CHAPTER 5

Cayley Digraphs of Brandt Semigroups Relative to Green's Equivalence Classes

In this chapter, we describe Cayley digraphs of Brandt semigroups relative to Green's equivalence classes $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{H}$. Moreover, we shall give isomorphism conditions for those Cayley digraphs. Note that, digraphs considered in this chapter are digraphs without multiple arcs and loops.

5.1 Cayley Digraphs of Brandt Semigroups Relative to $\mathcal{L}$-classes

In this section, we describe the Cayley digraph of a given Brandt semigroup S relative to the $\mathcal{L}$-class $S_{\_j}$. By Lemma 2.4.6(1), we have the following lemma.

Lemma 5.1.1. *Let $S = B(G, I)$ be a Brandt semigroup and $j \in I$. There is an arc from any nonzero vertex in $Cay(S, S_{\_j})$ to the vertex 0.*

Proof. Let $v = (l, g, k)$ be any nonzero vertex in S. There is $u = (q, h, j) \in S_{\_j}$ such that $q \neq k$, therefore $vu = 0$. This means that there is an arc from any nonzero vertex to the vertex 0. $\square$

Lemma 5.1.2. *Let $S = B(G, I)$ be a Brandt semigroup, $j \in I$, and u, v nonzero vertices in $Cay(S, S_{\_j})$. Then (u, v) is an arc in $Cay(S, S_{\_j})$ if and only if $u \in S_{k\_}$ and $v \in S_{kj}$ for all $k \in I$.*

Proof. ($\Rightarrow$) Let u, v be nonzero vertices in $Cay(S, S_{\_j})$ and $(u, v) \in E\big(Cay(S, S_{\_j})\big)$. For each $k \in I$, we take $u = (k, g, l) \in S_{k\_}$ for some $g \in G, l \in I$. Since $v \neq 0$, there is $a = (l, h, j) \in S_{\_j}$ for some $h \in G$ such that $v = ua = (k, g, l)(l, h, j) = (k, gh, j)$. It follows that $v \in S_{kj}$.

($\Leftarrow$) For each $k \in I$, let $u = (k, g, l) \in S_{k\_}$ and $v = (k, h, j) \in S_{kj}$ for some $g, h \in G, l \in I$. There is $(l, g^{-1}h, j) \in S_{\_j}$ such that $u(l, g^{-1}h, j) = (k, g, l)(l, g^{-1}h, j) = (k, h, j) = v$. Then (u, v) is an arc in $Cay(S, S_{\_j})$. $\square$

From the above lemma we have the following corollary.

Corollary 5.1.3. *Let $S = B(G, I)$ be a Brandt semigroup, $j, k \in I$, and $v \in S_{k-}$ be a vertex in $Cay(S, S_{-j})$. Then $\overrightarrow{d}(v) = 0$ if and only if $v \notin S_{kj}$.*

proof. ($\Rightarrow$) Let $\overrightarrow{d}(v) = 0$. Assume that $v \in S_{kj}$ and let u be any vertex in S_{k-}. By Lemma 5.1.2, we have (u, v) is an arc in $Cay(S, S_{-j})$. There is a contradiction because $\overrightarrow{d}(v) = 0$. Hence $v \notin S_{kj}$.

($\Leftarrow$) Let $v \notin S_{kj}$, then $v = (k, g, l)$ for some $l \neq j$. Assume that $\overrightarrow{d}(v) \neq 0$, there exists $u = (q, t, s)$ such that (u, v) is an arc in $Cay(S, S_{-j})$. It follows that there exists $a = (s, h, j) \in S_{-j}$ such that $v = ua = (q, t, s)(s, h, j) = (q, th, j)$. Therefore $l = j$, there is a contradiction because $l \neq j$. Hence $\overrightarrow{d}(v) = 0$. □

Let $S = B(G, I)$ be a Brandt semigroup and S_{-j} an $\mathcal{L}$-class of S for some $j \in I$. For any $k \in I$, we denote by (S_{kj}, E_{kj}) the strong subdigraph of $Cay(S, S_{-j})$ induced by S_{kj}.

Corollary 5.1.4. *Let $S = B(G, I)$ be a Brandt semigroup and $j, k \in I$. Then the strong subdigraph (S_{kj}, E_{kj}) of $Cay(S, S_{-j})$ is a complete digraph $(K_{|G|})$.*

proof. Let u, v be a vertices in (S_{kj}, E_{kj}). By Lemma 5.1.2, there is an arc between u and v. Then the strong subdigraph (S_{kj}, E_{kj}) is a complete digraph. Because $|S_{kj}| = |G|$, the strong subdigraph (S_{kj}, E_{kj}) is a complete digraph $K_{|G|}$. □

For any $k \in I$, we denote by Γ_k the strong subdigraph (S_{k-}, E_k) of $Cay(S, S_{-j})$ induced by S_{k-}. The following theorem shows that Γ_k is isomorphic to a Cayley digraph of right group $T = G \times R_n$ where R_n is a right zero semigroup such that $|n| = |I|$.

Theorem 5.1.5. *Let $S = B(G, I)$ be a Brandt semigroup and $k \in I$. Then $\Gamma_k \cong Cay(T, M)$ for some $M \subseteq T$.*

proof. Let $T = G \times R_n$ be a right group where $|n| = |I|$. Assume that $I = \{i_1, i_2, \ldots, i_n\}$, $R_n = \{r_1, r_2, \ldots, r_n\}$, and let $M = G \times \{r_l\} \subseteq T$ for some $r_l \in R_n$. Since Γ_k is the strong subdigraph of $Cay(S, S_{-j})$ for some $j \in I$, for convenience, we suppose that $j = i_z$ for some $z \in \{1, 2, \ldots, n\}$. For each $k \in I$, we define a map $f : V(\Gamma_k) \to T$ by

$$f(k, g, i_q) = \begin{cases} (g, r_l) & \text{if } i_q = i_z; \\ (g, r_z) & \text{if } i_q = i_l; \\ (g, r_q) & \text{otherwise.} \end{cases}$$

Obviously, f is a bijection. We will show that f and f^{-1} are digraph homomorphisms. Let $u = (k, g_1, i_s), v = (k, g_2, i_t)$ be any vertices in Γ_k. Suppose that (u, v) is an arc in Γ_k.

Therefore $\overrightarrow{d}(v) \neq 0$, it follows that $v \in S_{ki_z}$ by Corollary 5.1.3. Thus $f(v) = f(k, g_2, i_t) = f(k, g_2, i_z) = (g_2, r_l)$. We consider following three cases.

(case1) If $i_s = i_z$, then

$$\begin{aligned} f(v) &= (g_2, r_l) \\ &= (g_1 g_1^{-1} g_2, r_l) \\ &= (g_1, r_l)(g_1^{-1} g_2, r_l) \\ &= f(k, g_1, i_z)(g_1^{-1} g_2, r_l) \\ &= f(u)(g_1^{-1} g_2, r_l). \end{aligned}$$

Since $(g_1^{-1} g_2, r_l) \in M$, $\big(f(u), f(v)\big)$ is an arc in $Cay(T, M)$.

(case2) If $i_s = i_l$, then

$$\begin{aligned} f(v) &= (g_2, r_l) \\ &= (g_1 g_1^{-1} g_2, r_l) \\ &= (g_1, r_z)(g_1^{-1} g_2, r_l) \\ &= f(k, g_1, i_l)(g_1^{-1} g_2, r_l) \\ &= f(u)(g_1^{-1} g_2, r_l). \end{aligned}$$

Similarly to the case1, $\big(f(u), f(v)\big)$ is an arc in $Cay(T, M)$.

(case3) If $i_z \neq i_s \neq i_l$, similarly to the above two cases, we conclude that $\big(f(u), f(v)\big)$ is an arc in $Cay(T, M)$.

By above three cases we have f is a digraph homomorphism.

Suppose that $\big(f(u), f(v)\big)$ is an arc in $Cay(T, M)$, then there is $(g, r_l) \in M$ such that $f(v) = f(u)(g, r_l)$. We get that $p_2\big(f(v)\big) = r_l$, it follows that $p_2(v) = i_z$ and so $v = (k, g_2, i_z) \in S_{ki_z}$. By Lemma 5.1.2, there is an arc from u to v. Then f^{-1} is a digraph homomorphism. Therefore $\Gamma_k \cong Cay(T, M)$. □

Lemma 5.1.6. *(Lemma 2.3 [7]) Let $S = B(G, I)$ be a Brandt semigroup, and A a nonempty subset of S. For any $i, k \in I$, $\Gamma_i \cong \Gamma_k$, and there is no arc of $Cay(S, A)$ between Γ_i and Γ_k.*

By Lemma 5.1.6, Γ_i and Γ_k are isomorphic and there is no arc between Γ_i and Γ_k for any $i \neq k \in I$. Let $\Gamma = \dot\cup_{i \in I} \Gamma_i$ be the disjoint union of $|I|$ isomorphic strong subdigraphs of $Cay(S, S_{\_j})$. By Lemma 5.1.1 and Lemma 5.1.2, the following proposition is immediate.

Proposition 5.1.7. *Let $S = B(G, I)$ be a Brandt semigroup and $j \in I$. Then $Cay(S, S_{\_j}) = \Gamma \cup (S, E_0)$ where $E_0 = \{(u, o) | \forall u \in S \setminus \{0\}\}$.*

proof. Clearly, $V(Cay(S, S_{\_j})) = V(\Gamma \cup (S, E_0))$ we will show that $E(Cay(S, S_{\_j})) = E(\Gamma \cup (S, E_0))$. Let (u, v) be an arc in $Cay(S, S_{\_j})$. Consider the following two cases.

(case1) If $v = 0$, then $(u, v) \in E_0$. Therefore (u, v) is an arc in $\Gamma \cup (S, E_0)$.

(case2) If $v \neq 0$, in view of Lemma 5.1.2, we get that $u \in S_{k\_}$ and $v \in S_{kj}$ for some $k \in I$. Therefore (u, v) is an arc in Γ_k and this implies that it is an arc in $\Gamma \cup (S, E_0)$.

By above two cases we conclude that $E(Cay(S, S_{\_j})) \subseteq E(\Gamma \cup (S, E_0))$.

Suppose that (u, v) is an arc in $\Gamma \cup (S, E_0)$. We consider the following two cases.

(case1) If $(u, v) \in E(\Gamma)$, then $(u, v) \in E(\Gamma_k)$ for some $k \in I$. Since Γ_k is a strong subdigraph of $Cay(S, S_{\_j})$, $(u, v) \in E(Cay(S, S_{\_j}))$.

(case2) If $(u, v) \in E_0$, we have $u \in S \setminus \{0\}$ and $v = 0$. By Lemma 5.1.1, we thus get $(u, v) \in E(Cay(S, S_{\_j}))$.

By above two cases we conclude that $E(\Gamma \cup (S, E_0)) \subseteq E(Cay(S, S_{\_j}))$. This shows that $E(Cay(S, S_{\_j})) = E(\Gamma \cup (S, E_0))$. Hence $Cay(S, S_{\_j}) = \Gamma \cup (S, E_0)$. □

The following theorem shows that Γ is isomorphic to a Cayley digraph of a rectangular group $Y = G \times L_m \times R_n$ where L_m is a left zero semigroup and R_n is a right zero semigroup such that $|m| = |n| = |I|$.

Theorem 5.1.8. *Let $S = B(G, I)$ be a Brandt semigroup, $S_{\_j}$ an $\mathcal{L}$-class of S for some $j \in I$, and Γ the disjoint union of isomorphic strong subdigraphs of $Cay(S, S_{\_j})$. Then Γ is a rectangular group digraph.*

proof. Let $Y = G \times L_m \times R_n$ be a rectangular group where $|m| = |n| = |I|$. Assume that $I = \{i_1, i_2, \ldots, i_n\}$, $L_m = \{l_1, l_2, \ldots, l_n\}$ and $R_n = \{r_1, r_2, \ldots, r_n\}$. Let $C = G \times L \times \{r_t\} \subseteq Y$ for some $r_t \in R$. For convenience, we suppose that $j = i_k$ for some $k \in \{1, 2, \ldots, n\}$. We define a map $f : V(\Gamma) \to Y$ by

$$f(i_p, g, i_q) = \begin{cases} (g, l_p, r_t) & \text{if } i_q = i_k; \\ (g, l_p, r_k) & \text{if } i_q = i_t; \\ (g, l_p, r_q) & \text{otherwise.} \end{cases}$$

Obviously, f is a bijection. We will show that f and f^{-1} are digraph homomorphisms. Let $u = (i_w, g_1, i_s), v = (i_z, g_2, i_t)$ be any vertices in Γ and (u, v) is an arc in Γ. By Lemma 2.4.5 and Corollary 5.1.3, $w = z$ and $v \in S_{i_w i_k}$. Thus $f(v) = f(i_z, g_2, i_t) = f(i_z, g_2, i_k) = (g_2, l_z, r_t)$. We only need to consider three cases.

(case1) If $i_s = i_k$, then

$$\begin{aligned} f(v) &= (g_2, l_z, r_t) \\ &= (g_1 g_1^{-1} g_2, l_z, r_t) \\ &= (g_1, l_z, r_t)(g_1^{-1} g_2, l_z, r_t) \\ &= (g_1, l_w, r_t)(g_1^{-1} g_2, l_z, r_t) \\ &= f(r_w, g_1, i_k)(g_1^{-1} g_2, l_z, r_t) \\ &= f(u)(g_1^{-1} g_2, l_z, r_t). \end{aligned}$$

Since $(g_1^{-1} g_2, l_z, r_t) \in C$, $\big(f(u), f(v)\big)$ is an arc in $Cay(Y, C)$.

(case2) If $i_s = i_t$, then

$$\begin{aligned} f(v) &= (g_2, l_z, r_t) \\ &= (g_1 g_1^{-1} g_2, l_z, r_t) \\ &= (g_1, l_z, r_k)(g_1^{-1} g_2, l_z, r_t) \\ &= (g_1, l_w, r_k)(g_1^{-1} g_2, l_z, r_t) \\ &= f(r_w, g_1, i_t)(g_1^{-1} g_2, l_z, r_t) \\ &= f(u)(g_1^{-1} g_2, l_z, r_t). \end{aligned}$$

Similarly to the case1, $\big(f(u), f(v)\big)$ is an arc in $Cay(Y, C)$.

(case3) If $i_k \neq i_s \neq i_m$, similarly to the above two cases, we conclude that $\big(f(u), f(v)\big)$ is an arc in $Cay(Y, C)$.

This means that f is a digraph homomorphism. Similarly, f^{-1} is a digraph homomorphism. Therefore $\Gamma \cong Cay(Y, C)$. □

Example 5.1.9. Let $S = B(\mathbb{Z}_3, I)$ be a Brandt semigroup, where $\mathbb{Z}_3 = \{\bar{0}, \bar{1}, \bar{2}\}$, $I = \{1, 2\}$.

By the definition of S_{-1}, we have $S_{-1} = \{(1, \bar{0}, 1), (1, \bar{1}, 1), (1, \bar{2}, 1), (2, \bar{0}, 1), (2, \bar{1}, 1), (2, \bar{2}, 1)\}$ is an $\mathcal{L}$-class of S. Then the strong subdigraph $\Gamma = \Gamma_1 \dot{\cup} \Gamma_2$ of $Cay(S, S_{-1})$ are shown in Figure 5.1 and $Cay(S, S_{-1}) = \Gamma \cup (S, E_0)$ see Figure 5.2.

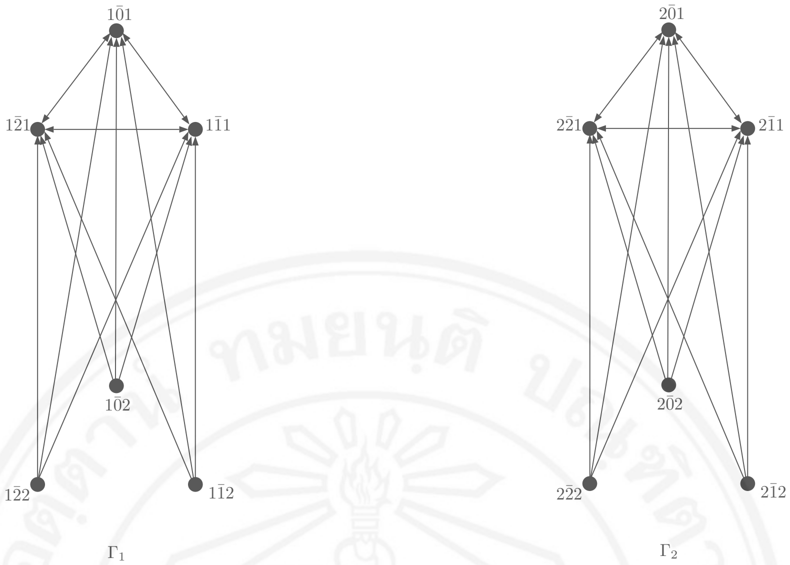

Figure 5.1: The strong subdigraph Γ of $Cay(B(\mathbb{Z}_3, \{1,2\}), B(\mathbb{Z}_3, \{1,2\})_{-1})$.

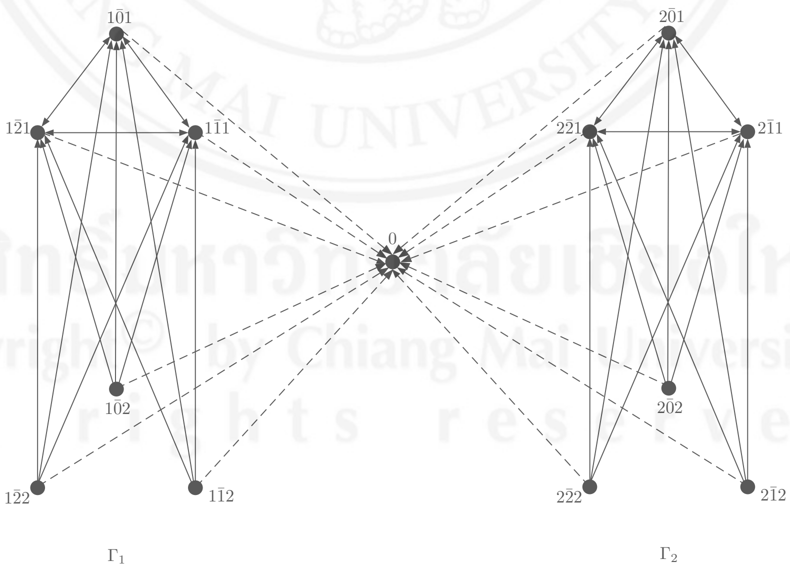

Figure 5.2: Cayley digraph $Cay(B(\mathbb{Z}_3, \{1,2\}), B(\mathbb{Z}_3, \{1,2\})_{-1})$.

Theorem 5.1.10. *Let $S = B(G, I)$ be a Brandt semigroup. Then $Cay(S, S_{\_i}) \cong Cay(S, S_{\_j})$ for all $i, j \in I$.*

proof. We define a map $f : S \to S$ by $f(0) = 0$ and

$$f(k,g,l) = \begin{cases} (k,g,j) & \text{if} \quad l = i; \\ (k,g,i) & \text{if} \quad l = j; \\ (k,g,l) & \text{otherwise.} \end{cases}$$

Obviously, f is a bijection. We will show that f and f^{-1} are digraph homomorphisms. Let $u, v \in S$ and (u, v) be an arc in $Cay(S, S_{\_i})$.

If $v = 0$, then $f(v) = 0$ and there is an arc from $f(u)$ to $f(v)$ by Lemma 5.1.1.

If $v \neq 0$, then we get that $u, v \in S_{k\_}$ for some $k \in I$. Since $\overrightarrow{d}(v) \neq 0$, $v \in S_{ki}$ by Corollary 5.1.3. Therefore $f(v) \in S_{kj}$ and $f(u) \in S_{k\_}$. By Lemma 5.1.2, $\big(f(u), f(v)\big)$ is an arc in $Cay(S, S_{\_j})$.

This means that f is a digraph homomorphism. Similarly, f^{-1} is a digraph homomorphism. Therefore $Cay(S, S_{\_i}) \cong Cay(S, S_{\_j})$. □

Example 5.1.11. Let $S = B(\mathbb{Z}_3, I)$ be a Brandt semigroup as in Example 5.1.9. By the definition of $S_{\_2}$, we have $S_{\_2} = \{(1,\bar{0},2), (1,\bar{1},2), (1,\bar{2},2), (2,\bar{0},2), (2,\bar{1},2), (2,\bar{2},2)\}$ is an $\mathcal{L}$-class of S. Consider Cayley digraph $Cay(S, S_{\_1})$ in Figure 5.2 and $Cay(S, S_{\_2})$ in Figure 5.3 It is easily seen that $Cay(S, S_{\_1}) \cong Cay(S, S_{\_2})$.

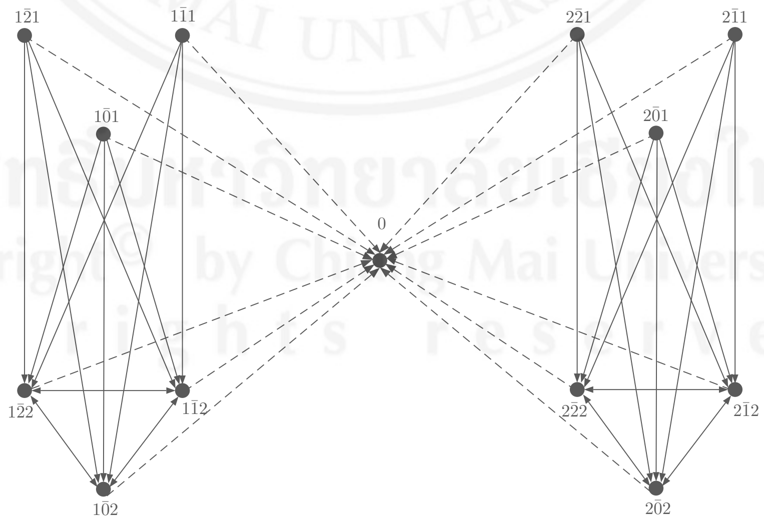

Figure 5.3: Cayley digraph $Cay(B(\mathbb{Z}_3, \{1,2\}), B(\mathbb{Z}_3, \{1,2\})_{\_2})$.

5.2 Cayley Digraphs of Brandt Semigroups Relative to $\mathcal{R}$-classes

In this section, we describe Cayley digraphs of a given Brandt semigroup S relative to the $\mathcal{R}$-class $S_{i\_}$.

Lemma 5.2.1. *Let $S = B(G, I)$ be a Brandt semigroup, $i \in I$, and $u, v \in S$. Then (u, v) is an arc in $Cay(S, S_{i\_})$ if and only if, for each $k \in I$, one of the following conditions hold:*

(1) $0 \neq u \notin S_{ki}$ *and* $v = 0$.

(2) $u \in S_{ki}$ *and* $v \in S_{k\_}$.

proof. ($\Rightarrow$) Let $u, v \in S$ and (u, v) be an arc in $Cay(S, S_{i\_})$. If $0 \neq u \notin S_{ki}$, then $u \in S_{kl}$ for some $l \neq i$. Let $u = (k, g, l)$ for some $g \in G$. Since, for all $(i, h, j) \in S_{i\_}$, $u(i, h, j) = (k, g, l)(i, h, j) = 0$, $v = 0$.

If $u \in S_{ki}$, it is easily seen that $v \neq 0$. Let $u = (k, g_1, i)$ and $v = (m, g_2, n)$ for some $g_1, g_2 \in G$ and $m, n \in I$. By Lemma 2.4.5, $k = m$. Hence $v = (k, g_2, n) \in S_{k\_}$.

($\Leftarrow$) If (1) holds, then $u = (k, g, l)$ for some $l \neq i$. By Lemma 2.4.6(2), $((k, g, l), 0)$ is an arc in $Cay(S, S_{i\_})$. Then (u, v) is an arc in $Cay(S, S_{i\_})$. If (2) holds, let $u = (k, g_1, i) \in S_{ki}$ and $v = (k, g_2, j) \in S_{k\_}$ for some $g_1, g_1 \in G$ and $j \in I$, then there exists $a = (i, g_1^{-1}g_2, j) \in S_{i\_}$ such that $ua = (k, g_1, i)(i, g_1^{-1}g_2, j) = (k, g_2, j) = v$. Hence (u, v) is an arc in $Cay(S, S_{i\_})$. □

From the above lemma the following corollaries are immediate.

Corollary 5.2.2. *Let $S = B(G, I)$ be a Brandt semigroup and $i \in I$. Then $\overrightarrow{d}(v) \neq 0$ for all the vertices v in $Cay(S, S_{i\_})$.*

Corollary 5.2.3. *Let $S = B(G, I)$ be a Brandt semigroup and $i, k \in I$. Then the strong subdigraph (S_{ki}, E_{ki}) of $Cay(S, S_{i\_})$ is a complete digraph $(K_{|G|})$.*

proof. Let u, v be any vertices in S_{ki}. By Lemma 5.2.1(2), we get that the both (u, v) and (v, u) are arcs in $Cay(S, S_{i\_})$. It follows that $(u, v), (v, u) \in E_{ki}$ for any $u, v \in S_{ki}$. Since $|S_{ki}| = |G|$, (S_{ki}, E_{ki}) is a complete digraph $(K_{|G|})$. □

Theorem 5.2.4. *Let $S = B(G, I)$ be a Brandt semigroup. Then $Cay(S, S_{i\_}) \cong Cay(S, S_{j\_})$ for all $i, j \in I$.*

proof. Let $i, j \in I$. We define a map $f : S \to S$ by $f(0) = 0$ and

$$f(k,g,l) = \begin{cases} (k,g,j) & \text{if} \quad l = i; \\ (k,g,i) & \text{if} \quad l = j; \\ (k,g,l) & \text{otherwise.} \end{cases}$$

Obviously, f is a bijection. We will show that f and f^{-1} are digraph homomorphisms. Let $u, v \in S$ and (u, v) be an arc in $Cay(S, S_{i\_})$.

If $v = 0$, then $0 \neq u \notin S_{ki}$ for each $k \in I$ by Lemma 5.2.1(1). It follows that $f(v) = 0$ and $0 \neq f(u) \notin S_{kj}$ and so $\big(f(u), f(v)\big)$ is an arc in $Cay(S, S_{j\_})$ by Lemma 5.2.1(1).

If $v \neq 0$, then $u, v \in S_{k\_}$ for some $k \in I$. By Lemma 5.2.1(2), $u \in S_{ki}$. Therefore $f(u) \in S_{kj}$ and $f(v) \in S_{k\_}$. Hence $\big(f(u), f(v)\big)$ is an arc in $Cay(S, S_{j\_})$ by Lemma 5.2.1(2).

This means that f is a digraph homomorphism. Similarly, f^{-1} is a digraph homomorphism. Therefore $Cay(S, S_{i\_}) \cong Cay(S, S_{j\_})$. □

Example 5.2.5. Let $S = B(\mathbb{Z}_3, I)$ be a Brandt semigroup as in Example 5.1.9. By the definition of $S_{1\_}$ and $S_{2\_}$, we have $S_{1\_} = \{(1,\bar{0},1), (1,\bar{1},1), (1,\bar{2},1), (1,\bar{0},2), (1,\bar{1},2), (1,\bar{2},2)\}$ and $S_{2\_} = \{(2,\bar{0},1), (2,\bar{1},1), (2,\bar{2},1), (2,\bar{0},2), (2,\bar{1},2), (2,\bar{2},2)\}$ are $\mathcal{R}$-classes of S. It is easily seen that $Cay(S, S_{1\_}) \cong Cay(S, S_{2\_})$ (see Figures 5.4 and 5.5).

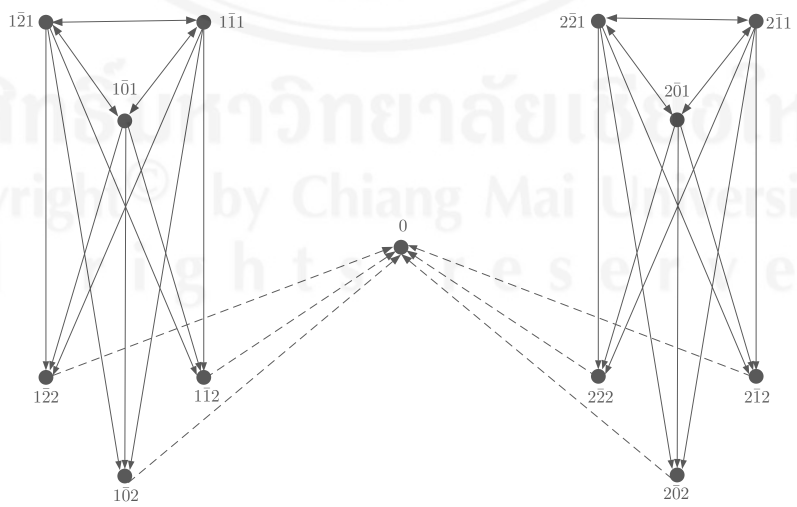

Figure 5.4: Cayley digraph $Cay(B(\mathbb{Z}_3, \{1,2\}), B(\mathbb{Z}_3, \{1,2\})_{1\_})$.

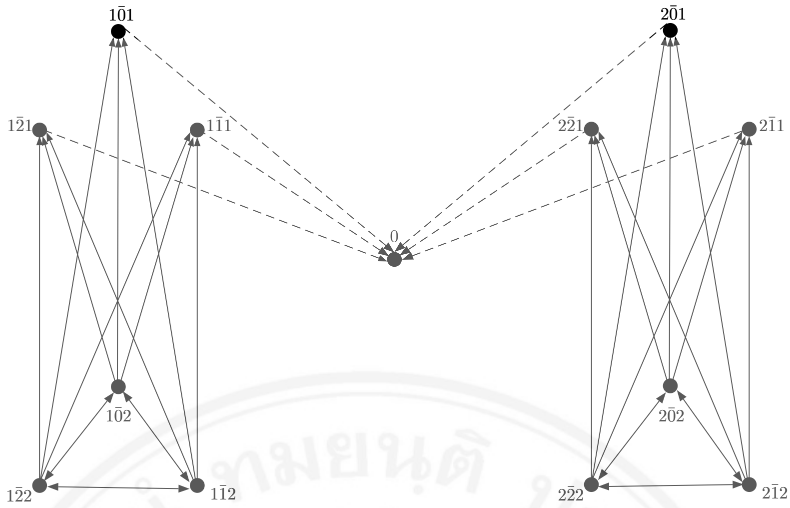

Figure 5.5: Cayley digraph $Cay(B(\mathbb{Z}_3, \{1, 2\}), B(\mathbb{Z}_3, \{1, 2\})_{2_-})$.

For each $i \in I$, we let $D_i = S_{-i} \cup S_{i-}$, where S_{-i} and S_{i-} are an $\mathcal{L}$-class and an $\mathcal{R}$-class of S, respectively. The next result shows that a strong subdigraph (S_{k_-}, E_k) of $Cay(S, D_i)$ is undirected.

Theorem 5.2.6. *Let $S = B(G, I)$ be a Brandt semigroup and $i \in I$. Then the strong subdigraph (S_{k_-}, E_k) of $Cay(S, D_i)$ is undirected for all $k \in I$.*

proof. For each $k \in I$, let $u, v \in S_{k_-}$ and let (u, v) be an arc in (S_{k_-}, E_k). Hence it is an arc in $Cay(S, D_i)$. By Lemma 5.1.2 and Lemma 5.2.1 we only need to consider two cases.

(case1) If $u \in S_{k_-}$ and $v \in S_{ki}$, then we assume that $u = (k, g, l)$ and $v = (k, h, i)$ for some $g, h \in G, l \in I$. There is $a = (i, h^{-1}g, l) \in S_{i_-} \subseteq D_i$ such that $va = (k, h, i)(i, h^{-1}g, l) = (k, g, l) = u$. Then (v, u) is an arc in $Cay(S, D_i)$. Since $u, v \in S_{k_-}$, (v,u) is an arc in the (S_{k_-}, E_k).

(case2) If $u \in S_{ki}$ and $v \in S_{k_-}$, then we assume that $u = (k, g, i)$ and $v = (k, h, l)$ for some $g, h \in G, l \in I$. There is $a = (l, h^{-1}g, i) \in S_{-i} \subseteq D_i$ such that $va = (k, h, l)(l, h^{-1}g, i) = (k, g, i) = u$. Then (v, u) is an arc in $Cay(S, D_i)$. Since $u, v \in S_{k_-}$, (v,u) is an arc in the (S_{k_-}, E_k).

We conclude that the strong subdigraph (S_{k_-}, E_k) of $Cay(S, D_i)$ is undirected. □

From Theorem 5.2.6, we have the strong subdigraph $(S \backslash \{0\}, E)$ of $Cay(S, D_i)$ is also undirected, because (S_{k_-}, E_k) and (S_{l_-}, E_l) are disjoint strong subdigraphs of $Cay(S, D_i)$.

Theorem 5.2.7. *Let $S = B(G, I)$ be a Brandt semigroup and $i \in I$. Then the strong subdigraph (S_{k_-}, E_k) of $Cay(S, D_i)$ is a complete n-partite digraph where $n = |G| + 1$ for all $k \in I$.*

proof. Assume that $G = \{g_1, g_2, \ldots, g_m\}$. Let $V_1 = \{(k, g_1, i)\}, V_2 = \{(k, g_2, i)\}, \ldots, V_m = \{(k, g_m, i)\}$ and $V_{m+1} = \{(k, g, j)|$ for all $g \in G, j \in I$ such that $j \neq i\}$. We will show that there is no arc between vertices in V_{m+1}.

Let $(k, g, j), (k, g', j') \in V_{m+1}$ and assume that $\big((k, g, j), (k, g', j')\big)$ is an arc in (S_{k_-}, E_k). Then $\big((k, g, j), (k, g', j')\big)$ is an arc in $Cay(S, D_i)$. There is $(l, h, q) \in D_i$ such that $(k, g, j)(l, h, q) = (k, g', j')$, so we have $l = j$ and $q = j'$. Since $(l, h, q) \in D_i$, $l = i$ or $q = i$. There is a contradiction because $l = j \neq i$ and $q = j' \neq i$. This means that there is no arc between vertices in V_{m+1}.

By Corollary 5.2.3, there is an arc between V_c and V_d for all $c \neq d$ in {1,2,…,m}. The following, we prove that there is an arc from all of the vertices in V_{m+1} to a vertex in V_t for $t = 1, 2, \ldots, m$.

Let $(k, g, j) \in V_{m+1}$ and $(k, g_t, i) \in V_t$. There is $(j, g^{-1}g_t, i) \in D_i$ such that $(k, g, j)(j, g^{-1}g_t, i) = (k, g_t, i)$. Then we have that $\big((k, g, j), (k, g_t, i)\big)$ is an arc in $Cay(S, D_i)$. It follows that there is an arc from any vertices in V_{m+1} to the vertex in V_t. Similarly, we can show that there is an arc from the vertex in V_t to all of vertices in V_{m+1}. We conclude that the strong subdigraph (S_{k_-}, E_k) of $Cay(S, D_i)$ is a complete n-partite digraph where $n = m + 1$. □

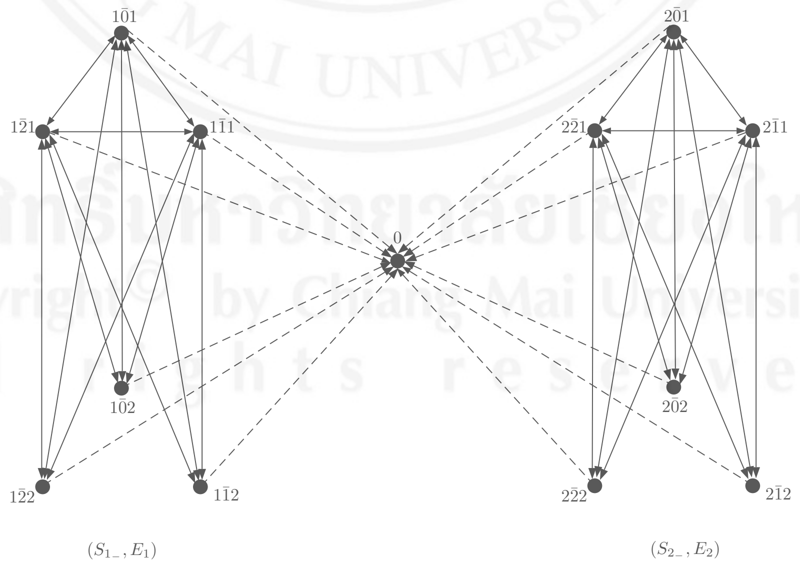

Figure 5.6: Cayley digraph $Cay\big(B(\mathbb{Z}_3, \{1, 2\}), B(\mathbb{Z}_3, \{1, 2\})_{-1} \cup B(\mathbb{Z}_3, \{1, 2\})_{1_-}\big)$.

Example 5.2.8. Let $S = B(\mathbb{Z}_3, I)$ be a Brandt semigroup as in Example 5.1.9. Then $D_1 = S_{-1} \cup S_{1-} = \{(1, \bar{0}, 1), (1, \bar{1}, 1), (1, \bar{2}, 1), (2, \bar{0}, 1), (2, \bar{1}, 1), (2, \bar{2}, 1)(1, \bar{0}, 2), (1, \bar{1}, 2), (1, \bar{2}, 2)\}$. The Cayley digraph $Cay(S, D_i)$ is shown in Figure 5.6.

The set of vertices of the strong subdigraph (S_{1-}, E_1) is partitioned into four disjoint subsets as follows: $V_1 = \{(1, \bar{0}, 1)\}$, $V_2 = \{(1, \bar{1}, 1)\}$, $V_3 = \{(1, \bar{2}, 1)\}$ and $V_4 = \{(1, \bar{0}, 2), (1, \bar{1}, 2), (1, \bar{2}, 2)\}$. It is easily seen that there is no arc between two vertices in one subset. Therefore the strong subdigraph (S_{1-}, E_1) is a completely 4-partite digraph. Similarly, the strong subdigraph (S_{2-}, E_2) is also a completely 4-partite digraph.

5.3 Cayley Digraphs of Brandt Semigroups Relative to $\mathcal{H}$-classes

In this section, we introduce the conditions for Cayley digraphs of a given Brandt semigroup S relative to the $\mathcal{H}$-class to be isomorphic to each other.

Lemma 5.3.1. *Let $S = B(G, I)$ be a Brandt semigroup, $i, j \in I$, and $u, v \in S$. Then (u, v) is an arc in $Cay(S, S_{ij})$ if and only if, for each $k \in I$, one of the following conditions hold:*

(1) $0 \neq u \notin S_{ki}$ *and* $v = 0$.

(2) $u \in S_{ki}$ *and* $v \in S_{kj}$.

proof. ($\Rightarrow$) Let $u, v \in S$ and (u, v) be an arc in $Cay(S, S_{ij})$. If $0 \neq u \notin S_{ki}$, then $u \in S_{kl}$ for some $l \neq i$. Let $u = (k, g, l)$ for some $g \in G$. Since, for all $(i, h, j) \in S_{ij}$, $u(i, h, j) = (k, g, l)(i, h, j) = 0$, $v = 0$.

If $u \in S_{ki}$, it is easily seen that $v \neq 0$. Let $u = (k, g_1, i)$ and $v = (m, g_2, n)$ for some $g_1, g_2 \in G$ and $m, n \in I$. By Lemma 2.4.5, $k = m$ and $(i, g_1^{-1}g_2, n) \in S_{ij}$. Hence $n = j$. Therefore $v = (k, g_2, j) \in S_{kj}$.

($\Leftarrow$) If (1) holds, then $u = (k, g, l)$ for some $l \neq i$. By Lemma 2.4.6(2), $((k, g, l), 0)$ is an arc in $Cay(S, S_{ij})$. Then (u, v) is an arc in $Cay(S, S_{ij})$. If (2) holds, let $u = (k, g_1, i) \in S_{ki}$ and $v = (k, g_2, j) \in S_{kj}$ for some $g_1, g_2 \in G$, then there exists $a = (i, g_1^{-1}g_2, j) \in S_{ij}$ such that $ua = (k, g_1, i)(i, g_1^{-1}g_2, j) = (k, g_2, j) = v$. Hence (u, v) is an arc in $Cay(S, S_{ij})$. □

By the above lemma, the following corollary is immediate.

Corollary 5.3.2. *Let $S = B(G, I)$ be a Brandt semigroup, $i, j, k \in I$, and $v \in S_{k-}$ be a vertex in $Cay(S, S_{ij})$. Then $\overrightarrow{d}(v) = 0$ if and only if $v \notin S_{kj}$.*

From (2) of Lemma 5.3.1, the next result shows that when the strong subdigraph (S_{kj}, E_{kj}) of $Cay(S, S_{ij})$ is a complete digraph.

Lemma 5.3.3. *Let $S = B(G, I)$ be a Brandt semigroup and $i, j, k \in I$. Then the strong subdigraph (S_{kj}, E_{kj}) of $Cay(S, S_{ij})$ is a complete digraph $(K_{|G|})$ if and only if $i = j$.*

proof. $(\Rightarrow)$ Let u and v be any vertices in the strong subdigraph (S_{kj}, E_{kj}). Since (S_{kj}, E_{kj}) is a complete digraph, there is an arc from u to v in $Cay(S, S_{ij})$. By (2) of Lemma 5.3.1, $u \in S_{ki}$. Therefore $i = j$.

$(\Leftarrow)$ Let $i = j$ and u, v be any vertices in the strong subdigraph (S_{kj}, E_{kj}). It follows that $u, v \in S_{ki}$. There is an arc between u and v by (2) of Lemma 5.3.1. This means that (S_{kj}, E_{kj}) is a complete digraph. Because $|S_{kj}| = |G|$, then the strong subdigraph (S_{kj}, E_{kj}) is a complete digraph $K_{|G|}$. □

5.4 Isomorphism Conditions for Cayley Digraphs of Brandt Semigroups Relative to $\mathcal{H}$-classes

In this section, we introduce the conditions for Cayley digraphs of a given Brandt semigroup S relative to $\mathcal{H}$-class to be isomorphic to each other.

Theorem 5.4.1. *Let $S = B(G, I)$ be a Brandt semigroup and $i, j, l, m \in I$. Then $Cay(S, S_{ij}) \cong Cay(S, S_{lm})$ if and only if one of the following conditions hold:*

(1) If $i = j$ then $l = m$.

(2) If $i \neq j$ then $l \neq m$.

proof. $(\Rightarrow)$ Let $Cay(S, S_{ij}) \cong Cay(S, S_{lm})$. (1) Let $i = j$. By Lemma 5.3.3, a strong subdigraph (S_{kj}, E_{kj}) of $Cay(S, S_{ij})$ is a complete digraph $K_{|G|}$ for all $k \in I$. Hence there is a complete strong subdigraph $K_{|G|}$ of $Cay(S, S_{lm})$. Assume that $l \neq m$. By Lemma 5.3.3, a strong subdigraph (S_{km}, E_{km}) of $Cay(S, S_{lm})$ is not a complete digraph. Then there exists a vertex u of the complete strong subdigraph $K_{|G|}$ of $Cay(S, S_{lm})$ and $u \notin S_{km}$. Also since u is a vertex of the complete digraph $K_{|G|}$, $\overrightarrow{d}(u) \neq 0$. By Corollary 5.3.2, $u \in S_{km}$. That is a contradiction and thus $l = m$.

(2) Similarly, if $i \neq j$ then $l \neq m$.

($\Leftarrow$) If (2) holds, we define a map $f : S \to S$ by $f(0) = 0$ and

$$f(k,g,r) = \begin{cases} (k,g,l) & \text{if} \quad r = i; \\ (k,g,m) & \text{if} \quad r = j; \\ (k,g,i) & \text{if} \quad r = l; \\ (k,g,j) & \text{if} \quad r = m; \\ (k,g,r) & \text{otherwise.} \end{cases}$$

Since $i \neq j$ and $l \neq m$, f is a bijection. We will show that f and f^{-1} are digraph homomorphisms. Let $u, v \in S$ and (u,v) is an arc in $Cay(S, S_{ij})$. By Lemma 5.3.1, we need only consider two cases.

If $v = 0$, then $0 \neq u \notin S_{ki}$ for each $k \in I$, it follows that $f(v) = 0$ and $0 \neq f(u) \notin S_{kl}$. By Lemma 5.3.1(1), $\big(f(u), f(v)\big)$ is an arc in $Cay(S, S_{lm})$.

If $v \neq 0$, then $u \in S_{ki}$ and $v \in S_{kj}$ for each $k \in I$. Hence $f(u) \in S_{kl}$ and $f(v) \in S_{km}$. By Lemma 5.3.1(2), $\big(f(u), f(v)\big)$ is an arc in $Cay(S, S_{lm})$.

This means that f is a digraph homomorphism. Similarly, f^{-1} is a digraph homomorphism. Therefore $Cay(S, S_{ij}) \cong Cay(S, S_{lm})$.

If (1) holds, then $S_{ij} = S_{ii}$ and $S_{lm} = S_{ll}$. We define a map $h : S \to S$ by $h(0) = 0$ and

$$h(k,g,r) = \begin{cases} (k,g,l) & \text{if} \quad r = i; \\ (k,g,i) & \text{if} \quad r = l; \\ (k,g,r) & \text{otherwise.} \end{cases}$$

Obviously, h is a bijection. With a similarly argument of the proof of (2) holds, we can show that h and h^{-1} are digraph homomorphisms. Hence $Cay(S, S_{ij}) \cong Cay(S, S_{lm})$. $\square$

Example 5.4.2. Let $S = B(\mathbb{Z}_3, I)$ be a Brandt semigroup as in Example 5.1.9. By the definition of S_{12}, S_{21}, S_{11} and S_{22}, we have $S_{12} = \{(1,\bar{0},2), (1,\bar{1},2), (1,\bar{2},2)\}$, $S_{21} = \{(2,\bar{0},1), (2,\bar{1},1), (2,\bar{2},1)\}$, $S_{11} = \{(1,\bar{0},1), (1,\bar{1},1), (1,\bar{2},1)\}$ and $S_{22} = \{(2,\bar{0},2), (2,\bar{1},2), (2,\bar{2},2)\}$ are $\mathcal{H}$-classes of S.

We see that $Cay(S, S_{11}) \cong Cay(S, S_{22}) \ncong Cay(S, S_{12}) \cong Cay(S, S_{21})$ (see in Figures 5.7 - 5.10).

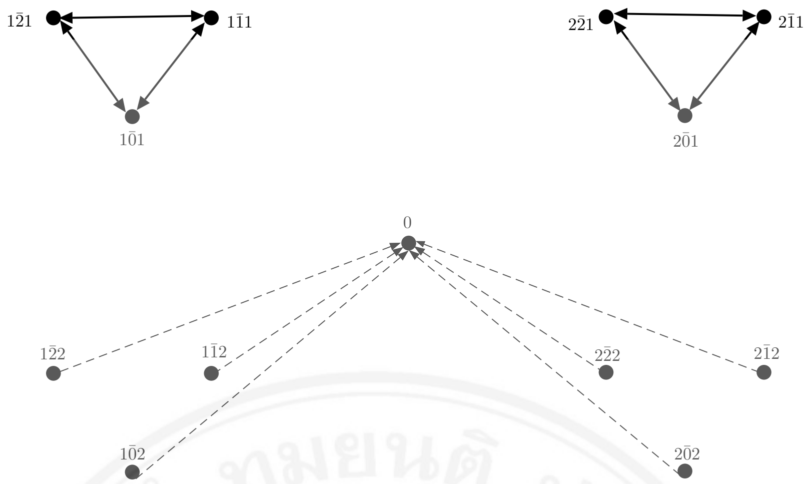

Figure 5.7: Cayley digraph $Cay(B(\mathbb{Z}_3,\{1,2\}),B(\mathbb{Z}_3,\{1,2\})_{11})$.

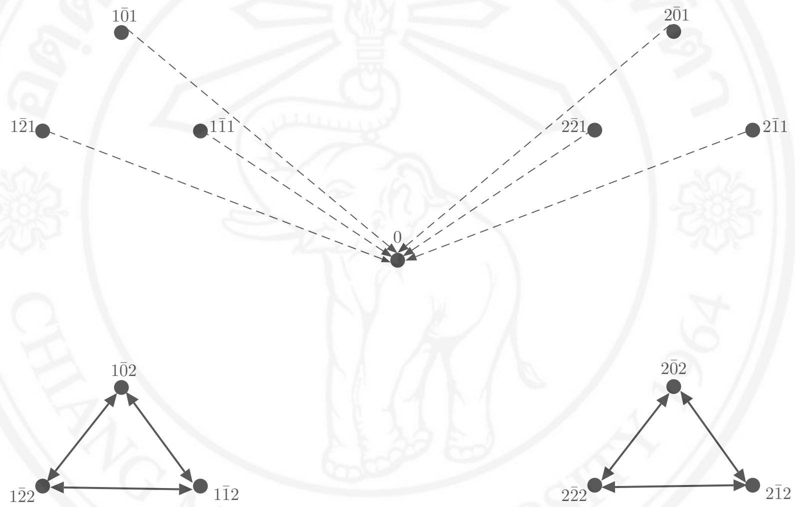

Figure 5.8: Cayley digraph $Cay(B(\mathbb{Z}_3,\{1,2\}),B(\mathbb{Z}_3,\{1,2\})_{22})$.

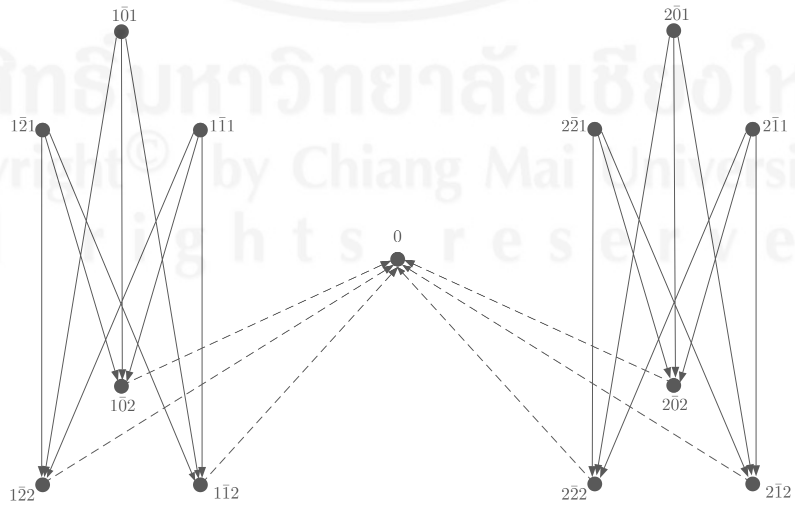

Figure 5.9: Cayley digraph $Cay(B(\mathbb{Z}_3,\{1,2\}),B(\mathbb{Z}_3,\{1,2\})_{12})$.

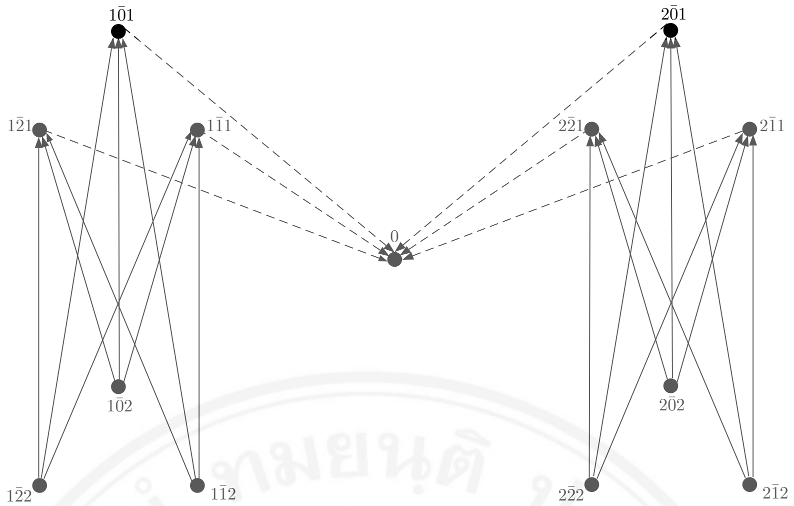

Figure 5.10: Cayley digraph $Cay(B(\mathbb{Z}_3, \{1, 2\}), B(\mathbb{Z}_3, \{1, 2\})_{21})$.

Next, we shall give the necessary conditions for Cayley digraphs of a Brandt semigroup are isomorphic. We begin with the following lemma.

Lemma 5.4.3. *Let $S = B(G, I)$ be a Brandt semigroup, $(i, g, j) \in S$, and v a nonzero vertex of $Cay(S, \{(i, g, j)\})$. Then $\overrightarrow{d}(v) \neq 0$ if and only if $v \in S_{kj}$ for all $k \in I$.*

proof. ($\Rightarrow$) For all $k \in I$, let $v = (k, h, l) \in S$ and $\overrightarrow{d}(v) \neq 0$. By Lemma 2.4.5, $l = j$. Hence $v = (k, h, j) \in S_{kj}$.

($\Leftarrow$) Assume that $v = (k, h, j) \in S_{kj}$ for all $k \in I$, then there is $(k, hg^{-1}, i) \in S$ such that $(k, hg^{-1}, i)(i, g, j) = (k, h, j) = v$. This means that $\overrightarrow{d}(v) \neq 0$. □

The next lemma shows the number of nonzero vertices with nonzero in-degree in $Cay(S, A)$.

Lemma 5.4.4. *Let $S = B(G, I)$ be a Brandt semigroup, A a nonempty subset of S, and $L_A = \{j|(i, g, j) \in A\}$. Then $\cup_{k\in I, j\in L_A} S_{kj}$ is a set of all nonzero vertices of $Cay(S, A)$ with nonzero indegree and $\left|\cup_{k\in I, j\in L_A} S_{kj}\right| = |I||L_A||G|$.*

proof. Let $v \in \cup_{k\in I, j\in L_A} S_{kj}$. Hence $v \in S_{im}$ for some $i \in I, m \in L_A$ and also there is $(l, h, m) \in A$. By Lemma 5.4.3, $\overrightarrow{d}(v) \neq 0$ this means that $\cup_{k\in I, j\in L_A} S_{kj}$ is a set of vertices of $Cay(S, A)$ with nonzero indegree. Let $0 \neq u \in S$ and $\overrightarrow{d}(u) \neq 0$, by Lemma 5.4.3, $u \in S_{kj}$ for some $k \in I, j \in L_A$ it follows that $u \in \cup_{k\in I, j\in L_A} S_{kj}$. Moreover, we have $\left|\cup_{k\in I, j\in L_A} S_{kj}\right| = |I||L_A||G|$, because $|S_{kj}| = |G|$. □

Lemma 5.4.5. *Let $S = B(G, I)$ be a Brandt semigroup, $(i, g, j) \in S$, and u a vertex of $Cay(S, \{(i, g, j)\})$. Then there is an arc from u to a nonzero vertex if and only if $u \in S_{ki}$ for all $k \in I$.*

proof. ($\Rightarrow$) Let $u = (k, h, m)$ for all $k \in I$, and assume that there is an arc from u to a nonzero vertex. By Lemma 2.4.5, $i = m$. Hence $u = (k, h, i) \in S_{ki}$.

($\Leftarrow$) Let $u \in S_{ki}$ for all $k \in I$. Assume that $u = (k, h, i)$, there is $0 \neq (k, hg, j) \in S$ such that $u(i, g, j) = (k, h, i)(i, g, j) = (k, hg, j)$. Hence there is an arc from u to a nonzero vertex. □

The next lemma shows the number of vertices which adjacent to a nonzero vertices in $Cay(S, A)$.

Lemma 5.4.6. *Let $S = B(G, I)$ be a Brandt semigroup, A a nonempty subset of S, and $R_A = \{i | (i, g, j) \in A\}$. Then $\cup_{k \in I, i \in R_A} S_{ki}$ is a set of all vertices of $Cay(S, A)$ with, for all $u \in \cup_{k \in I, i \in R_A} S_{ki}$, there exists a nonzero vertex $v \in S$ such that (u, v) is an arc in $Cay(S, A)$ and $\left| \cup_{k \in I, i \in R_A} S_{ki} \right| = |I||R_A||G|$.*

proof. Let $v \in \cup_{k \in I, i \in R_A} S_{ki}$. Hence $v \in S_{ki}$ for some $k \in I, i \in R_A$ and it implies that there is $(i, h, l) \in A$. By Lemma 5.4.5, there is an arc from v to a nonzero vertex. Let $u, v \in S, v \neq 0$ and (u, v) be an arc in $Cay(S, A)$. By Lemma 5.4.5, $u \in S_{ki}$ for some $k \in I, i \in R_A$ it follows that $u \in \cup_{k \in I, i \in R_A} S_{ki}$. Moreover, we have $\left| \cup_{k \in I, i \in R_A} S_{ki} \right| = |I||R_A||G|$ because $|S_{ki}| = |G|$. □

The following theorem gives the necessary conditions for Cayley digraphs of a given Brandt semigroup S to be isomorphic to each other.

Theorem 5.4.7. *Let $S = B(G, I)$ be a Brandt semigroup, A, B nonempty subsets of S, $L_A = \{j | (i, g, j) \in A\}$, $L_B = \{j | (i, g, j) \in B\}$, $R_A = \{i | (i, g, j) \in A\}$, and $R_B = \{i | (i, g, j) \in B\}$. If $Cay(S, A) \cong Cay(S, B)$, then $|L_A| = |L_B|$ and $|R_A| = |R_B|$.*

proof. Let $Cay(S, A) \cong Cay(S, B)$. We have that the numbers of nonzero vertices with nonzero indegree of $Cay(S, A)$ and $Cay(S, B)$ are equal. By Lemma 5.4.4, $|I||L_A||G| = |I||L_B||G|$, it follows that $|L_A| = |L_B|$. Similarly, the numbers of vertices which have an arc from them to a nonzero vertices of both $Cay(S, A)$ and $Cay(S, B)$ are equal. By Lemma 5.4.6, $|I||R_A||G| = |I||R_B||G|$, it follows that $|R_A| = |R_B|$. □

Example 5.4.8. Let $S = B(\mathbb{Z}_3, I)$ be a Brandt semigroup as in Example 5.1.9. Let $A = \{(1, \bar{1}, 1), (1, \bar{2}, 2)\}$, $B = \{(2, \bar{1}, 2), (2, \bar{2}, 1)\}$ be subsets of S. Then we have $L_A = \{1, 2\}$, $R_A = \{1\}$, $L_B = \{1, 2\}$ and $R_B = \{2\}$. It is easily seen that $|L_A| = 2 = |L_B|$, $|R_A| = 1 = |R_B|$ and $Cay(S, A) \cong Cay(S, B)$. (see in Figures 5.11 and 5.12).

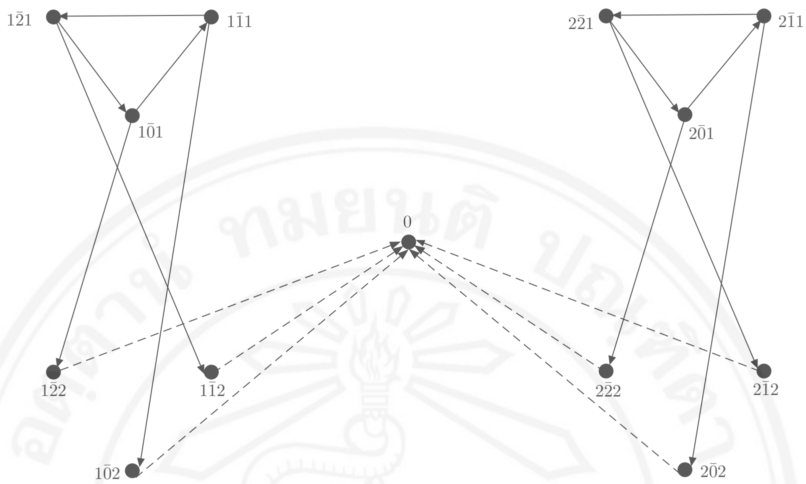

Figure 5.11: Cayley digraph $Cay(B(\mathbb{Z}_3, \{1, 2\}), \{(1, \bar{1}, 1), (1, \bar{2}, 2)\})$.

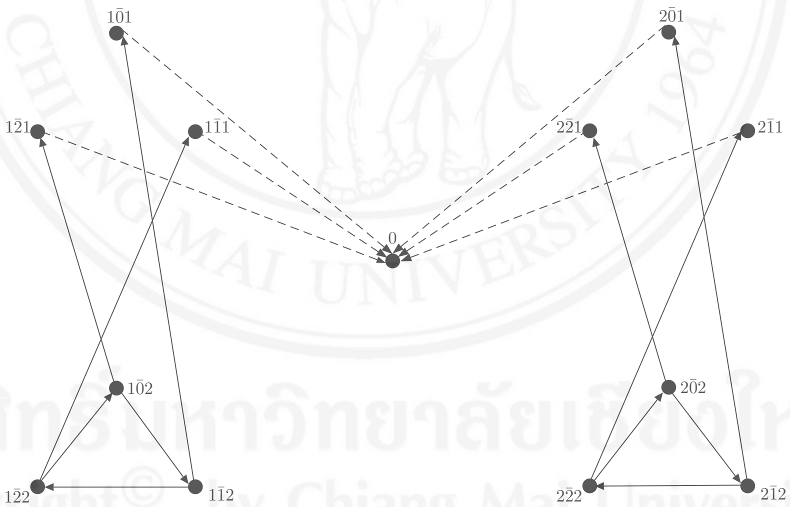

Figure 5.12: Cayley digraph $Cay(B(\mathbb{Z}_3, \{1, 2\}), \{(2, \bar{1}, 2), (2, \bar{2}, 1)\})$.

The converse of Theorem 5.4.7 does not hold. For example, let $A' = \{(1, \bar{0}, 1), (1, \bar{2}, 2)\}$, $B' = \{(1, \bar{1}, 1), (1, \bar{0}, 2)\}$ be subsets of S. Then we have $L_{A'} = \{1, 2\}$, $R_{A'} = \{1\}$, $L_{B'} = \{1, 2\}$ and $R_{B'} = \{1\}$. Therefore $|L_{A'}| = 2 = |L_{B'}|$ and $|R_{A'}| = 1 = |R_{B'}|$, but $Cay(S, A') \ncong Cay(S, B')$ (see Figures 5.13 and 5.14).

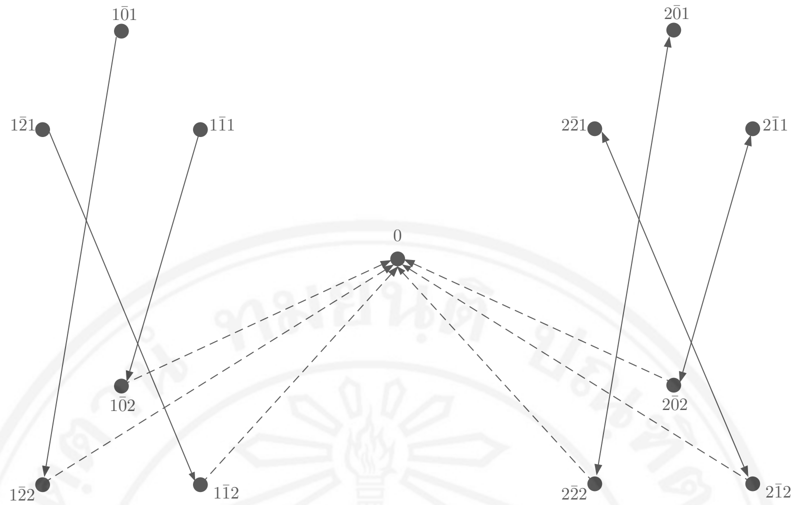

Figure 5.13: Cayley digraph $Cay(B(\mathbb{Z}_3, \{1, 2\}), \{(1, \bar{0}, 1), (1, \bar{2}, 2)\})$.

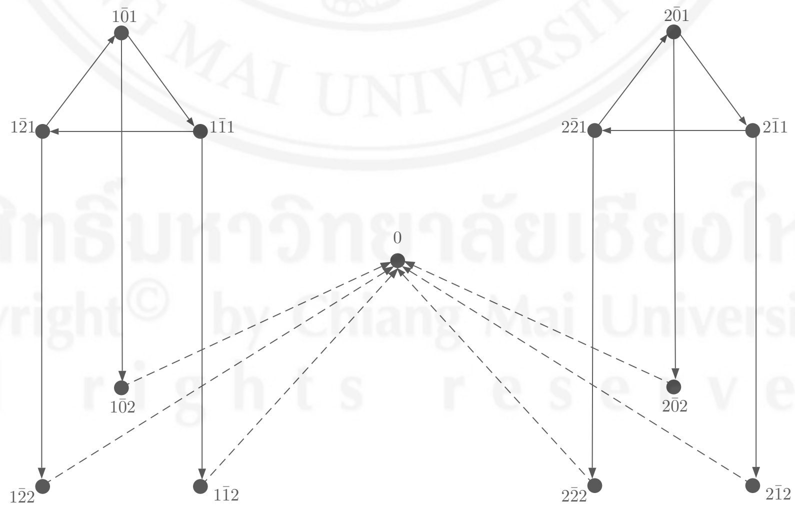

Figure 5.14: Cayley digraph $Cay(B(\mathbb{Z}_3, \{1, 2\}), \{(1, \bar{1}, 1), (1, \bar{0}, 2)\})$.