_w\langle p_1(A)\rangle\}$, and $(g_i\langle p_1(A)\rangle \times p_2(A), E_i)$ a strong subdigraph of $Cay(S,A)$. Then $(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \cong Cay(\langle A\rangle, A)$ for $i = 1, 2, ..., w$.*

proof. We define $f : (g_i\langle p_1(A)\rangle \times p_2(A), E_i) \to Cay(\langle A\rangle, A)$ by $(g_ia, r) \mapsto (a, r)$ for all $a \in \langle p_1(A)\rangle$ and $r \in p_2(A)$. Clearly, f is a bijection. We will prove that f and f^{-1} are digraph homomorphisms.

For $(g_ia, r), (g_ia', r') \in g_i\langle p_1(A)\rangle \times p_2(A)$, let $((g_ia, r), (g_ia', r'))$ be an arc in $(g_i\langle p_1(A)\rangle \times p_2(A), E_i)$. Since $(g_i\langle p_1(A)\rangle \times p_2(A), E_i)$ is a strong subdigraph of $Cay(S,A)$, we get that $((g_ia, r'), (g_ia', r'))$ is an arc in $Cay(S,A)$. There exists $(a'', r') \in A$ such that $g_ia' = g_iaa''$ so $a' = aa''$. Since $f(g_ia', r') = (a', r') = (aa'', r') = (a, r)(a'', r') = f(g_ia, r)(a'', r')$, we have $(f(g_ia, r), f(g_ia', r'))$ is an arc in $Cay(\langle A\rangle, A)$. Therefore f is a digraph homomorphism.

Let $(f(g_ia, r), f(g_ia', r'))$ be an arc in $Cay(\langle A\rangle, A)$. Then there exists $(a'', r'') \in A$ such that $f(g_ia', r') = f(g_ia, r)(a'', r'')$. Therefore $(a', r') = (a, r)(a'', r'') = (aa'', r'')$, $a' = aa''$, and $r' = r''$. Hence $(g_ia', r') = (g_iaa'', r'') = (g_ia, r)\ (a'', r'')$, so $((g_ia, r), (g_ia', r'))$ is an arc in $Cay(S,A)$. Since $(g_ia, r), (g_ia',\ r') \in g_i\langle p_1(A)\rangle \times p_2(A)$ and $(g_i\langle p_1(A)\rangle \times p_2(A), E_i)$ is a strong subdigraph of $Cay(S,A)$, we thus get $((g_ia, r), (g_ia', r'))$ is an arc in $(g_i\langle p_1(A)\rangle \times p_2(A), E_i)$. Therefore f^{-1} is a digraph homomorphism. This means that $(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \cong Cay(\langle A\rangle, A)$. □

The two following lemmas will be used in the proof of Lemma 4.2.8.

Lemma 4.2.6. *Let $S = G \times R_n$ be a right group, A a nonempty subset of S, $G/\langle p_1(A)\rangle = \{g_1\langle p_1(A)\rangle, g_2\langle p_1(A)\rangle, \ldots, g_w\langle p_1(A)\rangle\}$, and $(g_i\langle p_1(A)\rangle \times p_2(A), E_i)$ a strong subdigraph of $Cay(S, A)$. Then for all $v \in V(Cay(S, A))$, $\overrightarrow{d}(v) \neq 0$ if and only if $v \in \dot{\cup}_{i=1}^{w}\big(g_i\langle p_1(A)\rangle\big) \times p_2(A)$.*

proof. $(\Rightarrow)$ Let $v = (h_1, r_1) \in S$ and $\overrightarrow{d}(v) \neq 0$. Then there exists $u = (h_2, r_2) \in S$ such that (u, v) is an arc in $Cay(S, A)$. Hence there exists $a = (g', r') \in A$ such that $v = ua$. Therefore $(h_1, r_1) = (h_2, r_2)(g', r') = (h_2 g', r')$, which implies that $r_1 = r' \in p_2(A)$. Since $h_1 \in G = \dot{\cup}_{i=1}^{w}\big(g_i\langle p_1(A)\rangle\big)$, we have $v = (h_1, r_1) \in \dot{\cup}_{i=1}^{w}\big(g_i\langle p_1(A)\rangle\big) \times p_2(A)$.

$(\Leftarrow)$ Let $v = (h_1, r) \in \dot{\cup}_{i=1}^{w}\big(g_i\langle p_1(A)\rangle\big) \times p_2(A)$. We get that $h_1 \in G$ and $r \in p_2(A)$. We consider the two cases.

(case1) If $v \in A$, there exists $(e, r) \in S$, where e is the identity of G. Since $(e, r)(h_1, r) = (eh_1, r) = (h_1, r) = v$, there is an arc from (e, r) to v. Therefore $\overrightarrow{d}(v) \neq 0$.

(case2) If $v \notin A$, then there exists $(h_2, r) \in A$ for some $h_2 \in G$. Because G is a group and $h_1, h_2 \in G$, this implies that $h_2^{-1} \in G$ and $h_1 h_2^{-1} \in G$. Then we have $(h_1 h_2^{-1}, r) \in S$. Since $(h_1 h_2^{-1}, r)(h_2, r) = (h_1 h_2^{-1} h_2, r) = (h_1, r) = v$, there exists an arc from $(h_1 h_2^{-1}, r)$ to v. Therefore $\overrightarrow{d}(v) \neq 0$.

□

Lemma 4.2.7. *Let $S = G \times R_n$ be a right group, A a nonempty subset of S, $G/\langle p_1(A)\rangle = \{g_1\langle p_1(A)\rangle, g_2\langle p_1(A)\rangle, \ldots, g_w\langle p_1(A)\rangle\}$, and $(g_i\langle p_1(A)\rangle \times p_2(A), E_i)$ a strong subdigraph of $Cay(S, A)$. Then for any $i \in \{1, 2, \ldots, w\}$, $(g_i\langle p_1(A)\rangle \times p_2(A), E_i)$ is connected.*

proof. Let $(g_i x, \beta), (g_i y, \gamma) \in \big(g_i\langle p_1(A)\rangle \times p_2(A)\big)$. Then $(x, \beta), (y, \gamma) \in \langle p_1(A)\rangle \times p_2(A) = \langle A\rangle$. There are $a_1, a_2, \ldots, a_q \in A$ such that $(y, \gamma) = (x, \beta)a_1 a_2 \ldots a_q$ for some $q \leq |A|$. Hence $(g_i y, \gamma) = (g_i x, \beta)a_1 a_2 \ldots a_q$. This means that there is an arc from $(g_i x, \beta)a_1 a_2 \ldots a_{q-1}$ to $(g_i y, \gamma)$. Since $\big((g_i x, \beta), (g_i x, \beta)a_1\big)$, $\big((g_i x, \beta)a_1, (g_i x, \beta)a_1 a_2\big)$, $\ldots$, $\big((g_i x, \beta)a_1 a_2 \ldots a_{q-2}, (g_i x, \beta)a_1 a_2 \ldots a_{q-1}\big)$ are arcs in $Cay(S, A)$, there is a path $(g_i x, \beta)$, $(g_i x, \beta)a_1$, $(g_i x, \beta)a_1 a_2$, $\ldots$, $(g_i x, \beta)a_1 a_2 \ldots a_{q-1}$, $(g_i y, \gamma)$ in $Cay(S, A)$. We conclude that $(g_i\langle p_1(A)\rangle \times p_2(A), E_i)$ is connected. □

Since a strong subdigraph $(g_i\langle p_1(A)\rangle \times p_2(A), E_i)$ is connected, we have $(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \cup (g_i\langle p_1(A)\rangle \times R_n, E_i')$ is also connected for any $i \in \{1, 2, \ldots, w\}$, where E' is defined as in Theorem 4.2.3.

Lemma 4.2.8. *Let $S = G \times R_n$ be a right group, A and B be nonempty subsets of S, $G/\langle p_1(A)\rangle = \{g_1\langle p_1(A)\rangle, g_2\langle p_1(A)\rangle, \ldots, g_w\langle p_1(A)\rangle\}$, and $G/\langle p_1(B)\rangle = \{h_1\langle p_1(B)\rangle, h_2\langle p_1(B)\rangle, \ldots, h_z\langle p_1(B)\rangle\}$. If $\dot\bigcup_{i=1}^{w}(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \bigcup \dot\bigcup_{i=1}^{w} (g_i\langle p_1(A)\rangle \times R_n, E_i') \cong \dot\bigcup_{j=1}^{z}(h_j\langle p_1(B)\rangle \times p_2(B),\ E_j) \bigcup \dot\bigcup_{j=1}^{z}(h_j\langle p_1(B)\rangle \times R_n,\ E_j')$, then $w = z$ and $(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \cong (h_j\langle p_1(B)\rangle \times p_2(B), E_j)$ for all i, j.*

proof. Let $\dot\bigcup_{i=1}^{w}(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \bigcup \dot\bigcup_{i=1}^{w}(g_i\langle p_1(A)\rangle \times R_n, E_i') \cong \dot\bigcup_{j=1}^{z} (h_j\langle p_1(B)\rangle \times p_2(B), E_j) \bigcup \dot\bigcup_{j=1}^{z}(h_j\langle p_1(B)\rangle \times R_n, E_j')$. Then there exists an isomorphism $f : \dot\bigcup_{i=1}^{w}(g_i\langle p_1(A)\rangle \times p_2(A)) \bigcup \dot\bigcup_{i=1}^{w}(g_i\langle p_1(A)\rangle \times R_n) \to \dot\bigcup_{j=1}^{z} (h_j\langle p_1(B)\rangle \times p_2(B)) \bigcup \dot\bigcup_{j=1}^{z} (h_j\langle p_1(B)\rangle \times R_n)$. By Lemma 4.2.6, we get that $|\dot\bigcup_{i=1}^{w}\big(g_i\langle p_1(A)\rangle\big) \times p_2(A)| = |\dot\bigcup_{j=1}^{z} \big(h_j\langle p_1(B)\rangle\big) \times p_2(B)|$ and we have $f(\dot\bigcup_{i=1}^{w}\big(g_i\langle p_1(A)\rangle\big) \times p_2(A)) = \dot\bigcup_{j=1}^{z}\big(h_j\langle p_1(B)\rangle\big) \times p_2(B)$. Since f is an isomorphism, we thus get the restriction of f to $\dot\bigcup_{i=1}^{w}\big(g_i\langle p_1(A)\rangle\big) \times p_2(A)$ is a digraph isomorphism from $\dot\bigcup_{i=1}^{w} (g_i\langle p_1(A)\rangle \times p_2(A), E_i)$ to $\dot\bigcup_{j=1}^{z}(h_j\langle p_1(B)\rangle \times p_2(B), E_j)$. Therefore $\dot\bigcup_{i=1}^{w} (g_i\langle p_1(A)\rangle \times p_2(A), E_i) \cong \dot\bigcup_{j=1}^{z}(h_j\langle p_1(B)\rangle \times p_2(B), E_j)$. In view of Theorem 4.2.5 and Lemma 4.2.7, we get that $w = z$ and $(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \cong (h_j\langle p_1(B)\rangle \times p_2(B), E_j)$. $\square$

Lemma 4.2.9. *Let $S = G \times R_n$ be a right group, A and B nonempty subsets of S. If $Cay(S, A) \cong Cay(S, B)$, then $\big|p_2(A)\big| = \big|p_2(B)\big|$.*

proof. Let $G/\langle p_1(A)\rangle = \{g_1\langle p_1(A)\rangle, g_2\langle p_1(A)\rangle, \ldots, g_w\langle p_1(A)\rangle\}$ and $G/\langle p_1(B)\rangle = \{h_1\langle p_1(B)\rangle, h_2\langle p_1(B)\rangle, \ldots, h_z\langle p_1(B)\rangle\}$. Assume that $Cay(S,\ A) \cong Cay(S, B)$. By Theorem 4.2.3 and Lemma 4.2.8, we get that $\big|\dot\bigcup_{i=1}^{w} g_i\langle p_1(A)\rangle \times p_2(A)\big| = \big|\dot\bigcup_{j=1}^{w} h_j\langle p_1(B)\rangle \times p_2(B)\big|$ for all $g_i, h_j \in G$. Since $\dot\bigcup_{i=1}^{w} g_i\langle p_1(A)\rangle = G = \dot\bigcup_{j=1}^{w} h_j\langle p_1(B)\rangle$, we have $\big|G \times p_2(A)\big| = \big|G \times p_2(B)\big|$. Therefore $\big|G\big| \times \big|p_2(A)\big| = \big|G\big| \times \big|p_2(B)\big|$. Hence $\big|p_2(A)\big| = \big|p_2(B)\big|$. $\square$

Lemma 4.2.10. *Let $S = G \times R_n$ be a right group, A a nonempty subset of S, $G/\langle p_1(A)\rangle = \{g_1\langle p_1(A)\rangle, g_2\langle p_1(A)\rangle, \ldots, g_w\langle p_1(A)\rangle\}$, and $(g_i\langle p_1(A)\rangle \times p_2(A), E_i)$ a strong subdigraph of $Cay(S, A)$. Then for every $i \in \{1, 2, \ldots, w\}$, $\dot\bigcup_{i=1}^{w}(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \bigcup \dot\bigcup_{i=1}^{w}(g_i\langle p_1(A)\rangle \times R_n, E_i') = \dot\bigcup_{i=1}^{w}\big((g_i\langle p_1(A)\rangle \times p_2(A), E_i) \cup (g_i\langle p_1(A)\rangle \times R_n, E_i')\big)$, where E' is defined as in Theorem 4.2.3.*

proof. Let $D = \dot\bigcup_{i=1}^{w}(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \bigcup \dot\bigcup_{i=1}^{w}(g_i\langle p_1(A)\rangle \times R_n, E_i')$, and $H = \dot\bigcup_{i=1}^{w}\big((g_i\langle p_1(A)\rangle \times p_2(A), E_i) \cup (g_i\langle p_1(A)\rangle \times R_n, E_i')\big)$. Note that $G = \dot\bigcup_{i=1}^{w}(g_i\langle p_1(A)\rangle)$, then $\dot\bigcup_{i=1}^{w}\big(g_i\langle p_1(A)\rangle \times p_2(A)\big) \bigcup \dot\bigcup_{i=1}^{w}\big(g_i\langle p_1(A)\rangle \times R_n\big) = G \times R_n = \dot\bigcup_{i=1}^{w}\big((g_i\langle p_1(A)\rangle \times p_2(A)) \cup (g_i\langle p_1(A)\rangle \times R_n)\big)$. Therefore $V(D) = V(H)$. We will show that $E(D) = E(H)$.

Let $u, v \in V(D)$ and $(u, v) \in E(D)$. We consider two cases.

(case1) If (u,v) is an arc in $\dot\bigcup_{i=1}^{w}(g_i\langle p_1(A)\rangle \times p_2(A), E_i)$, then (u,v) is an arc in $(g_k\langle p_1(A)\rangle \times p_2(A), E_k)$ for some $k \in \{1,2,\ldots,w\}$. It follows that (u,v) is an arc in $\big((g_k\langle p_1(A)\rangle \times p_2(A), E_k) \cup (g_k\langle p_1(A)\rangle \times R_n, E'_k)\big)$ and so it is an arc in H.

(case2) If (u,v) is an arc in $\dot\bigcup_{i=1}^{w}(g_i\langle p_1(A)\rangle \times R_n, E'_i)$, then (u,v) is an arc in $(g_k\langle p_1(A)\rangle \times R_n, E'_k)$ for some $k \in \{1,2,\ldots,w\}$. Thus (u,v) is an arc in $\big((g_k\langle p_1(A)\rangle \times p_2(A), E_k) \cup (g_k\langle p_1(A)\rangle \times R_n, E'_k)\big)$. It follows that $(u,v) \in E(H)$.

We have $E(D) \subseteq E(H)$. Let $(u,v) \in E(H)$. Therefore it is an arc in $(g_j\langle p_1(A)\rangle \times p_2(A), E_j) \cup (g_k\langle p_1(A)\rangle \times R_n, E'_k)$ for some $j,k \in \{1,2,\ldots,w\}$. We consider two cases.

(case1) If (u,v) is an arc in $(g_j\langle p_1(A)\rangle \times p_2(A), E_j)$, then it is an arc in $\dot\bigcup_{i=1}^{w}(g_i\langle p_1(A)\rangle \times p_2(A), E_i)$. It follows that $(u,v) \in E(D)$.

(case2) If (u,v) is an arc in $(g_k\langle p_1(A)\rangle \times R_n, E'_k)$, it is an arc in $\dot\bigcup_{i=1}^{w}(g_i\langle p_1(A)\rangle \times R_n, E'_i)$. It follows that $(u,v) \in E(D)$.

Therefore $E(H) \subseteq E(D)$. Then $E(H) = E(D)$. We conclude that $D = H$. $\square$

It is know that a right group $S = G \times R_n$ is a special case of a completely simple semigroup $\mathcal{M}(G, I, \Lambda, P)$, where $|I| = 1$, $|\Lambda| = n$ and P is an identity matrix. By Theorem 3.2.7, we have the following corollary.

Corollary 4.2.11. *Let $S = G \times R_n$ be right group, and let $(g,\lambda), (h,\beta) \in S$, where $g, h \in G$ and $\lambda, \beta \in R_n$. Then $Cay(S, \{(g,\lambda)\}) \cong Cay(S, \{(h,\beta)\})$ if and only if $|\langle g\rangle| = |\langle h\rangle|$.*

Theorem 4.2.12. *Let $S = G \times R_n$ be a right group, A and B nonempty subsets of S. Let $A_r = \langle p_1(A)\rangle \times \{r\}$, $\hat{A}_r = A \cap A_r$ and $\hat{A} = \{\hat{A}_r | r \in p_2(A)\}$. $B_r, \hat{B}_r$ and $\hat{B}$ are defined similarly. If $Cay(\langle A\rangle, A) \cong Cay(\langle B\rangle, B)$, then $|\hat{A}| = |\hat{B}|$ and $|\langle p_1(A)\rangle| = |\langle p_1(B)\rangle|$.*

proof. Let $Cay(\langle A\rangle, A) \cong Cay(\langle B\rangle, B)$. By Lemma 4.2.9, $|p_2(A)| = |p_2(B)|$ and then $|\hat{A}| = |\hat{B}|$. Since $Cay(\langle A\rangle, A) \cong Cay(\langle B\rangle, B)$, we get that $|\langle A\rangle| = |\langle B\rangle|$. By Lemma 2.2.2,

$$
\begin{aligned}
\big|\langle p_1(A)\rangle \times p_2(A)\big| &= \big|\langle p_1(B)\rangle \times p_2(B)\big|; \\
\big|\langle p_1(A)\rangle\big| \times \big|p_2(A)\big| &= \big|\langle p_1(B)\rangle\big| \times \big|p_2(B)\big|; \\
\big|\langle p_1(A)\rangle\big| &= \big|\langle p_1(B)\rangle\big|.
\end{aligned}
$$

$\square$

Theorem 4.2.13. *Let $S = G \times R_n$ be a right group, A and B nonempty subsets of S. Let $A_r = \langle p_1(A)\rangle \times \{r\}$, $\hat{A}_r = A \cap A_r$ and $\hat{A} = \{\hat{A}_r | r \in p_2(A)\}$. $B_r, \hat{B}_r$ and $\hat{B}$ are defined similarly. Then $Cay(\langle A\rangle, A) \cong Cay(\langle B\rangle, B)$ if the following conditions hold:*

(1) $|\hat{A}| = |\hat{B}|$ and $|\langle p_1(A)\rangle| = |\langle p_1(B)\rangle|$;

(2) There exists a bijection $f : \hat{A} \to \hat{B}$ such that $|\hat{A}_r| = |f(\hat{A}_r)|$ for all $\hat{A}_r \in \hat{A}$;

(3) For each $\hat{A}_r \in \hat{A}$, there exists a bijection $\varphi_r : \hat{A}_r \to f(\hat{A}_r)$ such that $|\langle p_1(a)\rangle| = |\langle p_1(\varphi_r(a))\rangle|$ for all $a \in \hat{A}_r$.

proof. By (1), $|\langle A\rangle| = |\langle B\rangle|$. By Corollary 4.2.11 and (3), we get that $Cay(\langle A\rangle, \{a\}) \cong Cay(\langle B\rangle, \{\varphi_r(a)\})$ for all $a \in \hat{A}_r$. Then $Cay(\langle A\rangle, \hat{A}_r) = \oplus_{a\in\hat{A}_r} Cay(\langle A\rangle, \{a\}) \cong \oplus_{a\in\hat{A}_r} Cay(\langle B\rangle, \{\varphi_r(a)\}) = Cay(\langle B\rangle, \varphi_r(\hat{A}_r))$.

By (2), $Cay(\langle A\rangle, \hat{A}_r) \cong Cay(\langle B\rangle, f(\hat{A}_r))$ for all $\hat{A}_r \in \hat{A}$. Then

$$\begin{aligned} \oplus_{\hat{A}_r\in\hat{A}} Cay\big(\langle A\rangle, \hat{A}_r\big) &\cong \oplus_{\hat{A}_r\in\hat{A}} Cay\big(\langle B\rangle, f(\hat{A}_r)\big); \\ Cay\big(\langle A\rangle, \cup_{\hat{A}_r\in\hat{A}} \hat{A}_r\big) &\cong Cay\big(\langle B\rangle, \cup_{\hat{A}_r\in\hat{A}} f(\hat{A}_r)\big); \\ Cay\big(\langle A\rangle, A\big) &\cong Cay\big(\langle B\rangle, B\big). \end{aligned}$$

□

Example 4.2.14. Let $S = S_3 \times R_3$ be a right group, where $S_3 = \{(1), \sigma, \sigma^2, \tau, \tau\sigma^2, \tau\sigma\}$ is the symmetric group with (1) an identity, $\sigma = (123), \sigma^2 = (132), \tau = (12), \tau\sigma^2 = (13), \tau\sigma = (23)$. Let $A = \{((1), r_1), (\tau, r_1), (\tau, r_2)\}$ and $B = \{(\tau\sigma, r_1), ((1), r_2), (\tau\sigma, r_2)\}$. It is easily seen that $Cay(\langle A\rangle, A) \cong Cay(\langle B\rangle, B)$ (see Figures 4.3 and 4.4).

We have $\hat{A}_{r_1} = \{((1), r_1), (\tau, r_1)\}$, $\hat{A}_{r_2} = \{(\tau, r_2)\}$, $\hat{B}_{r_1} = \{(\tau\sigma, r_1)\}$, and $\hat{B}_{r_2} = \{((1), r_2), (\tau\sigma, r_2)\}$. Therefore $\hat{A} = \{\hat{A}_{r_1}, \hat{A}_{r_2}\}$, $\hat{B} = \{\hat{B}_{r_1}, \hat{B}_{r_2}\}$, and thus $|\hat{A}| = |\hat{B}|$. Since $\langle p_1(A)\rangle = \{(1), \tau\}$ and $\langle p_1(B)\rangle = \{(1), \tau\sigma\}$, then $|\langle p_1(A)\rangle| = |\langle p_1(B)\rangle|$.

We thus get $|\hat{A}_{r_1}| = 2 = |\hat{B}_{r_2}|$ and $|\hat{A}_{r_2}| = 1 = |\hat{B}_{r_1}|$. There exists a bijective function f from $\hat{A}$ to $\hat{B}$ such that $f(\hat{A}_{r_1}) = \hat{B}_{r_2}$ and $f(\hat{A}_{r_2}) = \hat{B}_{r_1}$.

Moreover, there are bijective functions

$$\begin{aligned} \varphi_{r_1} : \hat{A}_{r_1} \to \hat{B}_{r_2} \quad \text{such} \quad \text{that} \quad \varphi_{r_1}((1), r_1) &= ((1), r_2) \\ \varphi_{r_1}(\tau, r_1) &= (\tau\sigma, r_2) \\ \varphi_{r_2} : \hat{A}_{r_2} \to \hat{B}_{r_1} \quad \text{such} \quad \text{that} \quad \varphi_{r_2}(\tau, r_2) &= (\tau\sigma, r_1) \end{aligned}$$

and $|\langle p_1(a)\rangle| = |\langle p_1(\varphi_{r_1}(a))\rangle|$ and $|\langle p_1(b)\rangle| = |\langle p_1(\varphi_{r_2}(b))\rangle|$ for all $a \in \hat{A}_{r_1}$ and $b \in \hat{A}_{r_2}$. According to Theorem 4.2.13, $Cay(\langle A\rangle, A) \cong Cay(\langle B\rangle, B)$.

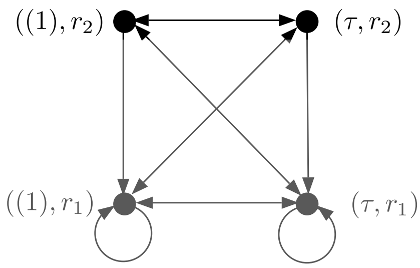

Figure 4.3: Cayley digraph $Cay(\langle A\rangle, A)$.

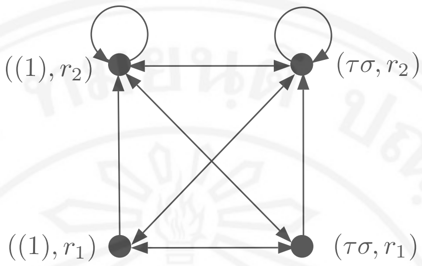

Figure 4.4: Cayley digraph $Cay(\langle B\rangle, B)$.

Theorem 4.2.15. *Let $S = G \times R_n$ be a right group, A and B be nonempty subsets of S. Then $Cay(S, A) \cong Cay(S, B)$ if and only if $Cay(\langle A\rangle, A) \cong Cay(\langle B\rangle, B)$.*

proof. Let $G/\langle p_1(A)\rangle = \{g_1\langle p_1(A)\rangle, g_2\langle p_1(A)\rangle, \ldots, g_w\langle p_1(A)\rangle\}$ and $G/\langle p_1(B)\rangle = \{h_1\langle p_1(B)\rangle, h_2\langle p_1(B)\rangle, \ldots, h_z\langle p_1(B)\rangle\}$.

($\Rightarrow$) Let $Cay(S, A) \cong Cay(S, B)$. Then there exists a digraph isomorphism $f : Cay(S, A) \to Cay(S, B)$. By Theorem 4.2.3, we get that $\dot\bigcup_{i=1}^{w}(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \bigcup \dot\bigcup_{i=1}^{w} (g_i\langle p_1(A)\rangle \times R_n, E_i') \cong \dot\bigcup_{j=1}^{z}(h_j\langle p_1(B)\rangle \times p_2(B), E_j) \bigcup \dot\bigcup_{j=1}^{z} (h_j\langle p_1(B)\rangle \times R_n, E_j')$. In view of Lemma 4.2.8, $(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \cong (h_j\langle p_1(B)\rangle \times p_2(B), E_j)$. By Theorem 4.2.5, we get $Cay(\langle A\rangle, A) \cong Cay(\langle B\rangle, B)$.

($\Leftarrow$) Let $Cay(\langle A\rangle, A) \cong Cay(\langle B\rangle, B)$. By Theorem 4.2.12, $|\langle p_1(A)\rangle| = |\langle p_1(B)\rangle|$ and thus $w = |G|/|\langle p_1(A)\rangle| = |G|/|\langle p_1(B)\rangle| = z$. By Theorem 4.2.5, we get $(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \cong (h_j\langle p_1(B)\rangle \times p_2(B), E_j)$ for all $i, j \in \{1, 2, \ldots, w\}$. It follows that $\dot\bigcup_{i=1}^{w} (g_i\langle p_1(A)\rangle \times p_2(A), E_i) \cong \dot\bigcup_{j=1}^{z}(h_j\langle p_1(B)\rangle \times p_2(B), E_j)$. There exists a digraph isomorphism $f : \dot\bigcup_{i=1}^{w}(g_i \langle p_1(A)\rangle \times p_2(A), E_i) \to \dot\bigcup_{j=1}^{z} (h_j\langle p_1(B)\rangle \times p_2(B), E_j)$. Therefore $|\dot\bigcup_{i=1}^{w}(g_i \langle p_1(A)\rangle \times p_2(A)| = |\dot\bigcup_{j=1}^{z} (h_j\langle p_1(B)\rangle \times p_2(B)|$. Since $\dot\bigcup_{i=1}^{w} g_i\langle p_1(A)\rangle = G = \dot\bigcup_{j=1}^{z} h_j\langle p_1(B)\rangle$, then $|G \times p_2(A)| = |G \times p_2(B)|$. Hence $|G| \times |p_2(A)| = |G| \times |p_2(B)|$ and thus $|p_2(A)| = |p_2(B)|$. Suppose that $R_n \setminus p_2(A) = \{q_1, q_2, \ldots, q_m\}$ and $R_n \setminus p_2(B) = \{q_1', q_2', \ldots, q_m'\}$. Let $r \in p_2(A)$. Define $T : \dot\bigcup_{i=1}^{w}(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \quad \bigcup \quad \dot\bigcup_{i=1}^{w}(g_i\langle p_1(A)\rangle \times R_n, E_i') \to$

$\dot{\bigcup}_{j=1}^{z}(h_j\langle p_1(B)\rangle \times p_2(B), E_j) \bigcup \dot{\bigcup}_{j=1}^{z}(h_j\langle p_1(B)\rangle \times R_n, E'_j)$ by

$$T(s, r_l) = \begin{cases} f(s, r_l) & \text{if } r_l \in p_2(A); \\ (p_1(f(s, r)), q'_k) & \text{if } r_l = q_k \text{ for some } q_k \in R_n \setminus p_2(A). \end{cases}$$

Clearly, T is well defined and is surjective. To show T is injective, let $x_1 = (u_1, \lambda_1), x_2 = (u_2, \lambda_2) \in S$ and $T(x_1) = T(x_2)$. We need to consider the following two cases.

(case1) If $\lambda_1, \lambda_2 \in p_2(A)$, then $T(x_1) = f(x_1)$ and $T(x_2) = f(x_2)$. Since $T(x_1) = T(x_2)$, we have $f(x_1) = f(x_2)$ and $x_1 = x_2$ because f is a digraph isomorphism.

(case2) If $\lambda_1, \lambda_2 \notin p_2(A)$, assume that $\lambda_1 = q_l$ and $\lambda_2 = q_k$. Thus $T(x_1) = (p_1(f(u_1, r)), q'_l)$ and $T(x_2) = (p_1(f(u_2, r)), q'_k)$. Since $T(x_1) = T(x_2)$, we have $(p_1(f(u_1, r)), q'_l) = (p_1(f(u_2, r)), q'_k)$. It follows that $q'_l = q'_k$ and $p_1(f(u_1, r)) = p_1(f(u_2, r))$. Hence by the definition of T, $\lambda_1 = \lambda_2$. Since f is a digraph isomorphism, we have $u_1 = u_2$. Therefore $x_1 = x_2$.

By the above two cases, we conclude that T is an injection. We will prove that T and T^{-1} are digraph homomorphisms.

Assume that $((x, r_c), (y, r_d))$ is an arc in $\dot{\bigcup}_{i=1}^{w}(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \bigcup \dot{\bigcup}_{i=1}^{w}(g_i\langle p_1(A)\rangle \times R_n, E'_i)$. Thus $(y, r_d) = (x, r_c)(a, r_t)$ for some $(a, r_t) \in A$. Hence $(y, r_d) = (xa, r_t)$ and thus $r_d = r_t \in p_2(A)$ and $y = xa$. We have the following two cases.

(case1) $r_c \in p_2(A)$. Then $\big(T(x, r_c), T(y, r_d)\big) = \big(f(x, r_c), f(y, r_d)\big)$ is an arc in $\dot{\bigcup}_{j=1}^{z}(h_j\langle p_1(B)\rangle \times p_2(B), E_j) \bigcup \dot{\bigcup}_{j=1}^{z}(h_j\langle p_1(B)\rangle \times R_n, E'_j)$ since f is a digraph isomorphism.

(case2) $r_c \in R_n \setminus p_2(A)$. Then $r_c = q_k$ for some $k \in \{1, 2, \ldots, m\}$. Hence $((x, r_c), (y, r_d))$ is an arc in $\dot{\bigcup}_{i=1}^{w}(g_i\langle p_1(A)\rangle \times R_n, E'_i)$. Then $((x, r_d), (y, r_d))$ is an arc in $\dot{\bigcup}_{i=1}^{w}(g_i\langle p_1(A)\rangle \times p_2(A), E_i)$. By Lemma 4.2.1, $((x, r), (y, r_d))$ is also an arc in $\dot{\bigcup}_{i=1}^{w}(g_i\langle p_1(A)\rangle \times p_2(A), E_i)$. This follows that $(f(x, r), f(y, r_d))$ is an arc in $\dot{\bigcup}_{j=1}^{z}(h_j\langle p_1(B)\rangle \times p_2(B), E_j)$. Let $f(x, r) = (x', r')$ and $f(y, r_d) = (y', r'_d)$. Therefore $\big((x', r'), (y', r'_d)\big)$ is an arc in $\dot{\bigcup}_{j=1}^{z}(h_j\langle p_1(B)\rangle \times p_2(B), E_j)$ and thus $\big((x', r'_d), (y', r'_d)\big)$ is also an arc in $\dot{\bigcup}_{j=1}^{z}(h_j\langle p_1(B)\rangle \times p_2(B), E_j)$. Therefore $((x', q'_k), (y', r'_d))$ is an arc in $\dot{\bigcup}_{j=1}^{z}(h_j\langle p_1(B)\rangle \times R_n, E'_j)$. This means that $((x', q'_k), (y', r'_d)) = \big((p_1(f(x, r)), q'_k), (y', r'_d)\big) = (T(x, r_c), T(y, r_d))$ is an arc in $\dot{\bigcup}_{j=1}^{z}(h_j\langle p_1(B)\rangle \times R_n, E'_j)$. Hence $(T(x, r_c), T(y, r_d))$ is an arc in $\dot{\bigcup}_{j=1}^{z}(h_j\langle p_1(B)\rangle \times p_2(B), E_j) \bigcup \dot{\bigcup}_{j=1}^{z}(h_j\langle p_1(B)\rangle \times R_n, E'_j)$.

Thus T is a digraph homomorphism.

Assume that $(T(x, r_c), T(y, r_d))$ is an arc in $\dot{\cup}_{j=1}^{z} (h_j\langle p_1(B)\rangle \times p_2(B), E_j) \bigcup \dot{\cup}_{j=1}^{z}(h_j\langle p_1(B)\rangle \times R_n, E'_j)$. We have the following two cases.

(case1) If $(T(x, r_c), T(y, r_d))$ is an arc in $\dot{\cup}_{j=1}^{z}(h_j\langle p_1(B)\rangle \times p_2(B), E_j)$, then we get that $T(x, r_c) = f(x, r_c)$ and $T(y, r_d) = f(y, r_d)$. Therefore $(f(x, r_c),\ f(y, r_d))$ is an arc in $\dot{\cup}_{j=1}^{z}(h_j\langle p_1(B)\rangle \times p_2(B),\ E_j)$. Since f is a digraph isomorphism from $\dot{\cup}_{i=1}^{w}(g_i\langle p_1(A)\rangle \times p_2(A), E_i)$ to $\dot{\cup}_{j=1}^{z}(h_j\langle p_1(B)\rangle \times p_2(B), E_j)$, we get that $((x, r_c), (y, r_d))$ is an arc in $\dot{\cup}_{i=1}^{w}\ (g_i\langle p_1(A)\rangle \times p_2(A), E_i)$ and it is also an arc in $\dot{\cup}_{i=1}$ $(g_i\langle p_1(A)\rangle \times\ p_2(A),\ E_i) \bigcup \dot{\cup}_{i=1}^{w}(g_i\langle p_1(A)\rangle \times R_n, E'_i)$.

(case2) Suppose that $(T(x, r_c), T(y, r_d))$ is an arc in $\dot{\cup}_{j=1}^{z}(h_j\langle p_1(B)\rangle \times R_n, E'_j)$. Then $r_c = q_k$ for some $k \in \{1, 2, \ldots, m\}$. Let $T(y, r_d) = f(y, r_d) = (y', r'_d)$. Then $((p_1(f(x, r)), q'_k), (y', r'_d))$ is an arc in $\dot{\cup}_{j=1}^{z}\ (h_j\langle p_1(B)\rangle \times R_n, E'_j)$ and so $((p_1(f(x, r)), r'_d),\ (y', r'_d))$ is an arc in $\dot{\cup}_{j=1}^{z}(h_j\langle p_1(B)\rangle \times p_2(B), E_j)$. Hence there exists $(b, r'_d) \in B$ such that $(y', r'_d) = (p_1(f(x, r)), r'_d)(b, r'_d)$. Let $f(x, r) = (x', r')$. Then $f(y, r_d) = (x', r'_d)(b, r'_d) = (x'b, r'_d) = (x', r')(b, r'_d) = f(x, r)$ (b, r'_d). This means that $(f(x, r), f(y, r_d))$ is an arc in $\dot{\cup}_{j=1}^{z}\ (h_j\langle p_1(B)\rangle \times$ $p_2(B),\ E_j)$. Then $((x, r), (y, r_d))$ is an arc in $\dot{\cup}_{i=1}^{w}\ (g_i\langle p_1(A)\rangle \times p_2(A), E_i)$. Therefore $((x, r_c), (y, r_d))$ is an arc in $\dot{\cup}_{i=1}^{w}(g_i\langle p_1(A)\rangle \times R_n, E'_i)$ and it is also an arc in $\dot{\cup}_{i=1}^{w}(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \bigcup \dot{\cup}_{i=1}^{w}(g_i\langle p_1(A)\rangle \times R_n, E'_i)$.

Thus T^{-1} is a digraph homomorphism. Hence $\dot{\cup}_{i=1}^{w}(g_i\langle p_1(A)\rangle \times p_2(A), E_i) \bigcup \dot{\cup}_{i=1}^{w}$ $(g_j\langle p_1(A)\rangle \times R_n, E'_i) \cong \dot{\cup}_{j\in I}^{z}(h_j\langle p_1(B)\rangle \times p_2(B), E_j) \bigcup \dot{\cup}_{j=1}^{z}(h_j\langle p_1(B)\rangle \times R_n, E'_j)$. By Theorem 4.2.3, we have $Cay(S, A) \cong Cay(S, B)$. $\square$

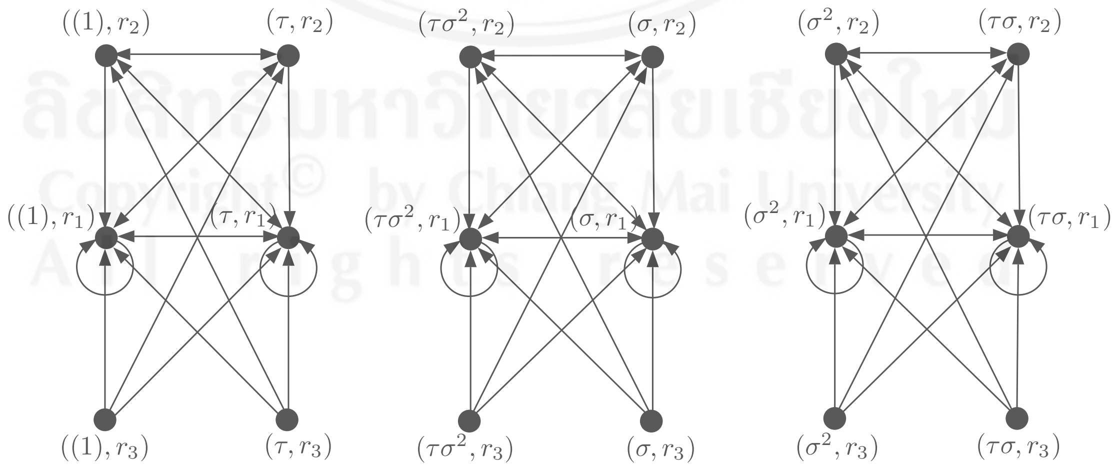

Figure 4.5: Cayley digraph $Cay(S_3 \times R_3, \{((1), r_1), (\tau, r_1), (\tau, r_2)\})$.

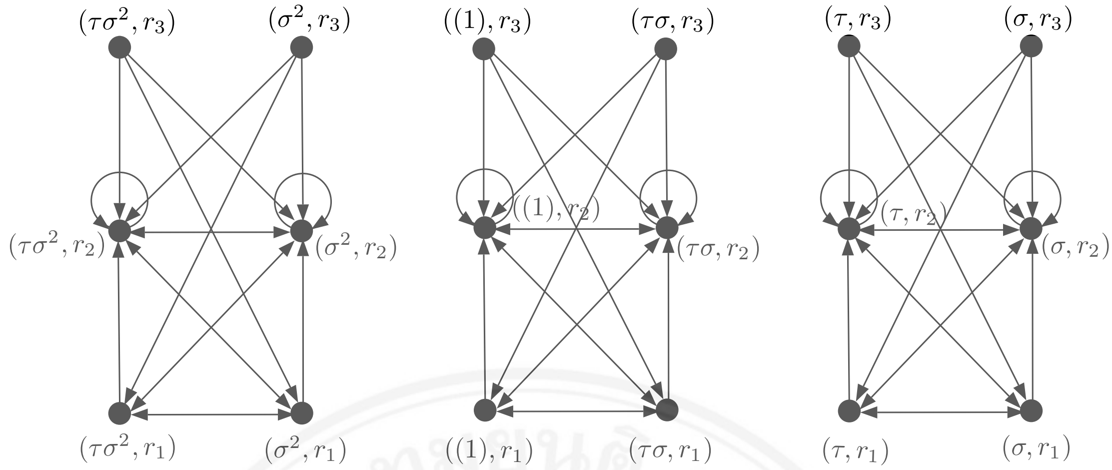

Figure 4.6: Cayley digraph $Cay(S_3 \times R_3, \{(\tau\sigma, r_1), ((1), r_2), (\tau\sigma, r_2)\})$.

Example 4.2.16. Let $S = S_3 \times R_3$ be a right group as in Example 4.2.14, let $A = \{((1), r_1), (\tau, r_1), (\tau, r_2)\}$ and $B = \{(\tau\sigma, r_1), ((1), r_2), (\tau\sigma, r_2)\}$ be subsets of S. By Example 4.2.14, we have $Cay(\langle A\rangle, A) \cong Cay(\langle B\rangle, B)$ and it is easily seen that $Cay(S, A) \cong Cay(S, B)$ (see Figures 4.5 and 4.6).

4.3 Isomorphism Conditions for Cayley Digraphs of Rectangular Groups

In this section, we give the necessary and sufficient conditions for the Cayley digraphs of a given rectangular groups to be isomorphic to each other. By definition of rectangular groups, we have the following lemma.

Lemma 4.3.1. *Let $S = G \times L_m \times R_n$ be a rectangular group, A a nonempty subset of S, and $(g_1, l_1, r_1), (g_2, l_2, r_2) \in S$. Then $\big((g_1, l_1, r_1), (g_2, l_2, r_2)\big)$ is an arc in $Cay(S, A)$ if and only if there exists $(a, l, r_2) \in A$ such that $g_2 = g_1 a$ and $l_1 = l_2$.*

proof. $(\Rightarrow)$ Let $\big((g_1, l_1, r_1), (g_2, l_2, r_2)\big)$ is an arc in $Cay(S, A)$. Then there is $(a, l, r) \in A$ such that $(g_2, l_2, r_2) = (g_1, l_1, r_1)(a, l, r) = (g_1 a, l_1, r)$. We have $g_2 = g_1 a, l_2 = l_1$ and $r_2 = r_1$.

$(\Leftarrow)$ Let $(a, l, r_2) \in A$, $g_2 = g_1 a$ and $l_1 = l_2$. Thus $(g_1, l_1, r_1)(a, l, r_2) = (g_1 a, l_1, r_2) = (g_2, l_2, r_2)$. Therefore $\big((g_1, l_1, r_1), (g_2, l_2, r_2)\big)$ is an arc in $Cay(S, A)$. □

As a direct consequence of Lemma 4.3.1, we have the following lemma.

Lemma 4.3.2. *Let $S = G \times L_m \times R_n$ be a rectangular group, A a nonempty subset of S. Then $Cay(S, A)$ is the disjoint union of m isomorphic strong subdigraphs $(G \times \{l_i\} \times R_n, E_i)$ for some $i = 1, 2, ..., m$.*

proof. For $i = 1, 2, ..., m$, let $V_i = G \times \{l_i\} \times R_n$ and $E_i = E\big(Cay(S, A)\big) \cap (V_i \times V_i)$. Hence (V_i, E_i) is a strong subdigraph of $Cay(S, A)$. For all $i \neq j$, $V_i \cap V_j = \emptyset$, then we have $S = \dot\cup_{i=1}^{n} V_i$. Since $E_i \subseteq E\big(Cay(S, A)\big)$, $\dot\cup_{i=1}^{m} E_i \subseteq E\big(Cay(S, A)\big)$. Let $\big((g, l_j, r),$ $(g', l_k, r')\big) \in E(Cay(S, A)$. By Lemma 4.3.1, $l_j = l_k$ and thus $\big((g, l_j, r), (g', l_k, r')\big) \in E_k$. Then $\big((g, l_j, r), (g', l_k, r')\big) \in \dot\cup_{i=1}^{m} E_i$. Hence $E\big(Cay(S, A)\big) \subseteq \dot\cup_{i=1}^{m} E_i$ and so $E\big(Cay(S,$ $A)\big) = \dot\cup_{i=1}^{m} E_i$. Therefore $Cay(S, A) = \dot\cup_{i=1}^{m} (V_i, E_i)$.

Let $p, q \in \{1, 2, ..., m\}$ and $p \neq q$. We will show that $(V_p, E_p) \cong (V_q, E_q)$. Define $f : V_p \to V_q$ by $f\big((g, l_p, r)\big) = (g, l_q, r)$. Since $|V_p| = |V_q|$, f is a bijection. To prove that f and f^{-1} are digraph homomorphisms. Let $(g, l_p, r), (g', l_p, r') \in V_p$ and $\big((g, l_p, r), (g', p, r')\big) \in E_p$. Since $E_p \subseteq E(Cay(S,\ A)), \big((g, l_p, r), (g', l_p, r')\big)$ is an arc in $Cay(S, A)$. By Lemma 4.3.1, there exists $(a, l, r'') \in A$ such that $g' = ga$, $r' = r''$, and thus $(g', l_q, r') = (ga, l_q, r'') = (g, l_q, r)\ (a, l, r'')$. Then $\big((g, l_q, r), (g', l_q, r')\big)$ is an arc in $Cay(S, A)$. It follows that $\big((g, l_q, r),\ (g', l_q, r')\big) \in E_q$. This shows that f is a digraph homomorphism. Similarly, f^{-1} is a digraph homomorphism. Hence f is a digraph isomorphism. Therefore $(V_p, E_p) \cong (V_q, E_q)$. □

Lemma 4.3.3. *Let $S = G \times L_m \times R_n$ be a rectangular group, A a nonempty subset of S, $G/\langle p_1(A)\rangle = \{g_1\langle p_1(A)\rangle, g_2\langle p_1(A)\rangle, \ldots, g_w\langle p_1(A)\rangle\}$, and $(g_k\langle p_1(A)\rangle \times \{l_i\} \times R_n, E_{ik})$ a strong subdigraph of $Cay(S, A)$. Then the following conditions hold:*

(1) $(G \times \{l_i\} \times R_n, E_i) = \dot\cup_{k=1}^{w} (g_k\langle p_1(A)\rangle \times \{l_i\} \times R_n, E_{ik})$;

(2) $(g_k\langle p_1(A)\rangle \times \{l_i\} \times R_n, E_{ik}) = Cay(g_k\langle p_1(A)\rangle \times \{l_i\} \times R_n, A^i)$ *where* $A^i = \{(g, l_i, r)|$ $(g, l, r) \in A$ *for all* $l \in L_m\}$.

proof. (1) Note that $G = \dot\cup_{k=1}^{w} g_k\langle p_1(A)\rangle$, then $G \times \{l_i\} \times R_n = \dot\cup_{k=1}^{w}\ (g_k\langle p_1(A)\rangle \times$ $\{l_i\} \times R_n)$. Let $\big((g, l_i, r), (g', l_i, r')\big) \in E_i$. By Lemma 4.3.1, there exists $(a, l, r') \in A$ such that $g' = ga$. Hence $g \in g_p\langle p_1(A)\rangle$, $g' \in g_q\langle p_1(A)\rangle$ for some $p, q \in \{1, 2, ..., w\}$. A simple computation shows that $p = q$ and then $(g, l_i, r), (g', l_i, r') \in (g_p\langle p_1(A)\rangle \times \{l_i\} \times R_n)$. Because $(g_p\langle p_1(A)\rangle \times \{l_i\} \times R_n, E_{ip})$ is the strong subdigraph of $Cay(S, A)$, therefore $\big((g, l_i, r), (g', l_i, r')\big) \in \dot\cup_{k=1}^{w} E_{ik}$. Hence $E_i \subseteq \dot\cup_{k=1}^{w} E_{ik}$. Similarly, we can prove that $\dot\cup_{k=1}^{w} E_{ik} \subseteq E_i$, and then $E_i = \dot\cup_{k=1}^{w} E_{ik}$. We conclude that $(G \times \{l_i\} \times R_n, E_i) =$ $\dot\cup_{k=1}^{w} (g_k\langle p_1(A)\rangle \times \{l_i\} \times R_n, E_{ik})$.

(2) Let $D = Cay(g_k\langle p_1(A)\rangle \times \{l_i\} \times R_n, A^i)$, we will prove that $E_{ik} = E(D)$. Let $\big((g, l_i, r), (g', l_i, r')\big) \in E_{ik}$. By Lemma 4.3.1, there exists $(a, l, r') \in A$ such that $g' = ga$. By Lemma 4.3.1 again, we get that $\big((g, l_i, r), (g', l_i, r')\big) \in E(D)$. This shows that $E_{ik} \subseteq E(D)$. Let $\big((g, l_i, r), (g', l_i, r')\big) \in E(D)$. By Lemma 4.3.1, there exists $(a, l_i, r') \in A^i$ such that

$g' = ga$. We get that $(a, j, r') \in A$ for some $j \in L_m$. Then by Lemma 4.3.1 again, $((g, l_i, r), (g', l_i, r')) \in E_{ik}$. This shows that $E(D) \subseteq E_{ik}$. Therefore $E_{ik} = E(D)$. We conclude that $(g_k\langle p_1(A)\rangle \times \{l_i\} \times R_n, E_{ik}) = Cay(g_k\langle p_1(A)\rangle \times \{l_i\} \times R_n, A^i)$. □

Theorem 4.3.4. *Let $S = G \times L_m \times R_n$ be a rectangular group, A, B nonempty subsets of S. Let $S' = G \times R_n$. Then $Cay(S, A) \cong Cay(S, B)$ if and only if $Cay(S', A') \cong Cay(S', B')$, where $A' = \{(g, r) \mid (g, l, r) \in A$ for some $l \in L_m\}$ and $B' = \{(g, r) \mid (g, l, r) \in B$ for some $l \in L_m\}$.*

proof. Let $G/\langle p_1(A)\rangle = \{g_1\langle p_1(A)\rangle, g_2\langle p_1(A)\rangle, \ldots, g_w\langle p_1(A)\rangle\}$, $G/\langle p_1(B)\rangle = \{h_1\langle p_1(B)\rangle, h_2\langle p_1(B)\rangle, \ldots, h_z\langle p_1(B)\rangle\}$. We let $(G \times \{l_i\} \times R_n, E_i^A)$, $A_i^k = (g_k\langle p_1(A)\rangle \times \{l_i\} \times R_n, E_{ik})$ be a strong subdigraph of $Cay(S, A)$, and let $(G \times \{l_j\} \times R_n, E_j^B)$, $B_j^t = (h_t\langle p_1(B)\rangle \times \{l_j\} \times R_n, E_{jt})$ be strong subdigraph of $Cay(S, B)$. By Lemma 4.3.2 and Lemma 4.3.3(1), we have $Cay(S, A) \cong Cay(S, B)$

$\Leftrightarrow Cay(G \times L_m \times R_n, A) \cong Cay(G \times L_m \times R_n, B)$

$\Leftrightarrow \dot{\cup}_{i=1}^{m}(G \times \{l_i\} \times R_n, E_i^A) \cong \dot{\cup}_{j=1}^{m}(G \times \{l_j\} \times R_n, E_j^B)$

$\Leftrightarrow \dot{\cup}_{i=1}^{m}\dot{\cup}_{k=1}^{w} A_i^k \cong \dot{\cup}_{j=1}^{m}\dot{\cup}_{t=1}^{z} B_j^t$.

Since A_i^k and B_j^t are connected subdigraphs, we get that $w = z$. Then for each i, k, there exist j, t such that $A_i^k \cong B_j^t$. Let $D_k^A = (g_k\langle p_1(A)\rangle \times p_2(A'), E_k)$ and $D_t^B = (h_t\langle p_1(B)\rangle \times p_2(B'), E_t)$ be strong subdigraphs of $Cay(g_k\langle p_1(A)\rangle \times R_n, A')$ and $Cay(h_t\langle p_1(B)\rangle \times R_n, B')$, respectively. Let $A^i = \{(g, l_i, r) |\ (g, l, r) \in A\}$, $B^j = \{(h, l_j, r) |\ (h, l, r) \in B\}$. By Lemma 4.3.3(2) and Theorem 4.2.3, we have

$Cay(g_k\langle p_1(A)\rangle \times \{l_i\} \times R_n, A^i) \cong Cay(h_t\langle p_1(B)\rangle \times \{l_j\} \times R_n, B^j)$

$\Leftrightarrow Cay(g_k\langle p_1(A)\rangle \times R_n, A') \cong Cay(h_t\langle p_1(B)\rangle \times R_n, B')$

$\Leftrightarrow \dot{\cup}_{k=1}^{w} D_k^A \cup (g_k\langle p_1(A)\rangle \times R_n, E_{A'}) \cong \dot{\cup}_{t=1}^{z} D_t^B \cup (h_t\langle p_1(B)\rangle \times R_n, E_{B'})$.

By Lemma 4.2.8 and Theorem 4.2.5, we have $\dot{\cup}_{k=1}^{w} D_k^A \cong \dot{\cup}_{t=1}^{z} D_t^B \Leftrightarrow D_k^A \cong D_t^B \Leftrightarrow Cay(\langle A'\rangle, A') \cong Cay(\langle B'\rangle, B') \Leftrightarrow Cay(S', A') \cong Cay(S', B')$. □

Example 4.3.5. Let $S = S_3 \times L_2 \times R_3$ be a rectangular group, where $S_3 = \{(1), \sigma, \sigma^2, \tau, \tau\sigma^2, \tau\sigma\}$ is the symmetric group as in Example 4.2.16, and let $A = \{((1), l_1, r_1), (\tau, l_1, r_1), (\tau, l_2, r_2)\}$, $B = \{(\tau\sigma, l_2, r_1), ((1), l_2, r_2), (\tau\sigma, l_2, r_2)\}$ be subsets of S.

We have $A' = \{((1), r_1), (\tau, r_1), (\tau, r_2)\}$ and $B' = \{(\tau\sigma, r_1), ((1), r_2), (\tau\sigma, r_2)\}$. By Example 4.2.16, $Cay(S', A') \cong Cay(S', B')$. It is easily seen that $Cay(S, A) \cong Cay(S, B)$. (See Figures 4.7 and 4.8).

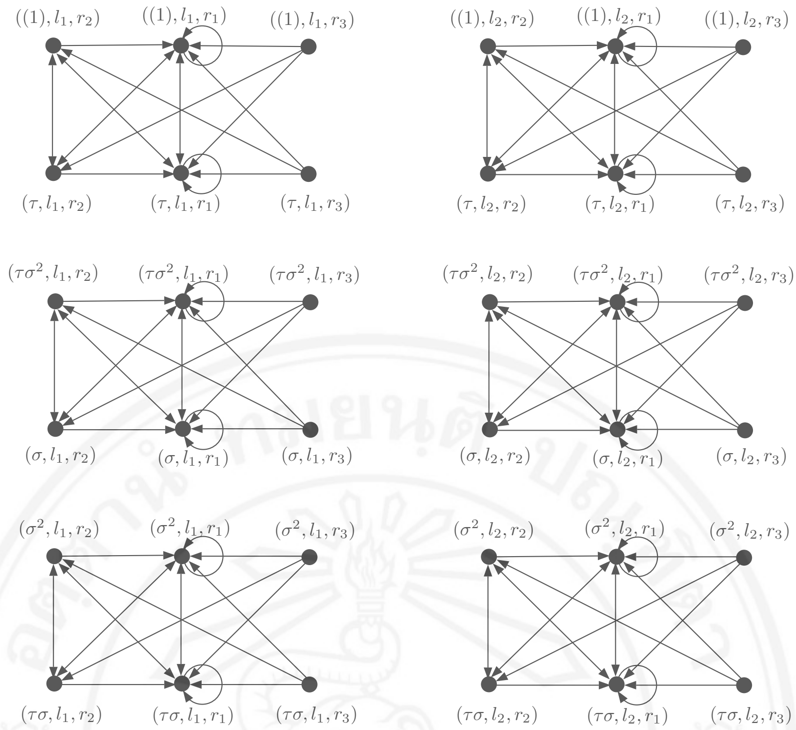

Figure 4.7: Cayley digraph $Cay(S_3 \times L_2 \times R_3, \{((1), l_1, r_1), (\tau, l_1, r_1), (\tau, l_2, r_2)\})$.

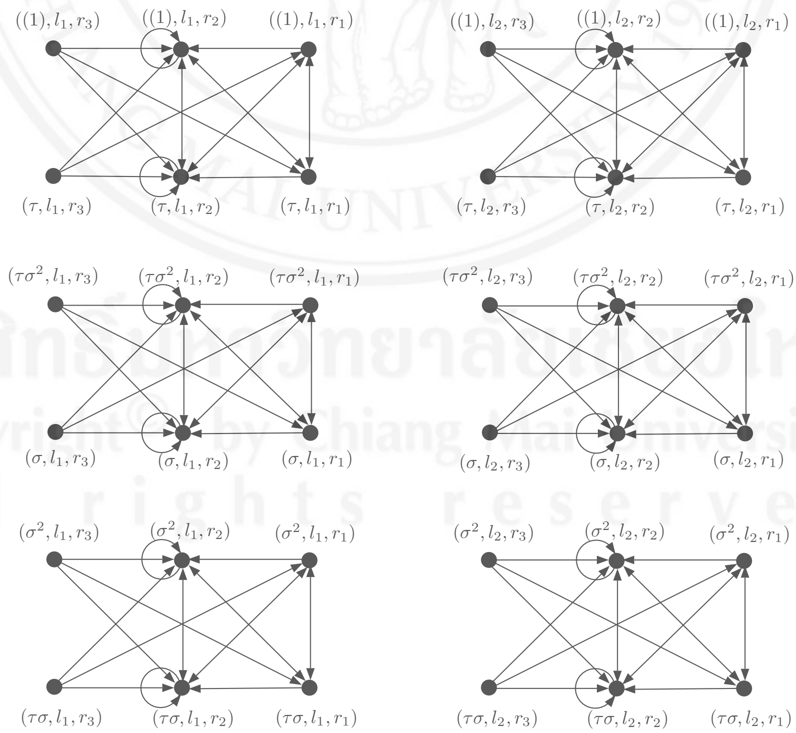

Figure 4.8: Cayley digraph $Cay(S_3 \times L_2 \times R_3, \{(\tau\sigma, l_2, r_1), ((1), l_2, r_2), (\tau\sigma, l_2, r_2)\})$.